PMTH333 COMPLEX ANALYSIS

The tutorial questions contained in this booklet have mostly been selected from *Complex Variables and Applications, 6th ed.* by J.W. Brown and R.V. Churchill, McGraw-Hill.

References to relevant sections from this text are included with each tutorial. If you have difficulty with particular questions, you should review these sections and attempt similar exercises from the text.

Sample worked solutions for the tutorial questions will appear at the course website. Resist the temptation to refer to the solution as soon as you encounter any difficulty. You will learn and remember more from perseverance, successful or not, than from simply following through a worked solution.

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TUTORIAL 1

Complex Variables and Applications, Sect. 1-8

1. Let z be any complex number. Prove that

$$(a) \quad \mathrm{Im}(\mathrm{i}\, z) = \mathrm{Re}\, z, \qquad (b) \quad \mathrm{Re}(\mathrm{i}\, z) = -\,\mathrm{Im}\, z.$$

2. Solve the equation $z^2 + z + 1 = 0$ by substituting $z = x + \mathrm{i}\, y$.

3. Verify that

$$(a) \quad |\bar{z}| = |z|, \qquad (b) \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2, \qquad (c) \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

4. Sketch (or describe) the set of points determined by

$$(a) \quad |z - 1 + \mathrm{i}\,| = 1, \qquad (b) \quad |2z - \mathrm{i}\,| \leq 4, \qquad (c) \quad \mathrm{Re}(\bar{z} - \mathrm{i}) = 2$$

5. Find the four roots of the equation $z^4 + 4 = 0$.

6. Sketch (or describe) the closure of the sets

$$(a) - \pi < \mathrm{Arg}\, z < \pi \quad (z \neq 0), \qquad (b) \quad |\,\mathrm{Re}\, z| < |z|.$$

TUTORIAL 2

Complex Variables and Applications, Sect. 9-14

1. Describe the natural domain of the function $f(z) = \frac{z}{z+\bar{z}}$.
2. Let $f(z) = x^2 - y^2 - 2y + \mathrm{i}(2x - 2xy)$, where $z = x + \mathrm{i}\,y$. Using $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2\,\mathrm{i}}$ or otherwise, express $f(z)$ in terms of z.
3. Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \frac{\pi}{4}$ is mapped by the transformations
$$(a) w = z^2, \qquad (b) \quad w = z^3, \qquad (c) \quad w = z^4.$$
4. Show that
$$(a) \lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4, \qquad (b) \quad \lim_{z\to 1} \frac{1}{(z-1)^3} = \infty.$$
5. Use the definitions to show that
$$(a) \lim_{z\to 0} \frac{1}{z} = \infty, \qquad (b) \quad \lim_{z\to\infty} \frac{1}{z} = 0.$$

TUTORIAL 3

Complex Variables and Applications, Sect. 15-19

1. Find $f'(z)$ where $f(z) = \frac{z-1}{2z+1}$, $(z \neq -\frac{1}{2})$.
2. Show that $f'(z)$ does not exist at any point if
$$(a) \quad f(z) = 2x + \mathrm{i}\,y^2, \qquad (b) \quad f(z) = \mathrm{e}^x\,\mathrm{e}^{-\mathrm{i}\,y}.$$
3. Determine where $f'(z)$ exists and find its value when
$$\begin{array}{ll} (a) & f(z) = x^2 + \mathrm{i}\,y^2, \\ (b) & f(z) = z\,\mathrm{Im}\,z, \\ (c) & f(z) = \mathrm{e}^{-\theta}\cos(\ln r) + \mathrm{i}\,\mathrm{e}^{-\theta}\sin(\ln r). \end{array}$$
4. Show that, for $f(z) = u(r,\theta) + \mathrm{i}\,v(r,\theta)$ where $z = r\,\mathrm{e}^{\mathrm{i}\,\theta}$, if $f'(z_0)$ exists and $z_0 \neq 0$, then
$$f'(z_0) = \frac{-\mathrm{i}}{z_0}(u_\theta + \mathrm{i}\,v_\theta).$$

TUTORIAL 4

Complex Variables and Applications, Sect. 20-25

1. Show that an analytic function $f(z)$ in a domain D which takes only real values for all $z \in D$ must be a constant in D.

2. Find all values of z such that $\mathrm{e}^z = \sqrt{3}\,\mathrm{i}$.

3. Show that $\mathrm{e}^{\bar{z}} = \overline{\mathrm{e}^z}$ and $\mathrm{e}^{\bar{z}}$ is not analytic anywhere.

4. Show that $\sin(\bar{z})$ is not analytic anywhere.

5. Find all roots of the equation $\cosh z = \frac{1}{2}$.

6. **Harmonic Functions**

   (a) Which of the following functions are harmonic on some domain in $\mathbb{C}$? Specify the domain.

$$f(x,y) = \frac{y}{x^2+y^2}$$
$$f(x,y) = \mathrm{e}^{x^2-y^2}$$

   (b) Using Laplace's equation, show directly that $\mathrm{Re}\,\frac{1}{z}$ is harmonic in a neighbourhood of any point z_0 except the origin.

   (c) Let

$$\phi(x,y) = 6x^2y^2 - x^4 - y^4 + y - x + 1.$$

   If ϕ is the real part of an analytic function $f(z)$, what is the imaginary part? Conversely, if ϕ is the imaginary part of $f(z)$, what is the real part? Are the answers equal?

   (d) Find the harmonic conjugate of

$$\phi(x,y) = \arctan\frac{x}{y}$$

   where $-\frac{\pi}{2} < \arctan\frac{x}{y} \le \frac{\pi}{2}$.

