Solutions Stat354exam UNE June 2008

Question 1 14

a.

$$\begin{aligned}
F_{XY}(x,y) &= f_X(x)f_Y(y) \\
&= (2\pi)^{-1}\exp\{-\frac{1}{2}(x^2+y^2)\}
\end{aligned}$$

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}$$

$$\begin{aligned}
r &= \sqrt{x^2+y^2} \quad 0 \le r < \infty \\
\theta &= \tan^{-1}(y/x) \quad 0 \le \theta \le 2\pi
\end{aligned}$$

$$\begin{aligned}
J &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \\
f_{R\Theta}(r,\theta) &= (2\pi)^{-1} \times r \times \exp\{-\frac{1}{2}(x^2+y^2)\} \\
&= (2\pi)^{-1} \times r \times \exp\{-\frac{1}{2}(r^2) \\
&= f_\Theta \times f_R \text{ where} f_\Theta = (2\pi)^{-1} \quad , f_R = r\exp(-r^2)
\end{aligned}$$

b.

$$\begin{aligned}
f_S(s) &= \int_0^s f_{X_1}(x)f_{X_2}(x)(s-x)dx \\
&= \int_0^s \lambda^2 e^{-\lambda x}e^{-\lambda(s-x)}dx \\
&= \lambda^2 e^{-\lambda s} \times s \quad s \ge 0
\end{aligned}$$

Question 2 is on page 2

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(a)

$$f_{Y_1Y_n}(y_1, y_n) = n(n-1)\left[F(y_n) - F(y_1)\right]^{n-2} f(y_n)f(y_1)$$

(b)

$$\begin{aligned} r &= y_n - y_1 \quad y_n = r + v \\ v &= y_1 \qquad y_1 = v \end{aligned}$$

$$J = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \quad a < y_1 < y_n < b$$

$$f_{R,V} = n(n-1)\left[F(r+v) - F(v)\right]^{n-2} f(r+v)f(v)$$
$$a < v < b - r \quad 0 < r < b - a$$

(c)

$$f_R(r) = \int_a^{b-r} n(n-1)\left[F(r+v) - F(v)\right]^{n-2} f(v)f(r+v)dv$$

(d)

$$\begin{aligned} f(x) &= 2e^{-2x} \times I_{(0,\infty)}(x) \\ F(x) &= \left[-e^{-2u}\right]_0^x = 1 - e^{-2x} \end{aligned}$$

$$\begin{aligned} F_R(r) &= \int_0^\infty n(n-1)\left[-e^{-2v} - e^{-2(r-v)}\right]^{n-2} \times 2e^{-2(r+v)} \times 2e^{-2v}dv \\ &= \int_0^\infty n(n-1)\left[e^{-2v}(1 - e^{-2(r-v)})\right]^{n-2} \times 4e^{-2r}e^{-4v}dv \\ &= 4n(n-1)(1 - e^{-2(r-v)})^{n-2} \int_0^\infty e^{-2nv}dv \\ &= 4n(n-1)(1 - e^{-2(r-v)})^{n-2}\left[-e^{-2nv}/2n\right]_0^\infty \\ &= 2(n-1)(1 - e^{-2(r-v)})^{n-2} \end{aligned}$$

$$\begin{aligned} Pr(R \leq 4) &= 2(n-1)\int_0^\infty (1 - e^{-2r})^{n-2}dr \\ &= 2(n-1) \end{aligned}$$

Question 3 is on page 3

Question 3 16

(a) (i) Weibull

$$\begin{aligned} B(\theta) &= \theta \\ h(x) &= \lambda x \times I_{(0,\infty)}(x) \\ p(\theta) &= -\theta \\ k(x) &= x^{\lambda} \end{aligned}$$

(ii) Negative Binomial

$$\begin{aligned} f(x) &= \binom{x+\lambda-1}{x} \frac{\theta^x}{(1+\theta)^{\lambda+x}} \\ &= \frac{(x+\lambda-1)!}{x!(\lambda-1)!} \exp\left\{ x \log\left(\frac{\theta}{1+\theta}\right) - \lambda(1+\theta) \right\} \end{aligned}$$

$$\begin{aligned} B(\theta) &= \exp\{-\lambda \log(1+\theta)\} \\ h(x) &= \frac{(x+\lambda-1)!}{x!(\lambda-1)!} \\ p(\theta) &= \log(\theta/(1+\theta)) \\ k(x) &= x \end{aligned}$$

(b)

$$\mathbf{X} \sim f(\mathbf{x}; \theta)$$

$T = t(\mathbf{X})$ is a sufficient statistic for θ if and only if we can find 2 functions g and h such that $f(\mathbf{x}; \theta) = g\left(t(\mathbf{x}); \theta\right) h(\mathbf{x})$ where for every fixed value of $t(\mathbf{x})$, $h(\mathbf{x})$ does not depend upon θ.

