Contents

CHAPTER 1

Complex Numbers

1. Basic properties

$x^2 + 1 = 0$ solved by $x = \pm\mathbf{i}$; say "$\mathbf{i} = \sqrt{-1}$"
$\therefore\ \mathbf{i}^2 = -1$ (cf. $x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{2}$)

Definition 1.1. *"**Complex number**" $z = a + \mathbf{i}b$ st $a, b \in \mathbb{R}$ say "$z \in \mathbb{C}$". If $w = c + \mathbf{i}d \in \mathbb{C}$, then define*

$$\begin{aligned}
z + w &= (a + c) + \mathbf{i}(b + d)\ (z + w = w + z) \\
z \cdot w &= (a + \mathbf{i}b)(c + \mathbf{i}d) = (ac - bd) + \mathbf{i}(ad + bc) \qquad (\textit{Check: } zw = wz) \\
\textit{Write } a &= \textit{Re}(z) \text{ ``\textit{real part}'' \textit{of} } z \\
b &= \textit{Im}(z) \text{ ``\textit{imaginary part}'' \textit{of} } z
\end{aligned}$$

Note: *if $e \in \mathbb{R}$, then $e + z = (e + a) + \mathbf{i}b$*

$$\left.\begin{aligned} w = e &\Rightarrow c = e, d = 0 \\ w = \mathbf{i}e &\Rightarrow c = 0, d = e \end{aligned}\right) \quad \begin{aligned} \mathbf{i}e + z &= a + \mathbf{i}(b + e) \\ e \cdot z &= ea + \mathbf{i}eb \\ \mathbf{i}e \cdot z &= -eb + \mathbf{i}ea \end{aligned} \quad \begin{aligned} a + \mathbf{i}0 &= a \\ 0 + \mathbf{i}b &= \mathbf{i}b \end{aligned}$$

Powers of i:

$$\begin{aligned}
\mathbf{i}^2 &= -1 \qquad \mathbf{i}^3 = \mathbf{i} \cdot \mathbf{i}^2 = -\mathbf{i} \\
\mathbf{i}^4 &= \mathbf{i} \cdot \mathbf{i}^3 = -(\mathbf{i} \cdot \mathbf{i}) = -(-1) = 1 \\
\mathbf{i}^5 &= \mathbf{i}\mathbf{i}^4 = \mathbf{i}
\end{aligned}$$

Example: Find Im $[(3+4\mathbf{i})(1-2\mathbf{i})]$

Solution:

$$\begin{aligned}
(3+4\mathbf{i})(1-2\mathbf{i}) &= 3.1-4.2(\mathbf{i}\cdot\mathbf{i})+4.1\mathbf{i}-3.2\mathbf{i}\\
&= 3+8+\mathbf{i}(4-6)\\
&= 11-2\mathbf{i}\\
\therefore\ Im[(3+4\mathbf{i})(1-2)\mathbf{i}] &= -2
\end{aligned}$$

Definition 1.2. *Given $z=a+\mathbf{i}b\in\mathbb{C}$, define the "***conjugate***" of z:*

$$\bar{z}=a-\mathbf{i}b$$

$$\begin{aligned}
\therefore\ z+\bar{z} &= (a+\mathbf{i}b)+(a-\mathbf{i}b)\\
&= (a+a)+\mathbf{i}(b-b)=2a+\mathbf{i}0=2a=2Re(z)\\
&\qquad\qquad (=2Re(\bar{z}))\\
z-\bar{z} &= (a+\mathbf{i}b)-(a-\mathbf{i}b)=(a+\mathbf{i}b)-a+\mathbf{i}b\\
&\qquad =(a-a)+\mathbf{i}(b+b)\\
&\qquad =0+2\mathbf{i}b=2\mathbf{i}b=2\mathbf{i}Im(z)\\
&\qquad =-2\mathbf{i}Im(\bar{z})
\end{aligned}$$

$$\begin{aligned}
z\bar{z}=(a+\mathbf{i}b)(a-\mathbf{i}b) &= (a\cdot a-b(-b))+\mathbf{i}(a\cdot(-b)+ba)\\
&= a^2+b^2+\mathbf{i}(-ab+ab)\\
&= a^2+b^2+\mathbf{i}0=a^2+b^2.
\end{aligned}$$

Division of Complex Nos: $z=a+\mathbf{i}b, w=c+\mathbf{i}d$

$$\begin{aligned}
\frac{z}{w} &= \frac{z}{w}\cdot\frac{\bar{w}}{\bar{w}}=\frac{(a+\mathbf{i}b)(c-\mathbf{i}d)}{(c+\mathbf{i}d)(c-\mathbf{i}d)}=\frac{(ac+bd)+\mathbf{i}(bc-ad)}{c^2+d^2}\\
&= \left(\frac{ac+bd}{c^2+d^2}\right)+\mathbf{i}\left(\frac{bc-ad}{c^2+d^2}\right)
\end{aligned}$$

Special Case:

$$z = 1 \Rightarrow a = 1,\ b = 0\ \therefore\ \frac{1}{w} = w^{-1} = \left(\frac{c}{c^2+d^2}\right) - \mathbf{i}\left(\frac{d}{c^2+d^2}\right)$$

Conjugation:

$$\left.\begin{array}{ll} \text{(i)} & \overline{z+w} = \bar{z} + \bar{w} \\ \text{(ii)} & \overline{z \cdot w} = \bar{z} \cdot \bar{w} \end{array}\right\} \text{ E}\mathit{xercise}!$$

(iii)

$$\begin{aligned} \overline{\left(\frac{1}{w}\right)} &= \overline{\left(\frac{c}{c^2+d^2}\right) + \mathbf{i}\left(\frac{-d}{c^2+d^2}\right)} \\ &= \frac{c}{c^2+(-d)^2} - \mathbf{i}\left(\frac{-d}{c^2+(-d)^2}\right) \\ &= \frac{1}{c+i(-d)} = \frac{1}{\bar{w}} \end{aligned}$$

(iv) $\overline{(\bar{z})} = z$

Corollary: $e \in \mathbb{R}, \overline{e \cdot z} = \bar{e} \cdot \bar{z} = e \cdot \bar{z}$

$$\overline{z-w} = \overline{z+(-w)} = \bar{z} + (\overline{-w}) = \bar{z} - \bar{w}$$

(using the above, with $e = -1$)

$$\overline{\left(\frac{z}{w}\right)} = \bar{z} \cdot \overline{\left(\frac{1}{w}\right)} = \bar{z} \cdot \frac{1}{\bar{w}} = \frac{\bar{z}}{\bar{w}}$$

Example: Write $\left[\frac{2+i}{3+4\mathbf{i}} - \frac{2\mathbf{i}}{3-4\mathbf{i}}\right]^2$ in the form $a + \mathbf{i}b$.