TUTORIAL 5

Complex Variables and Applications, Sect. 26-31

1. Show that
$$(a) \quad \mathrm{Log}(1-\mathrm{i}) = \frac{1}{2}\ln 2 - \frac{\pi}{4}\,\mathrm{i}$$
$$(b) \quad \mathrm{Log}(1+\mathrm{i})^2 = 2\,\mathrm{Log}(1+\mathrm{i}), \text{ but } \mathrm{Log}(-1+\mathrm{i})^2 \neq 2\,\mathrm{Log}(-1+\mathrm{i})$$

2. Find the principal value of (a) i^{i}, (b) $(1-\mathrm{i})^{4\,\mathrm{i}}$.

3. Calculate the following:
$$(a) \quad \int_0^{\frac{\pi}{6}} \mathrm{e}^{2\,\mathrm{i}\,t}\,dt, \qquad (b) \quad \int_0^{\infty} \mathrm{e}^{-zt}\,dt.$$

4. Evaluate $\int_0^{\pi} \mathrm{e}^x \cos x\,dx$ and $\int_0^{\pi} \mathrm{e}^x \sin x\,dx$ by using
$$\int_0^{\pi} \mathrm{e}^x \cos x\,dx + \mathrm{i}\int_0^{\pi} \mathrm{e}^x \sin x\,dx = \int_0^{\pi} \mathrm{e}^{(1+\mathrm{i})x}\,dx.$$

TUTORIAL 6

Complex Variables and Applications, Sect. 32-38

1. Let $z = z(t)$ be a smooth arc, and $f(z)$ be analytic at $z_0 = z(t_0)$. Show that
$$\frac{d}{dt} f(z(t)) = f'(z(t))z'(t)$$
at $t = t_0$.

2. Calculate $\int_C f(z)\,dz$ where C is the arc from $z = -1-\mathrm{i}$ to $z = 1+\mathrm{i}$ along the curve $y = x^3$ and
$$f(z) = \begin{cases} 1 & \text{when } y < 0 \\ 4y & \text{when } y > 0. \end{cases}$$

3. Show that $\int_C \frac{dz}{z-z_0} = 2\pi\,\mathrm{i}$ where C is given by $z = z_0 + R\,\mathrm{e}^{\mathrm{i}\,\theta}$, ($R > 0$ fixed, θ changes from $-\pi$ to π.)

4. Evaluate
$$(a) \quad \int_{\mathrm{i}}^{\frac{\mathrm{i}}{2}} \mathrm{e}^{\pi z}\,dz, \qquad (b) \quad \int_{\mathrm{i}}^{3} (z-2)^3\,dz.$$

5. Apply the Cauchy-Goursat Theorem to show that
$$\int_C \frac{dz}{z^2+2z+2} = 0$$
where C is the circle $|z| = 1$ oriented counter-clockwise.

TUTORIAL 7

Complex Variables and Applications, Sect. 39-42

1. Find $\int_C g(z)\,dz$ where C is the circle $|z - \mathrm{i}\,| = 2$ in the positive sense and
$$g(z) = \frac{1}{(z^2+4)^2}.$$

2. Let C be the circle $|z| = 3$ in the positive sense. Show that $g(2) = 8\pi\,\mathrm{i}$ and $g(4\,\mathrm{i}) = 0$ where
$$g(w) = \int_C \frac{2z^2 - z - 2}{z - w}\,dz, \quad (|w| \neq 3).$$

3. Let C be a simple closed contour oriented counter-clockwise. Define
$$g(w) = \int_C \frac{z^3 + 2z}{(z-w)^3} dz.$$
Show that $g(w) = 6\pi w$ when w is inside C and $g(w) = 0$ when C is outside C.

4. Show that if $f(z)$ is analytic whithin and on a simple closed contour C and z_0 is not on C, then
$$\int_C \frac{f'(z)}{z - z_0} dz = \int_C \frac{f(z)}{(z-z_0)^2} dz.$$

TUTORIAL 8

Complex Variables and Applications, Sect. 43-45

1. Show that $\mathrm{e}^z = \mathrm{e}\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$ $(z \in \mathbb{C})$.

2. Find the MacLaurin series expansion of the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\left[\frac{1}{1+\frac{z^4}{9}}\right].$$

3. Find the MacLaurin series expansion of the function $f(z) = \sin z^2$ and use it to show that $f^{(4)}(0) = f^{(2n+1)}(0) = 0$, $n = 0, 1, 2, \ldots$.

4. Derive the Taylor series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-\mathrm{i})^n}{(1-\mathrm{i})^{n+1}}, \qquad (|z-\mathrm{i}\,| < \sqrt{2}).$$

5. Show that when $0 < |z| < 4$,

$$\frac{1}{4z-z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

TUTORIAL 9

Complex Variables and Applications, Sect. 46-51

1. Represent $f(z) = \frac{z+1}{z-1}$ by
   (a) its MacLaurin series, and give the region of validity for the representation;
   (b) its Laurent series for the domain $1 < |z| < \infty$.

2. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = \frac{-1}{2(z-1} - 3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}}.$$

3. Differentiate $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $(|z| < 1)$, to obtain

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n, \qquad (|z| < 1)$$

$$\frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} (n+1)(n+2)z^n, \qquad (|z| < 1).$$

4. Prove that

$$f(z) = \begin{cases} \frac{\cos z}{z^2 - \left(\frac{\pi}{2}\right)^2}, & \text{when } z \neq \pm\frac{\pi}{2}, \\ -\frac{1}{\pi}, & \text{when } z = \pm\frac{\pi}{2} \end{cases}$$

   is entire.

5. Prove that if $f(z)$ is analytic at z_0 and

$$f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0,$$

   the the function

$$g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^{m+1}} & \text{when } z \neq z_0, \\ \frac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when } z = z_0, \end{cases}$$

   is analytic at z_0.

TUTORIAL 10

Complex Variables and Applications, Sect. 51, 53-55

1. Show that

$$(a) \qquad \frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots, \qquad (0 < |z| < 1)$$

$$(b) \qquad \frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots, \qquad (0 < |z| < 1).$$

2. Find the residues at $z = 0$ of the functions

$$(a) \quad \frac{1}{z+z^2}, \quad (b) \quad z\cos\frac{1}{z}, \quad (c) \quad \frac{z - \sin z}{z}.$$

3. Evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 2$ when $f(z)$ is

$$(a) \quad \frac{z^5}{1-z^3}, \quad (b) \quad \frac{1}{1+z^2}, \quad (c) \quad \frac{1}{z}.$$

TUTORIAL 11

Complex Variables and Applications, Sect. 56-63

1. Find $\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)} dx$.
2. Find $\int_{-\infty}^\infty \frac{x \sin ax}{x^4+4} dx$ where $a \in \mathbb{R}$.
3. Show that $\int_0^{2\pi} \frac{d\theta}{1+\frac{1}{2}\cos\theta} = \frac{4\pi}{\sqrt{3}}$.

