Solutions to the Math123 Exam, Summer 2007-2008

Question 1 [18 marks]

(a) Without using a calculator evaluate \( \frac{1}{4} - \frac{1}{2} \div \frac{3}{2} \).

Solution:
\[
\frac{1}{4} - \frac{1}{2} \div \frac{3}{2} = -\frac{1}{4} - \frac{1}{2} = -\frac{1}{2} = -0.5.
\]

(b) Calculate \( 2.134 \times 10^3 + 4.238 \times 10^{-3} \) and convert your answer to the nearest whole number.

Solution:
\[
2.134 \times 10^3 + 4.238 \times 10^{-3} = 2134 + 0.004238 \approx 2134.
\]

(c) One cubic metre of water weighs 1000 kg.

(i) What is the weight of one cubic centimetre of water?

Solution:
\[
1\text{cm} = 0.01\text{m},\ 1\text{cm}^3 = 0.01^3\text{m}^3 = 0.000001\text{m}^3.
\]

Hence 1 cm\(^3\) weighs 0.000001 \times 1000 = 0.001 kg.

(ii) If a water bottle weighs 0.45 kg, with 0.03 kg for the weight of the plastic bottle itself, how many cubic centimetres of water is contained in the bottle?

Solution
\[
(0.45 - 0.03)/0.001 = 420\text{cm}^3.
\]

The water bottle contains 420 cubic centimetres of water.

(d) The cost of an item after adding 10\% for GST is $10.99.

(i) What was the original price of the item?

Solution: Original price = $10.99/(1 + 0.1) = 9.99 dollars.

(ii) In order to collect $1000 of GST money from selling this particular item, how many must be sold?

Solution GST per item = $10.99 - 9.99 = 1 dollar. Hence 1000 of the item should be sold in order to collect $1000 of GST money.
Question 2 [16 marks]

(a) Expand (multiply out) \((3x^2 - 2x + 1)(2x^2 - 3)\).

**Solution:**
\[(3x^2 - 2x + 1)(2x^2 - 3) = 6x^4 - 9x^2 - 4x^3 + 6x + 2x^2 - 3 = 6x^4 - 4x^3 - 7x^2 + 6x - 3.\]

(b) Solve the following equations for \(x\):

(i) \((x + 1)(x + 2) = (x + 3)(x + 4)\).

**Solution:** The given equation is equivalent to
\[x^2 + 3x + 2 = x^2 + 7x + 12, \text{ or } 4x + 10 = 0.\]
Hence \(x = -10/4 = -2.5\).

(ii) \(\frac{1}{1+x} + \frac{1}{1-x} = 3\).

**Solution:** The given equation is equivalent to
\[\frac{1 - x + 1 + x}{(1+x)(1-x)} = 3, \text{ or } \frac{2}{1 - x^2} = 3.\]
Hence
\[3 - 3x^2 = 2, 3x^2 = 1, x^2 = \frac{1}{3}.\]
There are two solutions: \(x = \frac{1}{\sqrt{3}}\) and \(x = -\frac{1}{\sqrt{3}}\).

(c) The pressure, \(P\), of a gas at constant temperature is inversely proportional to its volume, \(V\), namely
\[P = \frac{k}{V}, \text{ } k \text{ is some positive constant.}\]

The gas has a volume of 8 cc at pressure 100 Newtons.

(i) What is the volume of the gas at pressure 62 Newtons?

**Solution:** \(100 = k/8\). Hence \(k = 100 \times 8 = 800\), and \(P = 800/V\).
When \(P = 62\), we have \(62 = 800/V\) and \(V = 800/62 = 12.90\) cc.

(ii) What is the pressure of the gas when its volume is 40 cc?

**Solution:** When \(V = 40\) cc, \(P = 800/40 = 20\) Newtons.
Question 3 [18 marks]

(a) Simplify \( \frac{2x^{1/2}y^{2/3}}{6x^{1/3}y^2} \).

Solution:
\[
\frac{2x^{1/2}y^{2/3}}{6x^{1/3}y^2} = \frac{1}{3} x^{1/2-1/3} y^{2/3-2} = \frac{1}{3} x^{1/6} y^{-4/3}.
\]

(b) Given that \( \log_2 x = 1.5850 \) find \( \log_2(2x^2) \).

Solution: \( \log_2(2x^2) = \log_2 2 + 2 \log_2 x = 1 + 2 \times 1.5850 = 4.17 \).

(c) Solve for \( x \): (i) \( \left( \frac{1}{3} \right)^x = \frac{1}{9} \) (ii) \( 2 \log_3 x = 6 \).

Solution: (i) \( x = 2 \). (ii) \( \log_3 x = 3 \), \( x = 3^3 = 27 \).

(d) Given the relationship \( N = 100 - 50e^{-0.001t} \).

(i) Find \( N \) when \( t = 300 \). (ii) Find \( t \) when \( N = 90 \).

Solution: (i) When \( t = 300 \), \( N = 100 - 50e^{-0.001 \times 300} = 63 \).

(ii) When \( N = 90 \), we have \( 90 = 100 - 50e^{-0.001t} \). Hence \( e^{-0.001t} = (100 - 90)/50 = 0.2 \), and \( -0.001t = \ln 0.2 \), \( t = \ln 2/(-0.001) = 1609.44 \).

Question 4 [18 marks]

(a) Without using a calculator find \( \sin(315^\circ) \) and \( \tan(240^\circ) \).