Solution:

(i) $$\frac{2+\mathbf{i}}{3+4\mathbf{i}} = \frac{(2+\mathbf{i})(3-4\mathbf{i})}{(3+4\mathbf{i})(3-4i)} = \frac{(6+4)+\mathbf{i}(3-8)}{3^2+4^2} = \frac{10}{25} - \frac{\mathbf{i}5}{25}\left(= \frac{2}{5} - \mathbf{i}\frac{1}{5}\right)$$

(ii) $$\frac{2\mathbf{i}}{3-4\mathbf{i}} = \frac{(2\mathbf{i})(3+4\mathbf{i})}{(3-4\mathbf{i})(3+4\mathbf{i})} = \frac{-8+6\mathbf{i}}{3^2+4^2} = -\frac{8}{25} + \mathbf{i}\frac{6}{25}$$

(iii) $$\begin{aligned} \frac{2+\mathbf{i}}{3+4\mathbf{i}} - \frac{2\mathbf{i}}{3-4\mathbf{i}} &= \frac{10}{25} - \mathbf{i}\frac{5}{25} - \left(\frac{-8}{25} + \mathbf{i}\frac{6}{25}\right) \\ &= \frac{18}{25} - \mathbf{i}\frac{11}{25} \end{aligned}$$

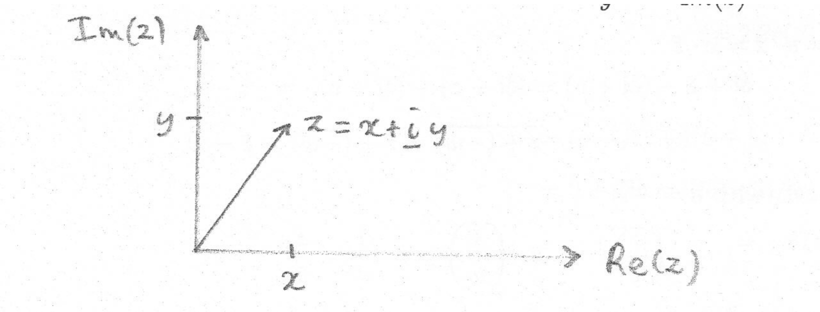
Therefore

(iv)

$$\begin{aligned}\left[\frac{2+\mathbf{i}}{3+4\mathbf{i}} - \frac{2\mathbf{i}}{3-4\mathbf{i}}\right]^2 &= \left(\frac{18}{25}+\mathbf{i}\frac{1}{25}\right)\left(\frac{18}{25}+\mathbf{i}\frac{1}{25}\right)\\ &= \frac{1}{25}(18+\mathbf{i})\,\frac{1}{25}(18+\mathbf{i})\\ &= \frac{1}{5^4}\left(18^2-1+\mathbf{i}(18+18)\right)\\ &= \frac{2^2 3^4 - 1}{5^4} + \mathbf{i}\frac{2^2 3^2}{5^4}\end{aligned}$$

2. The Argand Plane

$$z = x + \mathbf{i}y \leftrightarrow (x,y) \in \mathbb{R}^2,\ \begin{aligned} x &= \mathrm{R}e(z)\\ y &= \mathrm{I}m(z)\end{aligned}$$



Definition 2.1.

$$\begin{aligned}\text{“Modulus” (or length) of } z : |z| &= \sqrt{x^2+y^2}\\ &= \sqrt{z\bar{z}}\end{aligned}$$

Properties:

(1)

$$\begin{aligned}|zw| &= \sqrt{(zw)(\overline{zw})} = \sqrt{(z\bar{z})(w\bar{w})}\\ &= \left(\sqrt{z\bar{z}}\right)\left(\sqrt{w\bar{w}}\right) = |z||w|\end{aligned}$$

(2)

$$\left|\frac{z}{w}\right| = \left|z \cdot \frac{1}{w}\right| = |z|\left|\frac{1}{w}\right| = |z|\sqrt{\left(\frac{1}{w}\right)\overline{\left(\frac{1}{w}\right)}}$$

$$= |z|\sqrt{\frac{1}{w} \cdot \frac{1}{\bar{w}}} = |z|\sqrt{\frac{1}{|w|^2}} = \frac{|z|}{|w|}$$

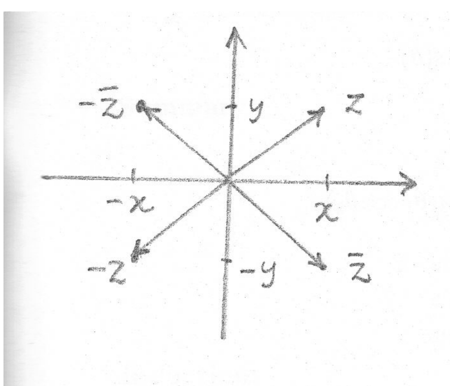
Example: Compute $\left|\frac{(3+4\mathbf{i})(1+\mathbf{i})}{3-4\mathbf{i}}\right|$

Solution:

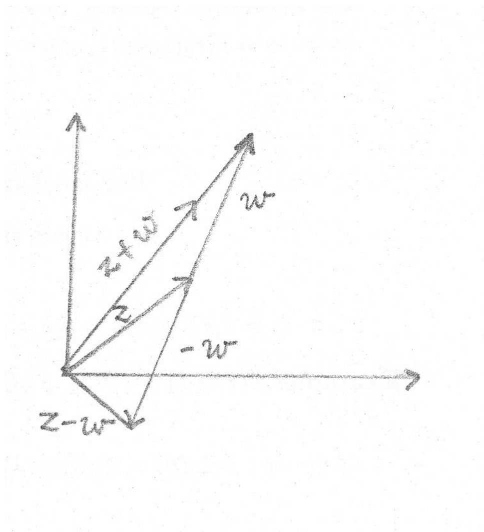
$$\begin{aligned}
|3+4\mathbf{i}| &= \sqrt{(3+4\mathbf{i})\overline{(3-4\mathbf{i})}} = \sqrt{3^2+4^2} = 5 \\
|1+\mathbf{i}| &= \sqrt{1^2+1^2} = \sqrt{2} \\
|3-4\mathbf{i}| &= \sqrt{3^2+4^2} = 5 \\
\therefore \left|\frac{(3+4\mathbf{i})(1+\mathbf{i})}{3-4\mathbf{i}}\right| &= \frac{|3+4\mathbf{i}||1+\mathbf{i}|}{|3-4\mathbf{i}|} = \frac{5\sqrt{2}}{5} = \sqrt{2}
\end{aligned}$$

Note:

$$|z| = |\bar{z}| = |-z| = |-\bar{z}|$$



$$\begin{aligned}
z &= x_1 + \mathbf{i}y_1 \leftrightarrow (x_1, y_1) \\
w &= x_2 + \mathbf{i}y_2 \leftrightarrow (x_2, y_2) \\
z + w &= \leftrightarrow (x_1 + x_2, y_1 + y_2)
\end{aligned}$$



Triangle Inequality

(i) $|z+w| \leq |z|+|w|$
(ii) $|z-w| \geq ||z|-|w||$

Proof of (ii)

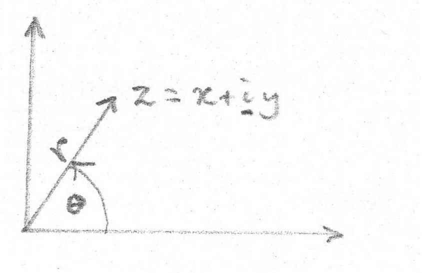
$$\begin{aligned}
|z-w|^2 = (z-w)\overline{(z-w)} &= z\bar{z} - z\bar{w} - w\bar{z} + w\bar{w} \\
(w\bar{z} = \overline{(z\bar{w})}) \qquad &= |z|^2 - 2\mathrm{Re}(z\bar{w}) + |w|^2 \\
\zeta = \alpha + \mathbf{i}\beta \qquad &\geq |z|^2 - 2|\mathrm{Re}(z\bar{w})| + |w|^2 \\
\Rightarrow |\mathrm{Re}(\zeta)| \leq |\zeta| \qquad &\geq |z|^2 - 2|z\bar{w}| + |w|^2 = (|z|-|w|)^2 \\
(|z\bar{w}| = |z||w|) \qquad \therefore\ |z-w| &\geq ||z|-|w||
\end{aligned}$$

Induction: $|z_1+z_2+z_3| \leq |z_1+z_2| + |z_3| \leq |z_1|+|z_2|+|z_3|$
$|z_1+\cdots+z_n| \leq |z_1|+|z_2|+\cdots+|z_n|$

Polar Representation of z

$z \sim (x,y) \sim (r,\theta)$

$r = |z| = \sqrt{x^2+y^2},$
$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x>0 \\ \tan^{-1}\left(\frac{y}{x}\right)+\pi & \text{if } x<0 \end{cases}$$



"**Principal Argument**" $\mathrm{A}rg(z) = \theta + 2k\pi \in (-\pi, \pi]$ (for a suitable integer k).

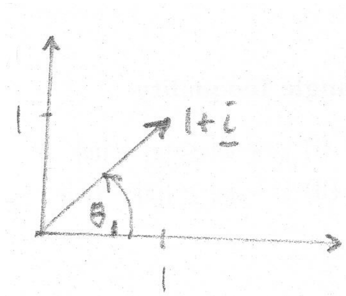
$$\left.\begin{array}{rl} \mathrm{N}ow & x = r\cos\theta \\ & y = r\sin\theta \end{array}\right] \Rightarrow z = r(\cos\theta + \mathbf{i}\sin\theta) := rcis(\theta)$$

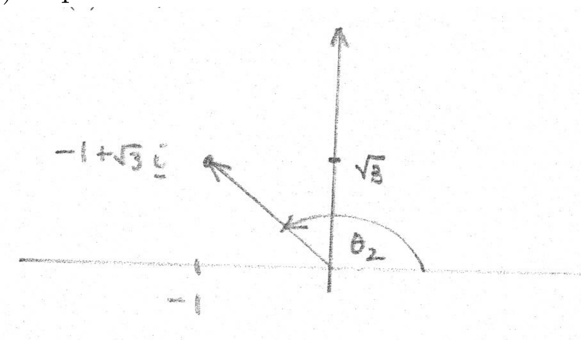
$z = r_1\cos\theta_1 + \mathbf{i}r_1\sin\theta_1,\ w = r_2\cos\theta_2 + \mathbf{i}r_2\sin\theta_2$
$\Rightarrow z \cdot w = r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \mathbf{i}r_1r_2(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$

$$\begin{array}{rcl} \mathrm{B}ut\ \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & = & \cos(\theta_1 + \theta_2) \\ \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 & = & \sin(\theta_1 + \theta_2) \end{array}$$

$\therefore\ z \cdot w = r_1r_2(\cos(\theta_1 + \theta_2) + \mathbf{i}\sin(\theta_1 + \theta_2)) = r_1r_2cis(\theta_1 + \theta_2)$

Example: Write $(1 + \mathbf{i})(-1 + \sqrt{3}\mathbf{i})$ in polar form.