COMPLEX ANALYSIS SOLUTIONS TUTORIAL PROBLEMS SET 1.

1. Let $z = x + \mathrm{i}\, y$. Then $\mathrm{i}\, z = -y + \mathrm{i}\, x$ and therefore, $\operatorname{Im} \mathrm{i}\, z = x = \operatorname{Re} z$ and $\operatorname{Re} \mathrm{i}\, z = -y = -\operatorname{Im} z$.

2. After the substitution we have

$$(x + \mathrm{i}\, y)^2 + (x + \mathrm{i}\, y) + 1 = (x^2 - y^2 + x + 1) + (2xy + y)\,\mathrm{i} = 0$$

which gives two real equations

$$\begin{aligned} x^2 - y^2 + x + 1 &= 0 \\ 2xy + y &= 0 \end{aligned}$$

From the second equation we find that $y = 0$ or $x = -\frac{1}{2}$. In the first case, the first equation won't have a real solution ($x = \operatorname{Re} z$ must be real!). In the second case, the first equation becomes

$$\frac{1}{4} - y^2 - \frac{1}{2} + 1 = 0.$$

Hence $y^2 = \frac{3}{4}$. This gives the two solutions $z = -\frac{1}{2} \pm \frac{3}{4}\,\mathrm{i}$.

3. (a) Let $z = x + \mathrm{i}\, y$. By definition

$$|\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|.$$

(b)

$$\begin{aligned} \overline{(x_1 + \mathrm{i}\, y_1)(x_2 + \mathrm{i}\, y_2)} &= \overline{x_1x_2 - y_1y_2 + (x_1y_2 + x_2y_1)\,\mathrm{i}} \\ &= x_1x_2 - y_1y_2 - (x_1y_2 + x_2y_1)\,\mathrm{i} = (x_1 - \mathrm{i}\, y_1)(x_2 - \mathrm{i}\, y_2) \\ &= \bar{z}_1\bar{z}_2 \end{aligned}$$

(c) This can be done analogously to (b) or as follows (by reduction to (b)): The assertion is equivalent to

$$\overline{\left(\frac{z_1}{z_2}\right)}\bar{z}_2 = \bar{z}_1.$$

Due to (b) the left hand side equals $\overline{(\frac{z_1}{z_2})z_2} = \bar{z}_1$ which proves (c).

4. (a) a circle centred at $1 - \mathrm{i}$ of radius 1 (the distance of z to $1 - \mathrm{i}$ equals 1).
(b) The equations is equivalent to $|z - \frac{\mathrm{i}}{2}| \leq 2$. This is a closed disc (disc including the boundaring circle) centred at $\frac{\mathrm{i}}{2}$ of radius 2.
(c) This equations is equivalent to $x = 2$, where $x = \operatorname{Re} z$. Thus the point set is a vertical line that intersects the real axis at $x = 2$.

5. From $z^4 + 4 =$ we get $z^2 = \pm 2\,\mathrm{i}$. This can be solved by passing to $z = x + \mathrm{i}\,y$. We find

$$x^2 - y^2 = 0$$
$$2xy = \pm 2$$

Hence, $x = \pm y$ and $x^2 = 1$. Therefore, the four solutions are $1+\mathrm{i}$, $1-\mathrm{i}$, $-1+\mathrm{i}$, $-1-\mathrm{i}$.

6. (a) The given set is the complex plane with the real half line $(-\infty, 0]$ deleted. The closure is the entire complex plane, since any point of the real half line is limiting point for the given set.
(b) The given formula is equivalent to $y^2 > 0$ which in turn is equivalent to $y \neq 0$. This is the complex plane with deleted real axis. Again, the closure is the entire complex plane.

TUTORIAL PROBLEMS SET 2.

1. The natural domain of this function consists of all complex points for which the denominator $z + \bar{z} = 2x$ does not vanish. Thus the natural domain is the complex plane with deleted imaginary axis.

2. $f(z) = \bar{z}^2 + 2\,\mathrm{i}\,z$

3. (a) The function $w = z^2$ squares the absolute value and doubles the argument. Therefore the image will consist of all point whose absolute value is betwee 0 and 1 and whose argument is between 0 and $\frac{\pi}{2}$. This is the (closed) upper right quadrant.
(b) This function cubes the absolute values and triples the argument. Thus, the image is the sector $r \leq 1$ and $0 \leq \theta \leq \frac{2\pi}{4}$.
(c) Here the image is the closed upper half plane.

4. (a)According to (5) in Sect. 14 we have

$$\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = \lim_{z\to 0} \frac{4/z^2}{(1/z-1)^2} = \lim_{z\to 0} \frac{4}{(1-z)^2}$$

Since the latter function is continuous at 0 the limit is the value of the function at 0 which is 4.
(b) We show that for any (not matter how big) number R there is a circle around 1 such that for all z within this circle the absolute value of $\left|\frac{1}{(z-1)^3}\right| > R$. In fact, just take a circle of any radius δ which is less that $1/\sqrt[3]{R}$. Then

$$\left|\frac{1}{(z-1)^3}\right| = \frac{1}{|z-1|^3} < \frac{1}{(1/\sqrt[3]{R})^3} = R.$$

(You find how to choose δ by starting with the last inequality and going backwards.)

5. (a) similarly to 4(b) we show that for any (not matter how big) number R there is a circle around 0 such that for all z within this circle the absolute value of $\left|\frac{1}{z}\right| > R$. Here it suffices to take $\delta < 1/R$.
(b) We have to show that for any (no matter how small) number ϵ there is a circle of radius δ such that *outside* that circle the absolute value of $\left|\frac{1}{z}\right| < \epsilon$. Take $\delta > 1/\epsilon$.

TUTORIAL PROBLEMS SET 3.

1. According to the quotient rule

$$f'(z) = \frac{2z+1-2(z-1)}{(2z+1)^2} = \frac{3}{(2z+1)^2}$$

for $z \neq -\frac{1}{2}$.

