STAT354 Exam A2005

Question 1
Two independent random variables $X$ and $Y$ are exponentially distributed, viz,

$$f(x) = \frac{e^{-x/\alpha}}{\alpha}, \quad x > 0,$$

and,

$$f(y) = \frac{e^{-y/\beta}}{\beta}, \quad y > 0.$$

(a) Use the change of variable method to find the distribution of $U = X + Y$.
Show all working.

(b) Check your solution to (a) using the marginal distribution.

(c) Briefly examine the case $\alpha = \beta$.

Question 2
The median is calculated for a sample size of $n$, where $n$ is an odd integer.

(a) Give the distribution function for the sample median.

(b) One of the simple trend estimation methods used in the early applications of spatial statistics was the spatial moving average. This measure used the average of the $r$ nearest neighbours, where $r$ was small in comparison to the size of the data set. An alternative would be to use a spatial moving median, the median of the $r$ nearest neighbours, especially if the data contained outliers.

Give the distribution function for such a spatial moving median based on the 3 nearest neighbours, assuming that the data are uniformly distributed $U(0,1)$ as the result of coordinate scaling.

(c) Is the spatial moving median a genuine trend estimation method? Give reasons.

(d) Briefly outline the extension of (b) to the general case of $r$ nearest neighbours.
Question 3
A random sample of size $n$ is taken from a population with distribution function

$$f(y; \theta) = (\theta + 1)y^\theta, \ 0 < y < 1, \ \theta > 0.$$ 

(a) Does this distribution belong to the exponential family? Show working.

(b) Does a sufficient statistic for $\theta$ exist? Give reasons.

(b) Show that the minimum variance bound for an estimator of $1/(1 + \theta)$ is independent of $\theta$.

Question 4
A random sample $X_1, \ldots, X_n$ is such that $X_i \sim N(\mu, \sigma_i^2)$ where the $\sigma_i^2$ are known.

(a) Derive a likelihood ratio test for testing

$$H_0 : \mu = \mu_0 \ vs \ H_1 : \mu \neq \mu_0$$

(b) Verify your solution to (a) for the case $\sigma_i^2 = \sigma^2 \ \forall \ i$. 
STA354 : FORMULAE

2. (a) 
   \[ f_{Y_r}(y_r) = \frac{n!}{(n-r)!(r-1)!} [F(y_r)]^{r-1} [1 - F(y_r)]^{n-r} f(y_r), \quad -\infty < y_r < \infty \]

3. (a) 
   \[ f(y; \theta) = B(\theta)h(x)e^{p(\theta)}K(x) \]

   (b) 
   \[ f(X; \theta) = g(t(X; \theta))h(X) \]

   (c) 
   \[
   V(T) \geq \frac{1}{I_X(\theta)}; \quad I_X(\theta) = E\left( \frac{\partial \log L(\theta)}{\partial \theta} \right)^2 = -E\left( \frac{\partial^2 \log L(\theta)}{\partial \theta^2} \right)^2, \quad E(T) = \theta
   \]
   \[
   V(T') \geq \frac{[\tau'(\theta)]^2}{I_X(\theta)}, \quad E[T'(\theta)] = \tau(\theta)
   \]