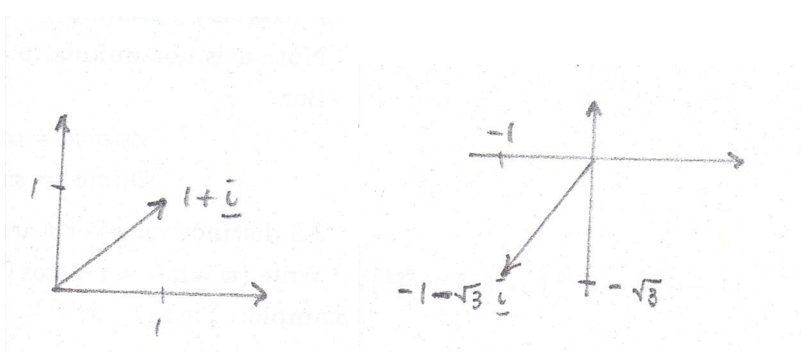


(Answer : $rcis(\theta) = 2\sqrt{2}cis(\frac{11\pi}{12})$)

$$\begin{aligned}
\frac{1}{z} = \frac{1}{r\cos\theta + \mathbf{i}r\sin\theta} &= \left(\frac{r\cos\theta}{r^2\sin^2\theta + r^2\cos^2\theta}\right) - \mathbf{i}\left(\frac{r\sin\theta}{r^2\sin^2\theta + r^2\cos^2\theta}\right) \\
&= \frac{\cos\theta}{r} - \mathbf{i}\left(\frac{\sin\theta}{r}\right) \\
&= \frac{1}{r}(\cos\theta - \mathbf{i}\sin\theta) = \frac{1}{r}\cos(-\theta) + \mathbf{i}\sin(-\theta) \\
&= \frac{1}{r}cis(-\theta)
\end{aligned}$$

Example: Express $\dfrac{(1+\mathbf{i})(-1-\mathbf{i}\sqrt{3})}{3cis(\pi/8)}$ as $rcis\theta$

$$\begin{array}{lll}
r_1 = \sqrt{2} & r_2 = 2 & r_3 = 3 \\
\theta_1 = \frac{\pi}{4} & \theta_2 = \tan^{-1}(\sqrt{3}) & \theta_3 = \pi/8 \\
 & \quad = -\frac{2\pi}{3} &
\end{array}$$



$$\begin{aligned}
\therefore \frac{(1+\mathbf{i})(-1-\mathbf{i}\sqrt{3})}{3cis(\pi/8)} &= \frac{r_1 r_2}{r_3}cis(\theta_1+\theta_2-\theta_3) \\
&= \frac{2\sqrt{2}}{3}cis(\frac{-13\pi}{24})
\end{aligned}$$

3. Powers of z

Summary:

$$\begin{aligned} z &= r_1(\cos\theta_1 + \mathbf{i}\sin\theta_1) \\ w &= r_2(\cos\theta_2 + \mathbf{i}\sin\theta_2) \\ z\cdot w &= r_1r_2(\cos(\theta_1+\theta_2) + \mathbf{i}\sin(\theta_1+\theta_2)) \\ \frac{z}{w} &= \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + \mathbf{i}\sin(\theta_1-\theta_2)) \end{aligned}$$

Induction:

$$\begin{aligned} z^n &= r^n(\cos(n\theta) + \mathbf{i}\sin(n\theta)) \\ w^{-n} &= \left(\frac{1}{w}\right)^n = \frac{1}{r^n}(\cos(n\theta) - \mathbf{i}\sin(n\theta)) \end{aligned}$$

$(r = 1 \Rightarrow)$

De Moivre: $(\cos\theta + \mathbf{i}\sin\theta)^n = \cos(n\theta) + \mathbf{i}\sin(n\theta)$

Fractional Powers:

$$\left.\begin{aligned} z = r(\cos\theta + \mathbf{i}\sin\theta) \\ w = \rho(\cos\phi + \mathbf{i}\sin\phi) \end{aligned}\right\} \text{Suppose } w^m = z$$

$\therefore\ \rho^m(\cos(m\phi) + \mathbf{i}\sin(m\phi)) = r(\cos(\theta) + \mathbf{i}\sin(\theta))$

Note w is **not unique**! $\rho = \sqrt[m]{r} = r^{\frac{1}{m}}$

But

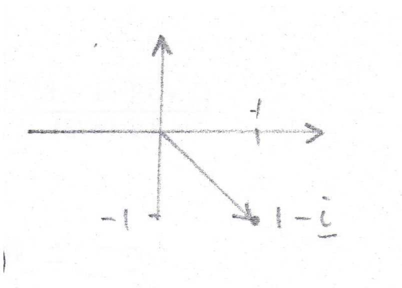
$$\left.\begin{aligned} \cos(m\phi) = \cos(\theta) \\ \sin(m\phi) = \sin(\theta) \end{aligned}\right] \Rightarrow \boxed{\phi = \tfrac{\theta}{m} + \tfrac{2\pi k}{m}}\ k = 0, \pm 1, \pm 2, \cdots$$

All **distinct** values of ϕ are covered if $k = 0, 1, \cdots, m-1 \therefore\ \exists m$ distinct values of w.

Write $(w =) z^{\frac{1}{m}} = r^{\frac{1}{m}}\left(\cos\left(\frac{\theta}{m} + \frac{2\pi k}{m}\right) + \mathbf{i}\sin\left(\frac{\theta}{m} + \frac{2\pi k}{m}\right)\right)\ k = 0, \cdots, m-1$

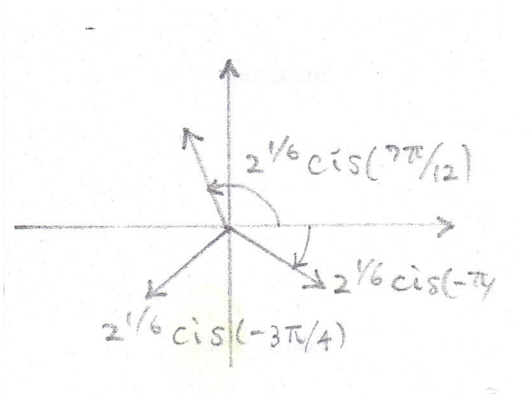
Example: Find $(1 - \mathbf{i})^{1/3}$

$$\begin{aligned} z &= 1 - \mathbf{i} \\ r &= \sqrt{2} \\ \theta &= \frac{-\pi}{4} \end{aligned}$$