2. (a) The matrix of partial derivatives of $\operatorname{Re} f$ and $\operatorname{Im} f$ with respect to $\operatorname{Re} z$ and $\operatorname{Im} z$ is

$$\begin{pmatrix} 2 & 0 \\ y^2 & 2xy \end{pmatrix}$$

Hence the Cauchy-Riemann equations imply $y = 0$ a then $2 = 0$ which is impossible.
(b) Here the matrix of partial derivatives is

$$\begin{pmatrix} \mathrm{e}^x \cos y & -\mathrm{e}^x \sin x \\ -e^x \sin x & -\mathrm{e}^x \cos x \end{pmatrix}$$

The Cauchy-Riemann equations imply $\sin x = \cos x = 0$ which is impossible.

3. (a) The matrix of partial derivatives is

$$\begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$

The Cauchy-Riemann equations are equivalent to $x = y$. Thus $f'(z)$ exists at the line $x = y$.
(b) The matrix of partial derivatives is

$$\begin{pmatrix} y & x \\ 0 & 2y \end{pmatrix}$$

The Cauchy-Riemann equations are equivalent to $x = y = 0$. Thus $f'(z)$ exists only at the origin.
(c) Since $\operatorname{Log} z = \ln r + \mathrm{i}\,\theta$ we have $f(z) = \mathrm{e}^{\mathrm{i}\operatorname{Log} z}$ which has a complex derivative for $z \neq 0$ and $\theta \neq \pi$.

4. If $f'(z)$ exists then

$$f'(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\,\frac{\partial f}{\partial y}\right)$$
$$0 = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\,\frac{\partial f}{\partial y}\right).$$

We have $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial \theta}{\partial x} = -\frac{y}{r^2}$, $\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$. According to the chain rule now

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(\frac{\partial f}{\partial r}\frac{x}{r} + \frac{\partial f}{\partial \theta}\frac{-y}{r^2}\right)$$
$$\frac{\partial f}{\partial y} = \frac{1}{2}\left(\frac{\partial f}{\partial r}\frac{y}{r} + \frac{\partial f}{\partial \theta}\frac{x}{r^2}\right).$$

Therefore,

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial r}\frac{\bar z}{r} + \frac{\partial f}{\partial \theta}\frac{-\mathrm{i}\,\bar z}{r^2}\right)$$
$$0 = \frac{1}{2}\left(\frac{\partial f}{\partial r}\frac{z}{r} + \frac{\partial f}{\partial \theta}\frac{\mathrm{i}\,z}{r^2}\right).$$

Subtracting $\bar z/z$ times the second equation from the first equation above yields

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \theta}\frac{-\mathrm{i}\,\bar z}{r^2} = \frac{-\mathrm{i}}{z}\frac{\partial f}{\partial \theta}.$$

TUTORIAL PROBLEMS SET 4.

1. If $f = u + \mathrm{i}\,v$ is analytic and takes only real values we have $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$ and therefore $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. But then u does not depend neither on x nor on y, thus u and therefore f is a constant.

2.
$$\mathrm{e}^z = e^x(\cos y + \mathrm{i}\sin y) = \sqrt{3}\,\mathrm{i}$$
is equivalent to $\mathrm{e}^x = \sqrt{3}$ and $\sin y = 1$. Hence $z = \frac{\ln 3}{2} + (\frac{\pi}{2} + 2k\pi)\,\mathrm{i}$ where k is any integer.

3.
$$\mathrm{e}^{\bar z} = e^x(\cos y + \mathrm{i}\sin(-y)) = e^x(\cos y - \mathrm{i}\sin y) = \overline{e^x(\cos y + \mathrm{i}\sin y)} = \overline{\mathrm{e}^z}.$$
Since $\mathrm{e}^{\bar z} = \mathrm{e}^x\,\mathrm{e}^{-Iy}$ it is nowhere analytic due to 2 (b) of Tutorial 2.

4. The function $f(z) = \sin \bar z = \sin(x - \mathrm{i}\,y)$ has a complex derivative where $\frac{\partial f}{\partial x} + \mathrm{i}\,\frac{\partial f}{\partial y} = 0$.
$$\cos(x - \mathrm{i}\,y) + \mathrm{i}(-\mathrm{i})\cos(x - \mathrm{i}\,y) = 2\cos(x - \mathrm{i}\,y) = 0$$
This happens for $\bar z = \frac{\pi}{2} + k\pi$, i.e. at isolated points. Therefore f is not analytic anywhere.

5. $\cosh z = \frac{1}{2}(\mathrm{e}^z + \mathrm{e}^{-z}) = \frac{1}{2}$ is equivalent to $\mathrm{e}^{2z} - \mathrm{e}^z + 1 = 0$. Substitute $\mathrm{e}^z = u$ and solve $u^2 - u + 1 = 0$. It follows

$$\mathrm{e}^z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\,\mathrm{i}$$
$$z = \log\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,\mathrm{i}\right)$$
$$z = \pm\frac{\pi}{3}$$

6. (a)

$$f(x,y) = \frac{y}{x^2+y^2} = \operatorname{Im}\frac{z}{|z|^2} = \operatorname{Im}\frac{1}{\bar{z}} = -\operatorname{Im}\frac{1}{z}$$

Hence f is harmonic outside the origin.
For $f(x,y) = \mathrm{e}^{x^2-y^2}$ we find $\Delta f = (4 + 8(x^2+y^2))\,\mathrm{e}^{x^2-y^2}$. Therefore f is not analytic anywhere.
(b) $f(z) = \operatorname{Re}\frac{1}{z} = \frac{1}{2}\left(\frac{1}{x+\mathrm{i}\,y} + \frac{1}{x-\mathrm{i}\,y}\right)$. Therefore

$$\frac{\partial^2 f}{(\partial x)^2} = \frac{1}{(x+\mathrm{i}\,y)^3} + \frac{1}{(x+\mathrm{i}\,y)^3}$$

and

$$\frac{\partial^2 f}{(\partial y)^2} = -\frac{1}{(x+\mathrm{i}\,y)^3} - \frac{1}{(x+\mathrm{i}\,y)^3}.$$

Hence $\Delta f = 0$.
(c) From $\frac{\partial\phi}{\partial x} = 12xy^2 - 4x^3 - 1$ and $\frac{\partial\phi}{\partial y} = 12x^2y - 4y^3 + 1$ we get for the desired imaginary part ψ:

$$\frac{\partial\psi}{\partial x} = -12x^2y + 4y^3 - 1, \qquad \frac{\partial\psi}{\partial y} = 12xy^2 - 4x^3 - 1.$$