Solution
\[
\sin(315^\circ) = \sin(360^\circ - 45^\circ) = -\sin(45^\circ) = -\frac{1}{\sqrt{2}}.
\]
\[
\tan(240^\circ) = \tan(360^\circ - 120^\circ) = -\tan(120^\circ) = -\tan(180^\circ - 60^\circ) = \tan(60^\circ) = \sqrt{3}.
\]
(b) A sphere has a surface area of 10 square metres. Find its volume.

**Solution:** \(10 = 4\pi r^2\). Hence \(r = \sqrt{10/(4\pi)} = 0.8921\) and the volume of the sphere is \(\frac{4}{3}\pi r^3 = 2.9735\).

(c) A triangle has sides of lengths 4 and 5 containing an angle of 37°. Find the other side and angles of the triangle.

**Solution:** Let \(a, b, c\) denote the three sides with \(b = 4\), \(c = 5\), and let \(A, B\) and \(C\) denote the angles facing the sides \(a, b\) and \(c\), respectively. Hence \(A = 37°\) and by the cosine law,

\[a^2 = b^2 + c^2 - 2bc \cos A = 16 + 25 - 40 \cos 37° = 9.05, \ a = 3.01.\]

Using the sine law, we obtain

\[
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},
\]

i.e.,

\[
\frac{3.01}{\sin 37°} = \frac{4}{\sin B} = \frac{5}{\sin C}.
\]

Hence

\[
\sin B = \frac{4}{3.01/\sin 37°} = 0.7998, \ B = 53.11°.\]

\[
\sin C = \frac{5}{3.01/\sin 37°} = 0.9997, \ C = 88.58°.\]

Question 5 [18 marks]

(a) Three balls are chosen at random from a bag containing 52 balls, with 4 red, 4 green, 4 yellow and the rest white.

(i) What is the probability that the first ball chosen is red or green?

**Solution:** \(\frac{8}{52} = \frac{2}{13}\).

(ii) What is the probability that the first two balls chosen are red?

**Solution:**

\[
\frac{C(4, 2)}{C(52, 2)} = \frac{4!/[2!(4 - 2)!]}{52!/[2!(52 - 2)!]} = \frac{6}{51 \times 26} = \frac{1}{221}.
\]
(iii) What is the probability that the three balls chosen are a red, a green and a yellow (in any order)?

**Solution:**
\[
\frac{\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{3}} = \frac{64}{22100} = 0.00289593.
\]

(iv) What is the probability that none of the balls chosen is red?

**Solution:** None of balls chosen is red is equivalent to all the balls chosen are from the 48 non-red balls. Thus the required probability is \(\frac{\binom{48}{3}}{\binom{52}{3}} = 0.782624434.

(b) Six students got the following marks in an examination:

50, 70, 28, 80, 66, 90

(i) What was the mean mark? (ii) What was the standard deviation?

**Solution:** (i) The mean mark is \(\frac{50 + 70 + 28 + 80 + 66 + 90}{6} = 64\).

(ii) The standard deviation is
\[
\sqrt{\frac{14^2 + 6^2 + 36^2 + 16^2 + 2^2 + 26^2}{5}} = 22.1991.
\]

(c) An experiment is performed in which the results are normally distributed with a mean of 10 and a standard deviation of 6.

(i) What is the probability that the result is between 11 and 12?

**Solution:** \(z = \frac{x - 10}{6}\). When \(11 < x < 12\), \(1/6 < z < 2/6\). Therefore the required probability is
\[
P(1/6 \leq z \leq 2/6) = P(0.17 \leq z \leq 0.33) = 0.1293 - 0.0675 = 0.0618.
\]

(ii) If 80% of the results are greater than \(x\), find \(x\).

**Solution:** We firstly observe that necessarily \(\frac{x - 10}{6} < 0\); this is because the area under the standard normal curve from \(\frac{x - 10}{6}\) to \(+\infty\) is 80%=0.8. Let \(A\) denote the area under the standard normal curve from \(\frac{x - 10}{6}\) to 0, and \(B\) denote the area from \(-\infty\) to \(\frac{x - 10}{6}\). Then using the symmetry of the standard normal curve we should have \(2A + 2B = 1\) and \(2A + B = 0.8\). It follows that \(B = 0.2\) and \(A = 0.3\). Hence \(\frac{x - 10}{6} = -0.84\) and \(x = 10 - 6 \times 0.84 = 4.96\).
Question 6 [12 marks]

Let \( A, B \) and \( C \) be the matrices

\[
A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 0 \\ 3 & -2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}
\]

Calculate

(i) \( A - B \).

(ii) \( AB \).

(iii) \( A^2 \).

(iv) \( C^{-1} \).

Solution:

(i)

\[
A - B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}.
\]

(ii)

\[
AB = \begin{bmatrix} 1 \cdot 0 + 2 \cdot 1 + 0 \cdot 3 & 1 \cdot 2 + 2 \cdot -1 + 0 \cdot -2 & 1 \cdot 1 + 2 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 0 + (-1) \cdot 1 + 2 \cdot 3 & 0 \cdot 2 + (-1) \cdot -1 + 2 \cdot -2 & 0 \cdot 1 + (-1) \cdot 0 + 2 \cdot 0 \\ 3 \cdot 0 + 0 \cdot 1 + 1 \cdot 3 & 3 \cdot 2 + 0 \cdot -1 + 1 \cdot -2 & 3 \cdot 1 + 0 \cdot 0 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 5 & -3 & 0 \\ 3 & 4 & 3 \end{bmatrix}.
\]

(iii)

\[
A^2 = \begin{bmatrix} 1 & 0 & 4 \\ 6 & 1 & 0 \\ 6 & 6 & 1 \end{bmatrix}.
\]

(iv)

\[
C^{-1} = \frac{1}{1 \cdot 3 - 2 \cdot 2} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}.
\]