$(1-i)^{1/3} = 2^{1/6}\left(\cos\left(\frac{-\pi}{12}+\frac{2\pi k}{3}\right)+\mathbf{i}\sin\left(\frac{-\pi}{12}+\frac{2\pi k}{3}\right)\right) \;\; k=0,1,2$

$$= \begin{cases} 2^{1/6}\left(\cos\left(\frac{\pi}{12}\right)-\mathbf{i}\sin\left(\frac{\pi}{12}\right)\right)(k=0) \\ 2^{1/6}\left(\cos\left(\frac{7\pi}{12}\right)+\mathbf{i}\sin\left(\frac{7\pi}{12}\right)\right)(k=1) \\ 2^{1/6}\left(\cos\left(\frac{15\pi}{12}\right)+\mathbf{i}\sin\left(\frac{15\pi}{12}\right)\right)(k=2) \end{cases}$$



What about $(1-\mathbf{i})^{-1/3}$? $w=(1-\mathbf{i})^{-1}=\frac{1}{\sqrt{2}}+\mathbf{i}\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}(1+\mathbf{i})$
$r=\frac{\sqrt{2}}{2}$, $\theta=\frac{\pi}{4}$ $\qquad w^{1/3}=2^{-1/6}\left(\cos\left(\frac{\pi}{12}+\frac{2\pi k}{3}\right)+\mathbf{i}\sin\left(\frac{\pi}{12}+\frac{2\pi k}{3}\right)\right)$
Note

$$\begin{aligned} \cos\left(\frac{\pi}{6}\right) &= \cos^2\left(\frac{\pi}{12}\right)-\sin^2\left(\frac{\pi}{12}\right)=2\cos^2\left(\frac{\pi}{12}\right)-1 \\ \Rightarrow \cos\left(\frac{\pi}{12}\right) &= \frac{\sqrt{2+\sqrt{3}}}{2}=\frac{1+\sqrt{3}}{2\sqrt{2}} \\ \sin\left(\frac{\pi}{12}\right) &= \frac{\sqrt{2-\sqrt{3}}}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}} \end{aligned}$$

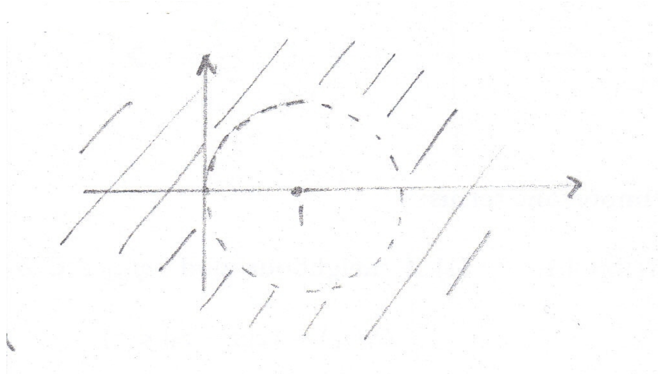
$$\begin{aligned} \therefore\;\; (1-\mathbf{i})^{1/3} &= 2^{-4/3}(1+\sqrt{3})-\mathbf{i}(\sqrt{3}-1)) \qquad (k=0) \\ &= 2^{-4/3}((1-\sqrt{3})-\mathbf{i}(1+\sqrt{3})) \qquad (k=1) \\ &= -2^{-1/3}(1+\mathbf{i}) \qquad (k=2) \end{aligned}$$

4. Loci and Regions in $\mathbb{C}$

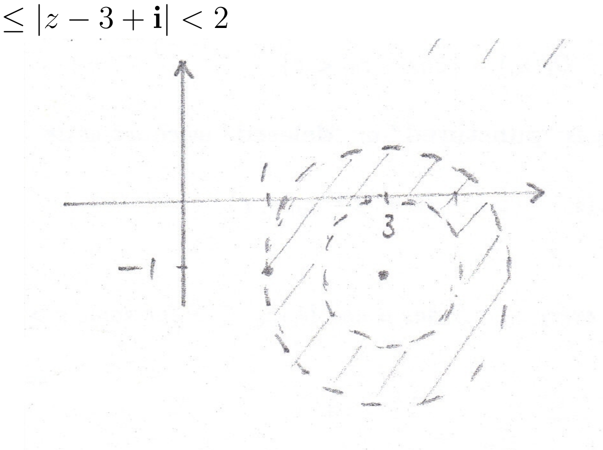
Example:

(1) Sketch $\{z \in \mathbb{C} | z\bar{z} > 2\mathrm{Re}(z)\} = \Omega$

$$x^2 + y^2 - 2x = 0 \Leftrightarrow (x-1)^2 + y^2 = 1$$



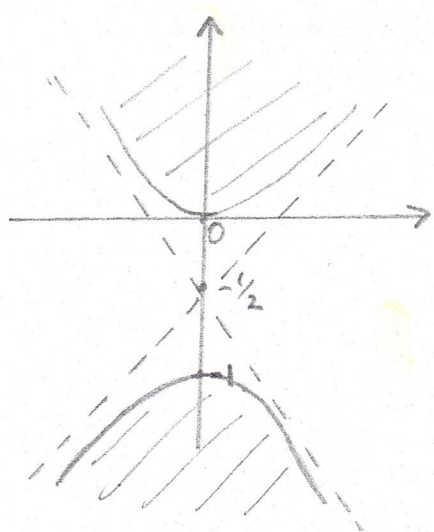
(2) $1 \leq |z - 3 + \mathbf{i}| < 2$



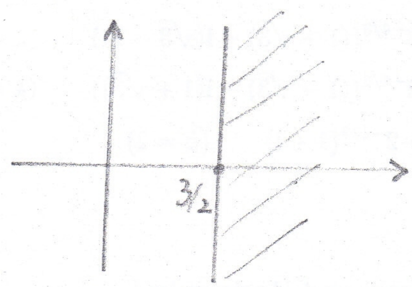
(3) $\mathrm{Im}(z) \geq \mathrm{Re}(z^2)$

$$z^2 = x^2 + 2\mathbf{i}xy - y^2$$

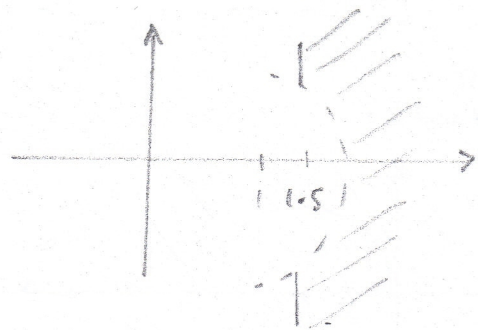
$$\begin{aligned} \mathrm{Re}(z^2) = x^2 - y^2 \quad \therefore \quad y = x^2 - y^2 \Rightarrow y^2 + y - x^2 &= 0 \\ \Rightarrow (y + \frac{1}{2})^2 - x^2 &= (\frac{1}{2})^2 \\ \therefore \mathrm{Im}(z) \geq \mathrm{Re}(z^2) \Leftrightarrow (y + \frac{1}{2})^2 - x^2 &\geq \frac{1}{4} \end{aligned}$$