It follows $\psi = -4x^3y + 4y^3x - x - g(y)$. Comparing the derivatives of ψ by y gives

$$-4x^3 + 12xy^2 - g'(y) = 12xy^2 - 4x^3 - 1$$

and hence $g'(y) = 1$. Therefore $g = y + C$ and the desired imaginary part has the form

$$\psi = -4x^3y + 4y^3x - x - y - C.$$

If ϕ was the imaginary part of an analytic function then the corresponding real part would have the form $\psi = 4x^3y - 4y^3x + x + y + C$. The two answers differ by sign.
(d) We know that $\operatorname{Log} z = \ln|z| + \mathrm{i}\operatorname{Arg} z = \ln|z| + \mathrm{i}\arctan\frac{y}{x}$. Hence

$$\operatorname{Log}(\mathrm{i}\,\bar{z}) = \ln|z| + \mathrm{i}\arctan\left(\frac{x}{y}\right)$$

and

$$-\operatorname{Log}(-\mathrm{i}\,z) = -\ln|z| + \mathrm{i}\arctan\left(\frac{x}{y}\right).$$

It follows that $-\ln|z|$ is a harmonic conjugate.

TUTORIAL PROBLEMS SET 5.

1. (a) Since $|1-\mathrm{i}\,| = \sqrt{2}$ and $\mathrm{Arg}(1-\mathrm{i}) = -\frac{\pi}{4}$ it follows $\mathrm{Log}(1-\mathrm{i}) = \frac{1}{2}\log 2 - \frac{\pi}{4}\,\mathrm{i}$.
(b)
$$\mathrm{Log}(1+\mathrm{i})^2 = \log 2 + \frac{\pi}{2}\,\mathrm{i} = 2\,\mathrm{Log}(1+\mathrm{i})$$
but
$$\mathrm{Log}(-1+\mathrm{i})^2 = \log 2 - \frac{\pi}{2}\,\mathrm{i} \neq \log 2 - \frac{3\pi}{2} = 2\,\mathrm{Log}(-1+\mathrm{i})$$

2. (a) $\mathrm{i}^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\,\mathrm{Log}\,\mathrm{i}} = \mathrm{e}^{\mathrm{i}\cdot\mathrm{i}\,\frac{\pi}{2}} = \mathrm{e}^{\frac{\pi}{2}}$
(b) $(1-\mathrm{i})^{4\,\mathrm{i}} = \mathrm{e}^{4\,\mathrm{i}\,\mathrm{Log}(1-\mathrm{i})} = \mathrm{e}^{2\,\mathrm{i}\log 2+\pi} = \mathrm{e}^{\pi}(\cos(2\log 2) + \mathrm{i}\sin(2\log 2))$

3. (a) $\int_0^{\frac{\pi}{6}} \mathrm{e}^{2\,\mathrm{i}\,t}\,dt = \left[\frac{1}{2\,\mathrm{i}}\,\mathrm{e}^{2\,\mathrm{i}\,t}\right]_0^{\frac{\pi}{6}} = \frac{\mathrm{e}^{\frac{\pi}{3}\,\mathrm{i}}-1}{2\,\mathrm{i}} = \frac{\sqrt{3}}{4} + \frac{1}{4}\,\mathrm{i}$
(b) $\int_0^{\infty} \mathrm{e}^{-zt}\,dt = \left[\frac{\mathrm{e}^{-zt}}{-z}\right]_0^{\infty} = \frac{1}{z}$ if $\mathrm{Re}\,z > 0$. The improper integral diverges if $\mathrm{Re}\,z \leq 0$.

4. $\int_0^{\pi} \mathrm{e}^{(1+\mathrm{i})x}\,dx = \left[\frac{\mathrm{e}^{(1+\mathrm{i})x}}{1+\mathrm{i}}\right]_0^{\pi} = -\frac{\mathrm{e}^{\pi}+1}{2} + \frac{\mathrm{e}^{\pi}+1}{2}\,\mathrm{i}$. Hence
$$\int_0^{\pi} \mathrm{e}^{x}\cos x\,dx = \mathrm{Re}\int_0^{\pi} \mathrm{e}^{(1+\mathrm{i})x}\,dx = -\frac{\mathrm{e}^{\pi}+1}{2}$$
$$\int_0^{\pi} \mathrm{e}^{x}\sin x\,dx = \mathrm{Im}\int_0^{\pi} \mathrm{e}^{(1+\mathrm{i})x}\,dx = \frac{\mathrm{e}^{\pi}+1}{2}.$$

TUTORIAL PROBLEMS SET 6.

1. Let $f = u + \mathrm{i}\,v$ and $z = x + \mathrm{i}\,y$. According to the chain rule
$$\frac{du(z(t))}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$
$$\frac{dv(z(t))}{dt} = \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}$$
Now, since $f(z)$ is analytic, we have $f'(z) = \frac{\partial u}{\partial x} - \mathrm{i}\,\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + \mathrm{i}\,\frac{\partial v}{\partial x}$. Hence
$$\frac{du(z(t))}{dt} = \mathrm{Re}\,f'(z)\frac{d}{dt}(x(t) + \mathrm{i}\,y(t))$$
$$\frac{dv(z(t))}{dt} = \mathrm{Im}\,f'(z)\frac{d}{dt}(x(t) + \mathrm{i}\,y(t))$$
and $\frac{df(z(t))}{dt} = f'(z(t))\frac{dz(t)}{dt}$.

