

Solutions to 1998 STAT354 exam

Question 1 1

(a)

$$\begin{aligned} F_{XY}(x, y) &= f_X(x)f_Y(y) \\ &= (2\pi)^{-1} \exp\left\{-\frac{1}{2}(x^2 + y^2)\right\} \end{aligned}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2} \quad 0 \leq r < \infty$$

$$\theta = \tan^{-1}(y/x) \quad 0 \leq \theta \leq 2\pi$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

$$\begin{aligned} f_{R\Theta}(r, \theta) &= (2\pi)^{-1} \times r \times \exp\left\{-\frac{1}{2}(x^2 + y^2)\right\} \\ &= (2\pi)^{-1} \times r \times \exp\left\{-\frac{1}{2}(r^2)\right\} \\ &= f_{\Theta} \times f_R \text{ where } f_{\Theta} = (2\pi)^{-1}, f_R = r \exp(-r^2) \end{aligned}$$

(b)

$$\begin{aligned} f_S(s) &= \int_0^s f_{X_1}(x)f_{X_2}(x)(s-x)dx \\ &= \int_0^s \lambda^2 e^{-\lambda x} e^{\lambda(s-x)} dx \\ &= \lambda^2 e^{-\lambda s} \times s \quad s \geq 0 \end{aligned}$$

(c)

$$\begin{aligned} \int f_X(x)dx &= 1 \\ C \int_{-\sqrt{5}}^{\sqrt{5}} \left(1 - \frac{x^2}{5}\right) &= C \left[x - \frac{x^3}{15}\right]_{-\sqrt{5}}^{\sqrt{5}} \\ C^{-1} &= 2 \left[\sqrt{5} - \frac{5\sqrt{5}}{15}\right] = 2\sqrt{5}\left(1 - \frac{1}{3}\right) = \frac{4}{3}\sqrt{5} \\ C &= \frac{3}{4\sqrt{5}} \end{aligned}$$

Question 2 1

(a)

$$f_X(x) = (2\pi)^{-p/2} |\Sigma|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (x - \mu)' \Sigma^{-1} (x - \mu) \right\}$$

(b) $\Sigma = PP'$

(i) $Y = P^{-1}(X - \mu)$

(ii) Σ is a square symmetric, positive definite matrix

(iii) A linear combination of normal random variables is also normal.

(iv)

$$\begin{aligned} E(Y) &= E(P^{-1}(X - \mu)) = P^{-1}[E(X) - \mu] = 0 \\ \text{var}(Y) &= P^{-1} \text{var}(X) (P')^{-1} \\ &= P^{-1} \Sigma (P')^{-1} = P^{-1} P P' (P')^{-1} = I \end{aligned}$$

(c)

$$Q_i \sim \sigma^2 \chi_{r_i}^2$$

(d)

$$\begin{aligned} (X_1, X_4) &\sim N \left(\begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2.25 & -0.5 \\ -0.5 & 9 \end{bmatrix} \right) \\ Y &\sim N(2, 7) \\ \text{corr}(X_1, X_3) &= \frac{0.75}{\sqrt{2.25} \sqrt{6.25}} = 0.2 \end{aligned}$$

Question 3 1

(a)

$$f_{Y_1 Y_n}(y_1, y_n) = n(n-1) [F(y_n) - F(y_1)]^{n-2} f(y_n) f(y_1)$$

(b)

$$\begin{aligned} r &= y_n - y_1 & y_n &= r + v \\ v & & y_1 &= v \end{aligned}$$

$$J \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \quad a < y_1 < y_n < b$$

$$\begin{aligned} f_{R,V} &= n(n-1) [F(r+v) - F(v)]^{n-2} f(r+v) f(v) \\ & \quad a < v < b-r \quad 0 < r < b-a \end{aligned}$$

(c)

$$f_R(r) = \int_a^{b-r} n(n-1) [F(r+v) - F(v)]^{n-2} f(v) f(r+v) dv$$

(d)

$$\begin{aligned} f(x) &= 2e^{-2x} \times I_{(0,\infty)}(x) \\ F(x) &= [-e^{-2u}]_0^x = 1 - e^{-2x} \end{aligned}$$

$$\begin{aligned} F_R(r) &= \int_0^\infty n(n-1) [-e^{-2v} - e^{-2(r-v)}]^{n-2} \times 2e^{-2(r+v)} \times 2e^{-2v} dv \\ &= \int_0^\infty n(n-1) [e^{-2v}(1 - e^{-2(r-v)})]^{n-2} \times 4e^{-2r} e^{-4v} dv \\ &= 4n(n-1)(1 - e^{-2(r-v)})^{n-2} \int_0^\infty e^{-2nv} dv \\ &= 4n(n-1)(1 - e^{-2(r-v)})^{n-2} [-e^{-2nv}/2n]_0^\infty \\ &= 2(n-1)(1 - e^{-2(r-v)})^{n-2} \end{aligned}$$

$$\begin{aligned} Pr(R \leq 4) &= 2(n-1) \int_0^\infty (1 - e^{-2r})^{n-2} dr \\ &= 2(n-1) \end{aligned}$$

Question 4 1

(a) (i) Weibull

$$\begin{aligned}
 B(\theta) &= \theta \\
 h(x) &= \lambda x \times I_{(0,\infty)}(x) \\
 p(\theta) &= -\theta \\
 k(x) &= x^\lambda
 \end{aligned}$$