(4) $\Sigma = \{z \in \mathbb{C} | \mathrm{Re}(z) \geq \frac{3}{2}\}$



(5) $\Omega \cap \Sigma = \{z \in \mathbb{C} | \frac{|z|^2}{2} > \mathrm{Re}(z) \geq \frac{3}{2}\}$



Some important terms:

Definition 4.1. (1) *A "***neighbourhood***" of* $z_0 \in \mathbf{C}$ *is a set*

$$D_r(z_0) = \{z||z - z_0| < r\}$$

(2) *A neighbourhood of* z_0 *is "***punctured***" or "***deleted***" when we write*

$$D_r(\hat{z}_0) = \{0 < |z - z_0| < r\}$$

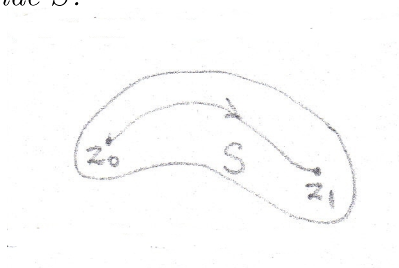
for some $r > 0$.

(3) *A set* S *is "***open***" if every* $z_0 \in S$ *has a nbd* $D_r(z_0) \subseteq S$ *for some* $r > 0$.

(4) $z_0 \in \mathbb{C}$ *is a "***boundary point***" (write* $z_0 \in \partial S$ *or* $z_0 \in Bd(S)$*) for* S *if for all* $r > 0$ $D_r(z_0) \cap S \neq \emptyset$, **and** $D_r(z_0) \cap \tilde{S} \neq \emptyset$ *(where* $\tilde{S}$ *denotes the complement of* S *in* $\mathbb{C}$*).*

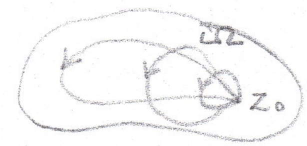
(5) $z_0 \in \mathbb{C}$ *is an "***interior***" (resp "***exterior***") point of* S *if there exists* $r > 0$ *such that* $D_r(z_0) \subseteq S$ *(resp.* $D_r(z_0) \subseteq \tilde{S}$*).*

(6) S *is "***connected***" if any two pts* $z_0, z_1 \in S$ *can be joined by a continuous* **path** *inside* S.

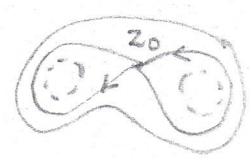


(7) $\Omega \subseteq \mathbb{C}$ *is a "***domain***" if open and connected.*

(8) $\Omega \subseteq \mathbb{C}$ *is a "***simply connected***" domain if every "***loop***" (i.e. every path from* $z_0 \in \Omega$ *to itself) inside* Ω *can be shrunk down to* z_0 *without going outside* Ω*. (cf. Examples 4,5)*



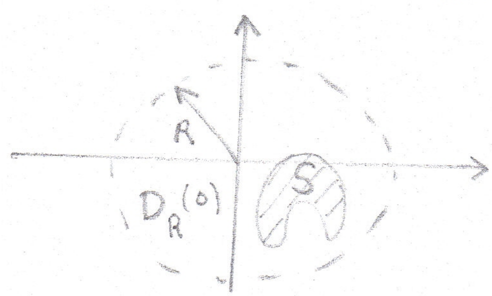
(9) Ω *is* "**multiply connected**" *if some path(s) cannot be shrunk (i.e., if* Ω *has one or more "holes" or punctures; cf. Ex.1,2)*



(10) $z_0 \in \mathbb{C}$ *is an* "**accumulation point**" *of* S *if for every* $r > 0$, $D_r(z_0) \cap S \neq \emptyset$ *($\therefore$ every boundary point of* S *is an accumulation point)*

(11) S *is a* "**region**" *if* $S = \Omega \cup \Sigma$ ($\Sigma \subseteq \partial\Omega$ *– cf. Ex.2,5), and a* "**closed**" *region if* $S = \Omega \cup \partial\Omega$ *for some* **domain** Ω.

(12) S *is* "**bounded**" *if there exists* $R > 0$ *st* $S \subset D_R(0)$



CHAPTER 2

Complex Functions

1. Continuity of complex functions

$$f(z) = w \in \mathbb{C} \therefore Dom(f) \subseteq \mathbb{C}\ ,\ Range(f) \subseteq \mathbb{C}$$

$$\left.\begin{array}{l} z = x + \mathbf{i}\ y \\ w = u + \mathbf{i}\ v \end{array}\right\} \Rightarrow f(z) = u(x,y) + \mathbf{i}\ v(x,y)$$

Example: $f(z) = |z|^2 + \mathbf{i}z$

$$\begin{aligned} w = |z|^2 + \mathbf{i}z &= x^2 + y^2 + \mathbf{i}(x + \mathbf{i}y) \\ &= x^2 + y^2 - y + \mathbf{i}x \\ \therefore u(x,y) &= x^2 + y^2 - y \\ v(x,y) &= x \end{aligned}$$

Conversely $\left.\begin{array}{l} z = x + \mathbf{i}\ y \\ \overline{z} = x - \mathbf{i}\ y \end{array}\right] \Rightarrow \left.\begin{array}{l} z + \overline{z} = 2x \\ z - \overline{z} = 2\mathbf{i}\ y \end{array}\right] \Rightarrow \begin{array}{l} x = \frac{1}{2}(z + \overline{z}) \\ y = \frac{-\mathbf{i}}{2}\ (z - \overline{z}) \end{array}$