2. The arc can be parametrised by $x = t$, $y = t^3$ for $t \in [-1, 1]$. Then $z = t + \mathrm{i}\,t^3$ and
$$\int_C f(z)\,dz = \int_{-1}^{0} 1\cdot(1 + 3\,\mathrm{i}\,t^2)dt + \int_0^1 4t^3\cdot(1 + 3\,\mathrm{i}\,t^2)dt$$
$$= \left[t + \mathrm{i}\,t^3\right]_{-1}^{0} + \left[t^4 + 2\,\mathrm{i}\,t^6\right]_0^1 = 2 + 3\,\mathrm{i}\,.$$

3. For $z = z_0 + R\,\mathrm{e}^{\mathrm{i}\,\theta}$ we get $dz = \mathrm{i}\,R\,\mathrm{e}^{\mathrm{i}\,\theta}\,d\theta$. Therefore

$$\int_C \frac{dz}{z-z_0} = \int_{-\pi}^{\pi} \frac{\mathrm{i}\,R\,\mathrm{e}^{\mathrm{i}\,\theta}\,d\theta}{R\,\mathrm{e}^{\mathrm{i}\,\theta}} = [\mathrm{i}\,\theta]_{-\pi}^{\pi} = 2\pi\,\mathrm{i}\,.$$

4. (a) $\int_{\mathrm{i}}^{\frac{\mathrm{i}}{2}} \mathrm{e}^{\pi z}\,dz = \left[\frac{\mathrm{e}^{\pi z}}{\pi}\right]_{\mathrm{i}}^{\frac{\mathrm{i}}{2}} = \frac{1-\mathrm{i}}{\pi}$.
   (b) $\int_{\mathrm{i}}^{3}(z-2)^3 dz = \left[\frac{(z-2)^4}{4}\right]_{\mathrm{i}}^{3} = 2 + 6\,\mathrm{i}$.

5. The integrand is singular for $z^2 + 2z + 2 = 0$, thus for $z = 1 \pm \mathrm{i}$. These points are outside the circle C. According to the Cauchy-Goursat theorem then the integral vanishes.

TUTORIAL PROBLEMS SET 7.

1. The only singularity within the circle is at $z = 2\,\mathrm{i}$. Therefore, the integral equals $2\pi\,\mathrm{i}\,\mathrm{res}_{2\,\mathrm{i}}\,g(z)$. Since the singularity is a pole of second order

$$\mathrm{res}_{2\,\mathrm{i}}\,g(z) = [(z-2\,\mathrm{i})g(z)]'(2\,\mathrm{i}) = \frac{1}{32\,\mathrm{i}}$$

   Hence the integral equals $\frac{\pi}{16}$.

2. For $w = 2$ we have exactly one singularity inside the circle. Therefore the integral $g(2) = 2\pi\,\mathrm{i}\,\mathrm{res}_2\,\frac{2z^2-z-2}{z-2} = 2\pi\,\mathrm{i}(8-2-2) = 8\pi\,\mathrm{i}$.
   For $w = 4\,\mathrm{i}$ the integrad has no singularities inside the circle and the integral vanishes.

3. When w is outside C the integrand has no singularities inside C and therefore the integral vanishes. If w is inside C then w is the only singularity and the integral equals $2\pi\,\mathrm{i}\,\mathrm{res}_w\,\frac{z^3+2z}{(z-w)^3}$. Since w is a 3rd order pole we find

$$\mathrm{res}_w\,\frac{z^3+2z}{(z-w)^3} = \frac{1}{2!}(z^3+2z)''|_{z=w} = 3w.$$

   This implies $g(w) = 6\pi\,\mathrm{i}\,w$ if w is inside C.

4. If z_0 outside C then both integrals vanish. If z_0 is inside C then z_0 is a simple pole of $\frac{f'}{(z-z_0)}$ and a second order pole of $\frac{f}{(z-z_0)^2}$. We have to prove

$$\mathrm{res}_{z_0}\,\frac{f'}{(z-z_0)} = \mathrm{res}_{z_0}\,fracf(z-z_0)^2.$$

   But both residues equal $f'(0)$.

TUTORIAL PROBLEMS SET 8.

1. From $\mathrm{e}^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$ we get for $u = z - 1$

$$\mathrm{e}^{z-1} = \frac{\mathrm{e}^z}{\mathrm{e}} = \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}.$$

Hence $\mathrm{e}^z = \mathrm{e} \sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$.

2.

$$f(z) = \frac{z}{9}\left[\frac{1}{1+\frac{z^4}{9}}\right] = \frac{z}{9}\sum_{n=0}^{\infty}\frac{z^{4n}}{9^n} = \sum_{n=0}^{\infty}\frac{z^{4n+1}}{9^{n+1}}$$

for $|z| < \sqrt{3}$.

3. From $\sin u = \sum_{n+0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}$ with $u = z^2$ we get

$$\sin z^2 = \sum_{n+0}^{\infty} (-1)^n \frac{z^{4n+2}}{(2n+1)!}.$$

In the expansion the coefficients at all odd powers and at powers divisible by 4 vanish. Thus for the corresponding derivatives at 0 must vanish.

4.

$$\frac{1}{z} = \frac{1}{1-\mathrm{i}-(z-\mathrm{i})} = \frac{1}{1-\mathrm{i}}\cdot\frac{1}{1-\frac{z-\mathrm{i}}{1-\mathrm{i}}} = \frac{1}{1-\mathrm{i}}\sum_{n=0}^{\infty}\left(\frac{z-\mathrm{i}}{1-\mathrm{i}}\right)^n = \sum_{n=0}^{\infty}\frac{(z-\mathrm{i})^n}{(1-\mathrm{i})^{n+1}}$$

for $|z-\mathrm{i}\,| < |1-\mathrm{i}\,| = \sqrt{2}$.

5.

$$\frac{1}{4z-z^2} = \frac{1}{4z}\cdot\frac{1}{1-\frac{z}{4}} = \frac{1}{4z}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n = \frac{1}{4z} + \frac{1}{4z}\sum_{n=1}^{\infty}\left(\frac{z}{4}\right)^n = \frac{1}{4z} + \sum_{n=1}^{\infty}\frac{z^{n-1}}{4^{n+1}}$$
$$= \frac{1}{4z} + \sum_{n=0}^{\infty}\frac{z^n}{4^{n+2}}$$

TUTORIAL PROBLEMS SET 9.

1. (a)

$$\frac{z+1}{z-1} = \frac{-z-1}{1-z} = -\sum_{n=0}^{\infty}(z^{n+1}+z^n) = -1-2\sum_{n=1}^{\infty} z^n$$

This representation is valid for $|z| < 1$.

