1 1998 exam

Question 1

(a) Let X and Y be independent $N(0,1)$ normal random variables. The pair (X,Y) defines a point in two dimensions by Cartesian coordinates. The transformation to polar-coordinates is given by

$$X = R\cos\theta$$
$$Y = R\sin\theta$$

(i) What is the joint density $f_{R\theta}(r,\theta)$. *[5 marks]*

(ii) What are the marginal densities $f_R(r)$ and $f_\theta(\theta)$. *[2 marks]*

(iii) What are the ranges for r and θ? *[2 marks]*

(b) Suppose X_1 and X_2 are independent exponential random variables, each with p.d.f. $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Let $S = X_1 + X_2$. What is the density of S. *[5 marks]*

(c) Let X be a random variable with density

$$f_X(x) = C \cdot \left(1 - \frac{x^2}{5}\right) \quad -\sqrt{5} \leq x \leq \sqrt{5}$$

where C is a constant.

Find C. *[3 marks]*

Question 2

Let $X_{p,1}$ be a random variable from a population that is multivariate normal with mean $\mu_{p,1}$ and covariance $\sum_{pp}$. *[3 marks]*

(a) Write the p.d.f. of X in matrix form.

(b) The random variable X can be transformed to another random variable whose distribution is $N(0, I_p)$.

(i) State the transformation. *[2 marks]*

(ii) What is the property of $\sum$ that permits the transformation. *[2 marks]*

(iii) State why the transformed variable is also normally distributed. *[2 marks]*

(iv) Demonstrate that the transformed variable has expected value zero and variance I_p. *[2 marks]*

Question 2(c) is on page 3

Question 2 continued.

(c) For random variables $X_{n,p} \sim N(0, \sigma^2 I_p)$, the sums of squares and products is decomposed into quadratic forms

$$X^T X = \sum_{i=1}^{k} Q_i, \quad Q_i = X B_i X$$

where rank $(B_i) = r_i$ and B_i's are positive definite. If $\sum_{i=1}^{k} r_i = p$, what is the distribution of Q_i? *[3 marks]*

(d) If $\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \sim N\left(\begin{bmatrix} 3 \\ 7 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2.25 & & & \\ 2.10 & 12.25 & & \\ 0.75 & 2.60 & 6.25 & \\ -1.50 & -2.10 & -0.75 & 9.00 \end{bmatrix} \right)$,

what is the marginal distribution of (X_1, X_4)? *[1 mark]*

what is the distribution of $Y = X_3 - X_1$ *[1 mark]*

what is the correlation between X_1 and X_3? *[1 mark]*

Question 3

A sample of size n is drawn from a population with probability density function $f(x)$ and the order statistics are designated $Y_1, Y_2, \ldots, Y_n$.

(a) What is the joint probability density function for (Y_1, Y_n). *[4 marks]*

(b) For $R = Y_n - Y_1$ and say $V = Y_1$, show the derivation of the joint density function $f_{RV}(r, v)$, including the domains of r, v. *[4 marks]*

(c) What is the probability density of the range? *[2 marks]*

(d) A sample of size n is drawn from a population with density $f(x) = 2e^{-2x}, \; x \geq 0$. Show that the probability that the range does not exceed 4 is

$$2(n-1) \int_0^4 \left(1 - e^{-2r}\right)^{n-2} dr.$$

[7 marks]

Question 4

(a) The exponential family of distributions are those distributions for which the density function can be written in the form

$$f(x;\theta) = B(\theta)h(x)\exp\{p(\theta)k(x)\}I(a,b)(x)I(\gamma,\delta)(\theta)$$

Show that the following distributions are members of the exponential family if the parameter λ is known, i.e. λ can be regarded as a constant. You are required to identify each component, $B(\theta), h(x), p(\theta), k(x)$.

(i) Weibull

$$f(x) = \lambda\theta x^{\lambda-1}\exp\left[-\theta x^{\lambda}\right] I_{(0,\infty)}(x) \quad \theta > 0,\ \lambda > 0$$

[3 marks]

(ii) Negative binomial.

$$f(x) = \binom{x+\lambda-i}{x}\frac{\theta^x}{(1+\theta)^{\lambda+x}}\ I_{(0,1,2,\ldots)}(x)$$

[5 marks]

(b) State the Factorization Theorem for finding a sufficient statistic. *[3 marks]*

(c) Let $X_1 \ldots X_n$ be a random sample from distribution with

$$f(x;\theta) = \theta x^{\theta-1}I_{(0,1)}(x).$$

Use the factorization theorem to find a sufficient statistic for θ. *[5 marks]*

Question 5 A random sample $X_1, \ldots X_n$ is drawn from the gamma density

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}\cdot x^{\alpha-1}\cdot e^{-\beta x}\cdot I_{(0,\infty)}(x)$$

(a) What is the Information matrix for (α,β)? *[8 marks]*

(b) For each of the distributions below, a sample of size n is taken in order to estimate θ. In each case, find the Cramér-Rao lower bound for the variance of an unbiased estimator of θ. Identify the estimator that has that variance.

(i) $f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}I_{(0,\infty)}(x)$

(ii) $f(x;\theta) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2}, I_{(-\infty,\infty)}(x)$ *[8 marks]*

Question 6

Let $X_1 \ldots X_n$ be a sample from

$$f_x(x;\theta) = \theta e^{-\theta x}I_{((0,\infty)}(x).$$

(a) Find $\hat{\theta}$, the m.l.e. for θ. *[3 marks]*

(b) Derive the generalized likelihood ratio test of

$$\begin{aligned} H_0 : \theta &\leq \theta_0 \text{ versus} \\ H_1 : \theta &> \theta_0. \end{aligned}$$

[10 marks]

(c) If the parameters of a distribution form a k-vector $\vec{\alpha}$, and sample size is large, how would you test the null hypothesis $H_0 : \alpha = \alpha_0$ against $H_1 : \alpha \neq \alpha_0$? *[4 marks]*

FORMULAE

a. If $U = X + Y, \quad f_U(u) = \int_{-\infty}^{+\infty} f_X(u - v) f_Y(v) dv.$

b. $f_{Yr}(yr) = \frac{n!}{(n-r)!(r-1)!} f(y_r) F(y_r)^{r-1} \left[1 - F(y_r)\right]^{n-r}.$

c. $f_{Y_r Y_s}(y_r, y_s) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} F(y_r)^{s-1} \left[F(y_s) - F(y_r)\right]^{s-r-1} \left[1 - F(y_s)\right]^{n-s}$

d. $\frac{d}{dx}\left(\Gamma(x)\right) = \Gamma'(x), \ \frac{d^2}{dx^2}\left(\Gamma(x)\right) = \Gamma''(x).$