Example: Consider $(u,v) : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$\begin{aligned} u(x,y) &= xy \\ v(x,y) &= x^2 + 2y^2. \end{aligned}$$

Find $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = u(x,y) + \mathbf{i}\ v(x,y)\ .$$

Solution:

$$xy = -\frac{\mathbf{i}}{4}(z + \overline{z})(z - \overline{z}) = -\frac{\mathbf{i}}{4}(z^2 - \overline{z^2})$$

$$\begin{aligned}
\mathbf{i}\ (x^2+2y^2) &= \mathbf{i}\left\{\frac{(z+\overline{z})^2}{4} - \frac{1}{2}(z-\overline{z})^2\right\} \\
&= \mathbf{i}\left\{z^2+2|z|^2\ +\overline{z}^2 - 2(z^2-2|z|^2+\overline{z}^2)\right\} \\
&= \frac{\mathbf{i}}{4}\left\{z^2+2|z|^2+\overline{z}^2 - 2(z^2-2|z|^2+\overline{z}^2)\right\} \\
&= \frac{\mathbf{i}}{4}\left\{-z^2+6|z|^2-\overline{z}^2\right\}
\end{aligned}$$

$$\therefore f(z) = u+\mathbf{i}v = -\frac{\mathbf{i}}{2}z^2 + \frac{3\mathbf{i}}{2}\ |z|^2 = \frac{\mathbf{i}}{2}\ (3|z|^2-z^2)$$

Definition 1.1. *Let $f:\mathbb{C}\to\mathbb{C}$ be a complex function, and $L\in\mathbb{C}$. We say*

$$\lim_{z\to z_0} f(z) = L$$

if for every $\varepsilon>0$ there is a $\delta>0$ such that $0<|z-z_0|<\delta \Rightarrow |\ f(z)-L|\ <\varepsilon$.

Example: $f(z) = \frac{\sin(|z|)}{|z|} + \mathbf{i}\ Re(z)^2$

Show $\lim\limits_{z\to 0} f(z) = 1$

Proof

$$\begin{aligned}
\sin(\theta) = \theta\ &-\ \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \\
\therefore \frac{\sin(\theta)}{\theta}\ &=\ 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \frac{\theta^6}{7!} + \cdots
\end{aligned}$$

Note $1-\frac{\theta^2}{6} < \frac{\sin(\theta)}{\theta} < 1+\frac{\theta^2}{6} \Rightarrow \left|\frac{\sin|z|}{|z|}-1\right|\ < \frac{|z|^2}{6}$

$$\begin{aligned}
\text{Now } |\ f(z)-1| &\le \left|\frac{\sin|z|}{|z|}-1\right| + |Re(z)|^2 && (\triangle \text{ inequality}) \\
&\le \left|\frac{\sin|z|}{|z|}-1\right| + |z|^2 && (|Re(z)|\ \le |z|\ ) \\
&< \frac{|z|^2}{6} + |z|^2 = \frac{7}{6}|z|^2
\end{aligned}$$

$$\therefore |z-0| = |z|\ < \sqrt{\frac{6}{7}\varepsilon} \Rightarrow |f(z)-1|\ < \varepsilon \therefore \text{ for any } \varepsilon>0 \text{ choose } \delta = \sqrt{\frac{6}{7}\varepsilon}$$

Example: $f(z) = |z|\ + \mathbf{i}\ \mathrm{A}rg(z)$. Show $\lim\limits_{z\to -1} f(z)$ **does not exist**

Remark

$\lim_{z \to z_0} f(z)$ **exists** if and only if $\lim_{(x,y)\to(x_0,y_0)} u(x,y)$ exists **and** $\lim_{(x,y)\to(x_0,y_0)} v(x,y)$ exists.

Recall $\mathrm{A}rg(z) \in (-\pi, \pi]$

$$\therefore \lim_{(x,y)\to(-1,0)} \arctan\left(\frac{y}{x}\right) = \begin{cases} -\pi \text{ if } (x,y) \in \text{ 3rd Quadrant} \\ \qquad\qquad (x<0, y<0) \\ \pi \text{ if } (x,y) \in \text{ 2nd Quadrant} \\ \qquad\qquad (x<0, y>0) \end{cases}$$

$$\therefore \lim_{z \to -1} \mathrm{A}rg(z) \text{ does not exist.}$$

Theorem: If $\lim_{z\to z_0} f(z) = L_1$ and $\lim_{z\to z_0} g(z) = L_2$, then

(i) $\lim_{z\to z_0} (f+g)(z) = L_1 + L_2$

(ii) $\lim_{z\to z_0} f(z)g(z) = L_1 L_2$

(iii) $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L_1}{L_2}$ if $L_2 \neq 0$.

Definition 1.2. *$f(z)$ is* **"continuous"** *at z_0 if* **all three** *conditions hold :*

(a) $f(z_0)$ *is* **defined**

(b) $\lim_{z\to z_0} f(z)$ **exists**

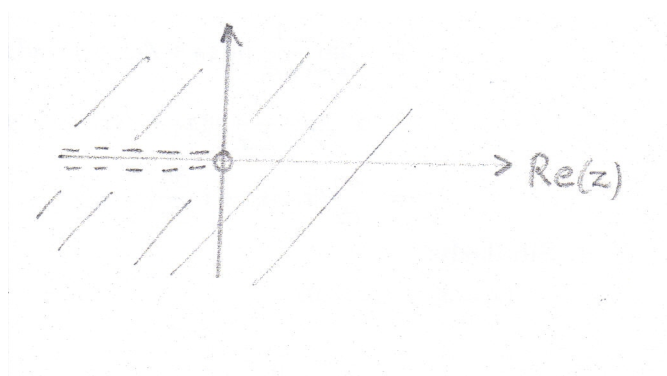
(c) $f(z_0) = \lim_{z\to z_0} f(z)$

Examples

(1) $f(z) = |z| + \mathbf{i}\, Arg(z)$ continuous at z_0 for all $z_0 \in \mathbb{C} \setminus \{Im(z) = 0,\ Re(z) \leq 0\}$

(2) $g(z) = \frac{\sin|z|}{|z|} + \mathbf{i}\, Re(z)^2$.