(b)

$$\frac{z+1}{z-1} = \frac{1+\frac{1}{z}}{1-\frac{1}{z}} = \sum_{n=0}^{\infty}\left(\frac{1}{z^n}+\frac{1}{z^{n+1}}\right) = \frac{1}{z} + 2\sum_{n=1}^{\infty}\frac{1}{z^n}$$

2.

$$\begin{aligned}\frac{z}{(z-1)(z-3)} &= \frac{z-1+1}{(z-1)(z-3)} = \frac{-1}{2-(z-1)} - \frac{1}{z-1}\cdot\frac{-1}{2-(z-1)} \\ &= \frac{-1}{2\left(1-\frac{z-1}{2}\right)} - \frac{1}{z-1}\cdot\frac{-1}{2\left(1-\frac{z-1}{2}\right)} \\ &= -\frac{1}{2}\sum_{0}^{\infty}\frac{(z-1)^n}{2^n} - \frac{1}{2}\sum_{0}^{\infty}\frac{(z-1)^{n-1}}{2^n} \\ &= -\frac{1}{2(z-1)} - 3\sum_{0}^{\infty}\frac{(z-1)^n}{2^{n+2}}\end{aligned}$$

3. Differentiating both sides of the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

yields

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty}(n+1)z^n,$$

and

$$\frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} n(n+1)z^{n-1} = \sum_{n=0}^{\infty}(n+1)(n+2)z^n.$$

4. We prove that $f(z)$ is continuous at $z = \pm\frac{\pi}{2}$. Then, according to Riemann's theorem, f is analytic at those points and therefore f is entire. Due to l'Hôpital's rule

$$\lim_{z\to\pm\frac{\pi}{2}} \frac{\cos z}{z^2 - \left(\frac{\pi}{2}\right)^2} = \lim_{z\to\pm\frac{\pi}{2}} \frac{-\sin z}{2z} = -\frac{\pm 1}{\pm\pi} = \frac{-1}{\pi}.$$

5. We show that g is continuous at z_0. Due to l'Hôpital's rule (applied $m+1$ times)

$$\lim_{z\to z_0} \frac{f(z)}{(z-z_0)^{m+1}} = \lim_{z\to z_0} \frac{f^{m+1}(z)}{m+1!} = \frac{f^{m+1}(z_0)}{m+1!}.$$

TUTORIAL PROBLEMS SET 10.

1. (a)

$$\begin{aligned}\frac{e^z}{z(z^2+1)} &= \frac{1}{z}(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots)(1 - z^2 + \cdots) \\ &= \frac{1}{z}(1 + z - \frac{z^2}{2} - \frac{5z^3}{6} + \cdots) = \frac{1}{z} + 1 - \frac{z}{2} - \frac{5z^2}{6} + \cdots\end{aligned}$$

(b)

$$\begin{aligned}\frac{1}{\mathrm{e}^z-1} =&\frac{1}{z}\cdot\frac{1}{1+\frac{z}{2}+\frac{z^2}{6}+\frac{z^3}{24}+\frac{z^4}{120}+\cdots}\\ =&\frac{1}{z}(1-\frac{z}{2}-\frac{z^2}{6}-\frac{z^3}{24}-\frac{z^4}{120}+\frac{z^2}{4}+2\frac{z^3}{12}+\frac{z^4}{36}+2\frac{z^4}{48}-\\ &\frac{z^3}{8}-3\frac{z^4}{24}+\frac{z^4}{16}+\cdots)\\ =&\frac{1}{z}-\frac{1}{2}+\frac{z}{12}-\frac{z^3}{720}+\cdots\end{aligned}$$

2. (a) This function has a pole of order 1 at 0, thus

$$\operatorname{res}_0\frac{1}{z+z^2}=\lim_{z\to 0}\frac{z}{z+z^2}=1.$$

(b)

$$z\cos\frac{1}{z}=z(1-\frac{1}{2z^2}+\cdots)$$

Therefore,

$$\operatorname{res}_0 z\cos\frac{1}{z}=-\frac{1}{2}.$$

(c) This function has a pole of order 1 at 0, thus

$$\operatorname{res}_0\frac{z-\sin z}{z}=\lim_{z\to 0}(z-\sin z)=0.$$

3. (a) The singularities of the function $f=\frac{z^5}{1-z^3}$ are 1, $\epsilon=-\frac{1}{2}+\frac{\sqrt{3}}{2}$, $\epsilon^2=-\frac{1}{2}-\frac{\sqrt{3}}{2}$. They are all inside the circle $C=\{|z|=2\}$. Therefore

$$\int_C f dz=2\pi\,\mathrm{i}(\operatorname{res}_1 f+\operatorname{res}_\epsilon f+\operatorname{res}_{\epsilon^2} f).$$

We have

$$\begin{aligned}\operatorname{res}_1 f&=\lim_{z\to 1}\frac{z^5(z-1)}{1-z^3}=\lim_{z\to 1}\frac{-z^5}{1+z+z^2}=-\frac{1}{3}\\ \operatorname{res}_\epsilon f&=\lim_{z\to\epsilon}\frac{z^5(z-\epsilon)}{1-z^3}=\frac{-\epsilon^2}{(\epsilon-1)(\epsilon-\epsilon^2)}\\ \operatorname{res}_{\epsilon^2} f&=\lim_{z\to\epsilon^2}\frac{z^5(z-\epsilon^2)}{1-z^3}=\frac{-\epsilon}{(\epsilon^2-1)(\epsilon^2-\epsilon)}\end{aligned}$$

This yields

$$\int_C f dz=2\pi\,\mathrm{i}\left(-\frac{1}{3}-\frac{\epsilon^2}{2\epsilon^2-1-\epsilon}-\frac{\epsilon}{2\epsilon-1-\epsilon^2}\right)$$

We take into account $1+\epsilon+\epsilon^2$. Thus

$$\int_C f dz=2\pi\,\mathrm{i}\left(-\frac{1}{3}-\frac{\epsilon^2}{3\epsilon^2}-\frac{\epsilon}{3\epsilon}\right)=-2\pi\,\mathrm{i}$$

(b) The singularities of the function $f = \frac{1}{z^2}$ are $\pm\mathrm{i}$ which are all inside C. Thus

$$\int_C f dz = 2\pi\,\mathrm{i}(\mathrm{res}_{\mathrm{i}} f + \mathrm{res}_{-\mathrm{i}} f).$$

$$\mathrm{res}_{\mathrm{i}} f = \lim_{z\to\mathrm{i}} \frac{z-\mathrm{i}}{z^2+1} = \frac{1}{2\,\mathrm{i}}$$

$$\mathrm{res}_{-\mathrm{i}} f = \lim_{z\to-\mathrm{i}} \frac{z+\mathrm{i}}{z^2+1} = -\frac{1}{2\,\mathrm{i}}$$

Hence $\int_C f dz = 0$.

