STAT354 Exam A2003

Question 1

Two independent random variables X and Y are exponentially distributed, viz,

$$f(x) = e^{-x}\,,\ x\ >\ 0,\ \text{and},\ f(y) = e^{-y}\,,\ y\ >\ 0.$$

(a) Determine the moment generating functions of :

(i) $U = X + Y$, and

(ii) $V = X - Y$.

(b) Hence, or otherwise determine the mean of U and the mean of V.

(c) Using the change of variable technique:

(i) Verify that the distribution of U is that obtained in (a) (i).

(ii) Determine the distribution of V.

(d) Use the results of (c) to corroborate the values for the means of U and V obtained in (b).

Question 2

(a) Show that if

$$\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

then for any linear combination $\boldsymbol{a}'\boldsymbol{X}$ it can be shown that

$$\boldsymbol{a}'\boldsymbol{X} \sim N_1(\boldsymbol{a}'\boldsymbol{\mu}, \boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a}).$$

(b) If

$$\boldsymbol{a}'\boldsymbol{X} \sim N_1(\boldsymbol{a}'\boldsymbol{\mu}, \boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a})$$

for every $\boldsymbol{a}$ then show that $\boldsymbol{X}$ *must* be $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

(c) (i) Determine the probability distribution function for the largest observation in a random sample from the uniform distribution.

(ii) A convoy of 10 trucks is to pass through a town with a low level underpass of height 3.8m. If the heights of the loads on each truck are uniformly distributed between cabin top height (3.0m) and the legal upper limit (4.0m), what is the probability that one truck will have to turn back. Provide an alternative derivation to convince a sceptic of the veracity of your solution.

Question 3

(a) A random sample $Y_1, \ldots, Y_n$ is taken from a distribution with df

$$f(y;\theta) = e^{-\alpha} y^{\theta-1}/\Gamma(\theta), \quad y > 0,\ \alpha > 0$$

where α is a constant.

Show that $T = \frac{1}{n}\sum_i \log Y_i$ is a sufficient estimator for θ.

(b) For a random sample from a population with df

$$f(y;\theta) = (1+\theta)(y+\theta)^{-2}, \ \ y > 1,\ \theta > -1,$$

(i) Obtain the minimum variance bound for an unbiased estimator of the parameter θ,

and

(ii) show that the minimum variance bound for an unbiased estimator of $\log(1+\theta)$ is independent of θ.

Question 4

(a) A random sample $Y_1, \ldots, Y_n$ is chosen from the two parameter Gamma distribution with pdf

$$f(y) = \frac{1}{\Gamma(\nu)}(\frac{\nu y}{\mu})^{\nu} e^{-\nu y/\mu} \cdot \frac{1}{y}, \ \ y,\ \nu,\ \mu > \ 0.$$

(i) Assuming that ν is known, determine $\widehat{\mu}$, the maximum likelihood estimator for μ.

(ii) Determine $V(\widehat{\mu})$, the variance of the estimator $\widehat{\mu}$, and hence give the approximate large sample distribution of $\widehat{\mu}$.

(iii) If ν is unknown, outline steps in the joint maximum likelihood estimation of ν and μ.

(b) In a demonstration experiment on Boyle's law

$$PV = \text{constant},$$

the volume is measured accurately but pressure measurements are subject to normally distributed random errors. Two sets of results are obtained, $(P_1,\ V_1)$ and $(P_2,\ V_2)$.

Derive a likelihood ratio test of the validity of Boyle's law.