(ii) negative Binomial

$$\begin{aligned}
 f(x) &= \binom{x + \lambda - 1}{x} \frac{\theta^x}{(1 + \theta)^{\lambda + x}} \\
 &= \frac{(x + \lambda - 1)!}{x!(\lambda - 1)!} \exp \left\{ x \log \left(\frac{\theta}{1 + \theta} \right) - \lambda(1 + \theta) \right\}
 \end{aligned}$$

$$B(\theta) = \exp \{-\lambda \log(1 + \theta)\}$$

$$h(x) = \frac{(x + \lambda - 1)!}{x!(\lambda - 1)!}$$

$$p(\theta) = \log(\theta/(1 + \theta))$$

$$k(x) = x$$

(b)

$$\begin{aligned}
 f(x; \theta) &= \theta x^{\theta-1} \times I_{(0,1)}(x) \\
 f(\mathbf{x}; \theta) &= \theta^n (x_1 x_2 \dots, x_n)^{\theta-1} \times I_{(0,1)}(\mathbf{x}) \\
 &= \theta^n (x_1 x_2 \dots, x_n)^\theta \times \frac{1}{x_1 x_2 \dots, x_n}
 \end{aligned}$$

Therefore X_1, X_2, \dots, X_n is a sufficient statistic for θ .

Question 5 1

(a)

$$\begin{aligned}\ell &= \sum_{i=1}^n \alpha \log \beta + (\alpha - 1) \log x_i - \beta x_i - \log(\Gamma(\alpha)) \\ &= n\alpha \log \beta + (\alpha - 1) \sum \log x_i - \beta \sum x_i - n \log(\Gamma(\alpha))\end{aligned}$$

$$\mathcal{I} = - \begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \beta \partial \alpha} \\ \frac{\partial^2 \ell}{\partial \beta \partial \alpha} & \frac{\partial^2 \ell}{\partial \beta^2} \end{pmatrix}$$

where

$$\frac{\partial \ell}{\partial \alpha} = n \log \beta + \sum \log x_i - \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)}$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = -n \left(\frac{\Gamma(\alpha)\Gamma''(\alpha) - \Gamma'(\alpha)^2}{\Gamma(\alpha)^2} \right)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n\alpha}{\beta} - \sum x_i$$

$$\frac{\partial^2 \ell}{\partial \beta^2} = \frac{-n\alpha}{\beta^2}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \alpha} = 0$$

(b) (i)

$$\log L(\theta) = -n \log \theta - \frac{\sum x_i}{\theta}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{-n}{\theta} + \frac{\sum x_i}{\theta^2} = \frac{n}{\theta^2}(\bar{x} - \theta)$$

$$E \left(\frac{\partial \ell}{\partial \theta} \right)^2 = E \left(\frac{n^2}{\theta^4} (\bar{x} - \theta)^2 \right) = \frac{n^2 \theta^2}{\theta^4 n} = \frac{n}{\theta^2}$$

$$\text{CRLB} = 1/I_x = \theta^2/n$$

$$\text{or } E \left(\frac{\partial^2 \ell}{\partial \theta^2} \right) = \frac{n}{\theta^2} - E \left(\frac{2n\bar{x}}{\theta^3} \right) = -\frac{n}{\theta^2}$$

(ii)

$$\ell = \frac{-n}{2} \log(2\pi) - \frac{1}{2} \sum (x_i - \theta)^2$$

$$\frac{\partial \ell}{\partial \theta} = n(\bar{x} - \theta)$$

$$E \left(\frac{\partial \ell}{\partial \theta} \right)^2 = n^2 \text{var}(\bar{x}) = n^2/n = n$$

$$\text{CRLB} = n^{-1}$$

\bar{x} has variance $1/n$

Question 6 1

$$f(\mathbf{x}; \theta) = \theta e^{-\theta x} \times I_{(0, \infty)}(x)$$

$$\ell = n \log \theta - \theta \sum x$$

(a)

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \sum x \Rightarrow \hat{\theta} = \frac{n}{\sum x_i}$$

(b)

$$L = \theta^n \exp(-\theta \sum x_i)$$

$$\sup_{\theta > 0} (L) = \sup_{\theta > 0} \left(\theta^n \exp(-\theta \sum x_i) \right)$$

$$= \left(\frac{n}{\sum x_i} \right)^n e^{-n}$$

$$\sup_{0 < \theta \leq \theta_0} (L) = \begin{cases} \left(\frac{n}{\sum x_i} \right)^n e^{-n} & \text{if } \frac{n}{\sum x_i} \leq \theta_0 \\ \theta_0^n \exp(-\theta_0 \sum x_i) & \text{if } \frac{n}{\sum x_i} > \theta_0 \end{cases}$$

$$\Lambda = \begin{cases} 1 & \text{if } \frac{n}{\sum x_i} \leq \theta_0 \\ \frac{\theta_0^n \exp(-\theta_0 \sum x_i)}{\left(\frac{n}{\sum x_i} \right)^n e^{-n}} & \text{if } \frac{n}{\sum x_i} > \theta_0 \end{cases}$$

(c)

$$-2 \log \Lambda \sim \chi_{k-1}^2$$