(c) $f(z) = \frac{1}{z}$ has a simple pole in C with residue 1. Thus $\int_C f dz = 2\pi\,\mathrm{i}$.

TUTORIAL PROBLEMS SET 11.

1. Since the integrand is an even function the integral equals $\frac{1}{2}\int_{-\infty}^{\infty} f(z)\,dz$. The integral along a half circle of (sufficiently big) radius R in the upper half plane consists of the straight part which tends to the desired integral and the circle part which tends to zero. On the other hand, this integral equals to the sum of residues in the upper half plane multiplied with $2\pi\,\mathrm{i}$. Thus

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{1}{2}2\pi\,\mathrm{i}\,(\mathrm{res}_{\mathrm{i}} f + \mathrm{res}_{2\,\mathrm{i}} f)$$
$$= \pi\,\mathrm{i}\left(\frac{-1}{2\,\mathrm{i}\cdot 3} + \frac{-4}{(-3)\cdot 4\,\mathrm{i}}\right) = \frac{\pi}{6}$$

2.

$$I(a) = \int_{-\infty}^{\infty} \frac{x\sin ax}{x^4+4}dx = \frac{1}{2\,\mathrm{i}}\int_{-\infty}^{\infty}\frac{x\,\mathrm{e}^{\mathrm{i}\,ax}}{x^4+4}dx - \frac{1}{2\,\mathrm{i}}\int_{-\infty}^{\infty}\frac{x\,\mathrm{e}^{-\mathrm{i}\,ax}}{x^4+4}dx$$

Since $I(a)$ is an odd function of a we may restict ourselves to $a > 0$. Then Jordan's lemma applies for the first integral in the upper half plane and for the second integral in the lower half plane. Thus the first integral equals $2\pi\,\mathrm{i}$ times the two residues at $\pm 1 + \mathrm{i}$ and the second integral equals $-2\pi\,\mathrm{i}$ times the two residues at $\pm 1 - \mathrm{i}$.

$$I(a) = 2\pi\,\mathrm{i}\Big(\mathrm{res}_{1+\mathrm{i}}\frac{1}{2\,\mathrm{i}}\int_{-\infty}^{\infty}\frac{x\,\mathrm{e}^{\mathrm{i}\,ax}}{x^4+4} + \mathrm{res}_{-1+\mathrm{i}}\frac{1}{2\,\mathrm{i}}\int_{-\infty}^{\infty}\frac{x\,\mathrm{e}^{\mathrm{i}\,ax}}{x^4+4} -$$
$$\mathrm{res}_{1-\mathrm{i}}\frac{1}{2\,\mathrm{i}}\int_{-\infty}^{\infty}\frac{x\,\mathrm{e}^{-\mathrm{i}\,ax}}{x^4+4} - \mathrm{res}_{-1-\mathrm{i}}\frac{1}{2\,\mathrm{i}}\int_{-\infty}^{\infty}\frac{x\,\mathrm{e}^{-\mathrm{i}\,ax}}{x^4+4}\Big)$$

For the first integral we get

$$I_1(a) = \pi\left(\frac{(1+\mathrm{i})\,\mathrm{e}^{(-1+\mathrm{i})a}}{2\cdot 2\,\mathrm{i}\cdot(2+2\,\mathrm{i})} + \frac{(-1+\mathrm{i})\,\mathrm{e}^{(-1-\mathrm{i})a}}{(-2)\cdot(-2+2\,\mathrm{i})\cdot 2\,\mathrm{i}}\right)$$
$$= \pi\left(\frac{\mathrm{e}^{-a}(\cos a + \mathrm{i}\sin a)}{8\,\mathrm{i}} - \frac{\mathrm{e}^{-a}(\cos a - \mathrm{i}\sin a)}{8\,\mathrm{i}}\right)$$
$$= \frac{\pi\,\mathrm{e}^{-a}\sin a}{4}$$

Analogously we get for the second integral

$$\begin{aligned} I_2(a) =& -\pi\left(\frac{(1-\mathrm{i})\,\mathrm{e}^{-(1+\mathrm{i})a}}{(-2\,\mathrm{i})\cdot(2-2\,\mathrm{i})\cdot 2}+\frac{(-1-\mathrm{i})\,\mathrm{e}^{-(1-\mathrm{i})a}}{(-2-2\,\mathrm{i})\cdot(-2\,\mathrm{i})\cdot(-2)}\right)\\ =&\pi\left(\frac{\mathrm{e}^{-a}(\cos a-\mathrm{i}\sin a)}{8\,\mathrm{i}}-\frac{\mathrm{e}^{-a}(\cos a+\mathrm{i}\sin a)}{8\,\mathrm{i}}\right)\\ =& -\frac{\pi\,\mathrm{e}^{-a}\sin a}{4} \end{aligned}$$

(Note that the orientation of the contour of integration is clockwise!)

Altogether we find for positive a

$$I(a) = I_1(a) - I_2(a) = \frac{\pi\,\mathrm{e}^{-a}\sin a}{2}.$$

Since $I(a)$ is odd we find $I(a) = -\frac{\pi\,\mathrm{e}^{a}\sin(-a)}{2} = \frac{\pi\,\mathrm{e}^{a}\sin a}{2}$ for negative a. Clearly, $I(0) = 0$. Hence

$$I(a) = \frac{\pi\sin a}{2\,\mathrm{e}^{|a|}}.$$

3. The integral equals $I = \int_0^{2\pi}\frac{d\theta}{1+\frac{1}{4}(\mathrm{e}^{\mathrm{i}\,\theta}+\mathrm{e}^{-\mathrm{i}\,\theta})}$. We substitute $z = \mathrm{e}^{\mathrm{i}\,\theta}$. This yields

$$I = \int_{|z|=1}\frac{dz}{\mathrm{i}\,z(1+\frac{1}{4}(z+\frac{1}{z}))} = \frac{4}{\mathrm{i}}\int_{|z|=1}\frac{dz}{z^2+4z+1} = 2\pi\,\mathrm{i}\cdot\frac{4}{\mathrm{i}}\,\mathrm{res}_{-2+\sqrt{3}}\,\frac{1}{z^2+4z+1} = \frac{4\pi}{\sqrt{3}}.$$