Question 1 15

a. Let X and Y be independent $N(0,1)$ normal random variables. The pair (X, Y) defines a point in two dimensions by Cartesian coordinates. The transformation to polar-coordinates is given by

$$\begin{aligned} X &= R\cos\theta \\ Y &= R\sin\theta \end{aligned}$$

(a) What is the joint density $f_{R\theta}(r, \theta)$. [*4 marks*]

(b) What are the marginal densities $f_R(r)$ and$f_\theta(\theta)$. [*4 marks*]

(c) What are the ranges for r and θ? [*2 marks*]

b. Suppose X_1 and X_2 are independent exponential random variables, each with p.d.f. $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Let $S = X_1 + X_2$. What is the density of S. [*5 marks*]

Question 2 15

A sample of size n is drawn from a population with probability density function $f(x)$ and the order statistics are designated $Y_1, Y_2, \ldots, Y_n$.

a. What is the joint probability density function for (Y_1, Y_n). [*4 marks*]

b. For $R = Y_n - Y_1$ and say $V = Y_1$, show the derivation of the joint density function $f_{RV}(r, v)$, including the domains of r, v. [*4 marks*]

c. What is the probability density of the range? [*2 marks*]

d. A sample of size n is drawn from a population with density $f(x) = 2e^{-2x}$, $x \geq 0$. Show that the probability that the range does not exceed 4 is

$$2(n-1)\int_0^4 \left(1 - e^{-2r}\right)^{n-2} dr.$$

[*5 marks*]

Question 3 is on page 2

Question 3 15

a. The exponential family of distributions are those distributions for which the density function can be written in the form

$$f(x;\theta) = B(\theta)h(x)\exp\{p(\theta)k(x)\}I(a,b)(x)I(\gamma,\delta)(\theta)$$

Show that the following distributions are members of the exponential family if the parameter λ is known, i.e. λ can be regarded as a constant. You are required to identify each component, $B(\theta), h(x), p(\theta), k(x)$.

(a) Weibull

$$f(x) = \lambda\theta x^{\lambda-1}\exp\left[-\theta x^{\lambda}\right] I_{(0,\infty)}(x) \quad \theta > 0,\ \lambda > 0$$

[3 marks]

(b) Negative binomial.

$$f(x) = \binom{x+\lambda-i}{x} \frac{\theta^x}{(1+\theta)^{\lambda+x}} I_{(0,1,2,\ldots)}(x)$$

[5 marks]

b. State the Factorization Theorem for finding a sufficient statistic. *[3 marks]*

c. Let $X_1 \ldots X_n$ be a random sample from distribution with

$$f(x;\theta) = \theta x^{\theta-1} I_{(0,1)}(x).$$

Use the factorization theorem to find a sufficient statistic for θ. *[5 marks]*

Question 4 15

A random sample $X_1, \ldots X_n$ is drawn from the gamma density

$$f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-\beta x} \cdot I_{(0,\infty)}(x)$$

a. What is the Information matrix for (α, β)? *[7 marks]*

b. For each of the distributions below, a sample of size n is taken in order to estimate θ. In each case, find the Cramér-Rao lower bound for the variance of an unbiased estimator of θ. Identify the estimator that has that variance.

(a) $f(x;\theta) = \frac{1}{\theta}e^{-x/\theta} I_{(0,\infty)}(x)$

(b) $f(x;\theta) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2}, I_{(-\infty,\infty)}(x)$ *[8 marks]*

FORMULAE

a. If $U = X + Y, \quad f_U(u) = \int_{-\infty}^{+\infty} f_X(u-v) f_Y(v) dv.$

b. $f_{Yr}(yr) = \frac{n!}{(n-r)!(r-1)!} f(y_r) F(y_r)^{r-1} \left[1 - F(y_r)\right]^{n-r}$.

c. $f_{Y_r Y_s}(y_r, y_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} F(y_r)^{r-1} \left[F(y_s) - F(y_r)\right]^{s-r-1} \left[1 - F(y_s)\right]^{n-s} f(y_r) f(y_s)$

d. $\frac{d}{dx}\left(\Gamma(x)\right) = \Gamma'(x), \; \frac{d^2}{dx^2}\left(\Gamma(x)\right) = \Gamma''(x).$