(c)

$$\begin{aligned} f(x; \theta) &= \theta x^{\theta-1} \times I_{(0,1)}(x) \\ f(\mathbf{x}; \theta) &= \theta^n (x_1 x_2 \ldots, x_n)^{\theta-1} \times I_{(0,1)}(\mathbf{x}) \\ &= \theta^n (x_1 x_2 \ldots, x_n)^{\theta} \times \frac{1}{x_1 x_2 \ldots, x_n} \end{aligned}$$

Therefore $X_1, X_2, \ldots X_n$ is a sufficient statistic for θ.

Question 4 is on page 4

Question 4 16

(a)

$$\begin{aligned}
\ell &= \sum_{i=1}^{n} \alpha \log \beta + (\alpha - 1)\log x_i - \beta x_i - \log(\Gamma(\alpha)) \\
&= n\alpha \log \beta + (\alpha - 1)\sum \log x_i - \beta \sum x_i - n\log(\Gamma(\alpha))
\end{aligned}$$

$$\begin{aligned}
\mathcal{I} &= -E\begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta^2} \end{pmatrix} \\
& \quad \text{where} \\
\frac{\partial \ell}{\partial \alpha} &= n \log \beta + \sum \log x_i - \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} \\
\frac{\partial^2 \ell}{\partial \alpha^2} &= -n\left(\frac{\Gamma(\alpha)\Gamma''(\alpha) - \Gamma'(\alpha)^2}{\Gamma(\alpha)^2}\right) \\
\frac{\partial \ell}{\partial \beta} &= \frac{n\alpha}{\beta} - \sum x_i \\
\frac{\partial^2 \ell}{\partial \beta^2} &= \frac{-n\alpha}{\beta^2} \\
\frac{\partial^2 \ell}{\partial \beta \partial \alpha} &= 0 \\
\rightarrow \mathcal{I} &= \begin{bmatrix} -n\left(\frac{\Gamma(\alpha)\Gamma''(\alpha) - \Gamma'(\alpha)^2}{\Gamma(\alpha)^2}\right) & 0 \\ 0 & \frac{-n\alpha}{\beta^2} \end{bmatrix}
\end{aligned}$$

(b) (i)

$$\begin{aligned}
\log L(\theta) &= -n\log\theta - \frac{\sum x_i}{\theta} \\
\frac{\partial \ell}{\partial \theta} &= \frac{-n}{\theta} + \frac{\sum x_i}{\theta^2} = \frac{n}{\theta^2}(\bar{x} - \theta) \\
E\left(\frac{\partial \ell}{\partial \theta}\right)^2 &= E\left(\frac{n^2}{\theta^4}(\bar{x} - \theta)^2\right) = \frac{n^2\theta^2}{\theta^4 n} = \frac{n}{\theta^2} \\
\text{CRLB} &= 1/I_x = \theta^2/n \\
\text{or } E\left(\frac{\partial^2 \ell}{\partial \theta^2}\right) &= \frac{n}{\theta^2} - E\left(\frac{2n\bar{x}}{\theta^3}\right) = -\frac{n}{\theta^2}
\end{aligned}$$

(ii)

$$\begin{aligned}
\ell &= \frac{-n}{2}\log(2\pi) - \frac{1}{2}\sum(x_i-\theta)^2 \\
\frac{\partial \ell}{\partial \theta} &= n(\bar{x}-\theta) \\
E\left(\frac{\partial \ell}{\partial \theta}\right)^2 &= n^2\mathrm{var}(\bar{x}) = n^2/n = n \\
\text{CRLB} &= n^{-1} \\
& \text{Therefore } \bar{x} \text{ has variance } 1/n
\end{aligned}$$