Question 1  15

a. Let $X$ and $Y$ be independent $N(0, 1)$ normal random variables. The pair $(X, Y)$ defines a point in two dimensions by Cartesian coordinates. The transformation to polar-coordinates is given by

\[
X = R \cos \theta \\
Y = R \sin \theta
\]

(a) What is the joint density $f_{R\theta}(r, \theta)$. \hfill [4 marks]

(b) What are the marginal densities $f_R(r)$ and $f_\theta(\theta)$. \hfill [4 marks]

(c) What are the ranges for $r$ and $\theta$? \hfill [2 marks]

b. Suppose $X_1$ and $X_2$ are independent exponential random variables, each with p.d.f. $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Let $S = X_1 + X_2$. What is the density of $S$. \hfill [5 marks]

Question 2  15

A sample of size $n$ is drawn from a population with probability density function $f(x)$ and the order statistics are designated $Y_1, Y_2, \ldots, Y_n$.

a. What is the joint probability density function for $(Y_1, Y_n)$. \hfill [4 marks]

b. For $R = Y_n - Y_1$ and say $V = Y_1$, show the derivation of the joint density function $f_{RV}(r, v)$, including the domains of $r, v$. \hfill [4 marks]

c. What is the probability density of the range? \hfill [2 marks]

d. A sample of size $n$ is drawn from a population with density $f(x) = 2e^{-2x}$, $x \geq 0$. Show that the probability that the range does not exceed 4 is

\[
2(n - 1) \int_0^4 (1 - e^{-2r})^{n-2} dr.
\] \hfill [5 marks]

Question 3 is on page 2
Question 3  15

a. The exponential family of distributions are those distributions for which the density function can be written in the form

\[ f(x; \theta) = B(\theta)h(x) \exp\{p(\theta)k(x)\} I(a, b)(x)I(\gamma, \delta)(\theta) \]

Show that the following distributions are members of the exponential family if the parameter \( \lambda \) is known, i.e. \( \lambda \) can be regarded as a constant. You are required to identify each component, \( B(\theta), h(x), p(\theta), k(x) \).

(a) Weibull

\[ f(x) = \lambda \theta x^{\lambda-1} \exp \left[-\theta x^\lambda\right] I_{(0, \infty)}(x) \quad \theta > 0, \lambda > 0 \]

(b) Negative binomial.

\[ f(x) = \left(\frac{x + \lambda - i}{x}\right) \frac{\theta^x}{(1 + \theta)^{\lambda + x}} I_{(1,2,\ldots)}(x) \]

b. State the Factorization Theorem for finding a sufficient statistic.  

3 marks

c. Let \( X_1 \ldots X_n \) be a random sample from distribution with

\[ f(x; \theta) = \theta x^{\theta-1} I_{(0,1)}(x). \]

Use the factorization theorem to find a sufficient statistic for \( \theta \).  

5 marks

Question 4  15

A random sample \( X_1, \ldots X_n \) is drawn from the gamma density

\[ f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \cdot e^{-\beta x} \cdot I_{(0, \infty)}(x) \]

a. What is the Information matrix for \((\alpha, \beta)\)?  

7 marks

b. For each of the distributions below, a sample of size \( n \) is taken in order to estimate \( \theta \). In each case, find the Cramér-Rao lower bound for the variance of an unbiased estimator of \( \theta \). Identify the estimator that has that variance.

(a) \( f(x; \theta) = \frac{1}{\theta} e^{-x/\theta} I_{(0, \infty)}(x) \)

(b) \( f(x; \theta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\theta)^2} I_{(-\infty, \infty)}(x) \)

8 marks
FORMULAE

a. If \( U = X + Y \), \( f_U(u) = \int_{-\infty}^{+\infty} f_X(u - v) f_Y(v) dv \).

b. \( f_{Yr}(yr) = \frac{n!}{(n-r)!(r-1)!} f(y_r) F(y_r)^{r-1} [1 - F(y_r)]^{n-r} \).

c. \( f_{Yr,Ys}(y_r, y_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} F(y_r)^{r-1} [F(y_s) - F(y_r)]^{s-r-1} [1 - F(y_s)]^{n-s} f(y_r)f(y_s) \)

d. \( \frac{d}{dx} (\Gamma(x)) = \Gamma'(x), \quad \frac{d^2}{dx^2} (\Gamma(x)) = \Gamma''(x). \)