Note $g(0)$ **not defined**, but if we consider



$$\hat{g}(z) = \begin{cases} g(z) & z \neq 0 \\ 1 & z = 0 \end{cases} \quad \textit{then } \hat{g} \text{ \underline{cts} at \underline{0}}$$

Theorem

(1) If $f(z)$ cts at z_0 and $g(z)$ cts at z_0, then
   (a) $(f \pm g)(z)$ cts at z_0
   (b) $f(z)g(z)$ cts at z_0
   (c) $f(z)/g_{(z)}$ cts at z_0 if $g(z_0) \neq 0$

(2) If $f(z)$ cts at z_0 and $g(z)$ cts at $f(z_0) = w_0$, then the composition $g \circ f(z) = g(f(z))$ is cts at z_0.

(3) $f(z) = u(x, y) + \mathbf{i}\ v(x, y)$; $f(z)$ cts at z_0 if and only if $u(x, y)$ cts at (x_0, y_0) and $v(x, y)$ cts at (x_0, y_0).

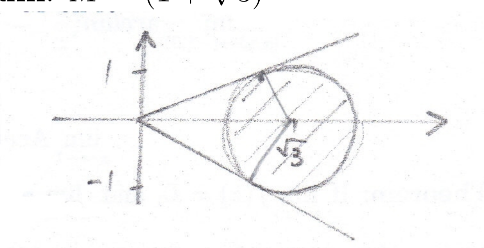
(4) If f cts at z_0 for all $z_0 \in R$ (a region in $\mathbb{C}$), then $|f(z)|$ also cts on R. If R **closed and bounded**, then $\exists\ M \in \mathbb{R}$ such that $|f(z)| \leq M$ for all $z \in R$, and $\exists z_0 \in R$ st $|f(z_0)| = M$.

Example $f(z) = |z| + \mathbf{i}\ \mathrm{A}rg(z)$. Find M st $|f(z)| \leq M$ on $R = \{z \in \mathbb{C} |\ |z - \sqrt{3}| \leq 1\}$

$|f(z)| = \sqrt{|z|^2 + (\mathrm{A}rg(z))^2}$

$\sqrt{3} - 1 \leq |z| \leq \sqrt{3} + 1; -\frac{\pi}{6} \leq \mathrm{A}rg\ (z) \leq \frac{\pi}{6}$

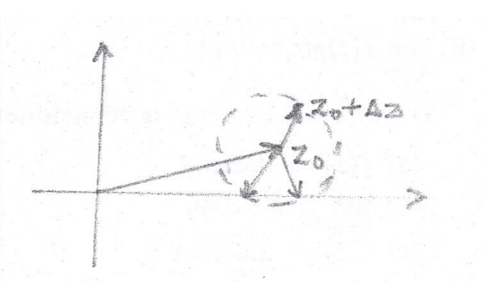
Claim: $M = (1 + \sqrt{3})$



2. The Complex Derivative

Definition 2.1. *Given $f(z)$ a complex function, we define the* **"derivative"** *at $z_0 \in \mathbb{C}$ by*

$$\boxed{f'(z_0) = \lim_{\triangle z \to 0} \frac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}}$$



Note: $\triangle z = \triangle x + \mathbf{i} \ \triangle y$

Special Case: $\begin{array}{l} y = y_0 \\ \triangle y = 0 \end{array}$ $\qquad f(z) = u(x, y) + \mathbf{i}v(x, y).$

$$\begin{aligned} f'(z_0) &= \lim_{\triangle x \to 0} \frac{1}{\triangle x} \left[u(x_0 + \triangle x, y_0) + \mathbf{i} \ v(x_0 + \triangle x, y_0) - u(x_0, y_0) - \mathbf{i} \ v(x_0, y_0)\right] \\ &= \lim_{\triangle x \to 0} \frac{1}{\triangle x} \left[u(x_0 + \triangle x, y_0) - u(x_0, y_0) + \mathbf{i}(v(x_0 + \triangle x, y_0) - v(x_0, y_0))\right] \\ &= \lim_{\triangle x \to 0} \frac{1}{\triangle x} \left[u(x_0 + \triangle x, y_0) - u(x_0, y_0)\right] \\ &+ \ \mathbf{i} \lim_{\triangle x \to 0} \frac{1}{\triangle x} \left[v(x_0 + \triangle x, y_0) - v(x_0, y_0)\right] \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + \mathbf{i} \ \frac{\partial v}{\partial x}(x_0, y_0) \end{aligned}$$

Similarly:

$(x = x_0 \ , \ \triangle x = 0)$

$$\begin{aligned} f'(z_0) &= \lim_{\triangle y \to 0} \frac{1}{\mathbf{i} \triangle y} \left[u(x_0, y_0 + \triangle y) - u(x_0, y_0) + \mathbf{i}(v(x_0, y_0 + \triangle y) - v(x_0, y_0))\right] \\ &= -\mathbf{i} \ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}(x_0, y_0) \end{aligned}$$

Conclusion: if $f'(z_0)$ exists, then

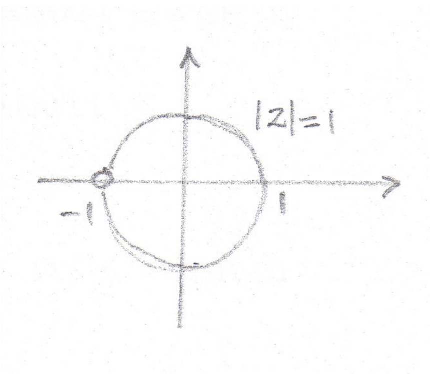
$$\left.\begin{array}{l} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \end{array}\right\} \textbf{Cauchy-Riemann Equations}$$

Remark: If $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at $z_0 \in \mathbb{C}$, and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, u, v$ are all cts on $D_r(z_0)$ for some $r > 0$, then $f'(z_0)$ **exists**.

Example:

(1) $f(z) = |z| \ + \mathbf{i} \ \mathrm{A}rg(z)$

$$u(x, y) = \sqrt{x^2 + y^2} \qquad v(x, y) = \arctan(\frac{y}{x})$$



$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2} \\
\frac{\partial u}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} \qquad \frac{\partial v}{\partial y} = \frac{x}{x^2+y^2} \\
\therefore \quad \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \Rightarrow \sqrt{x^2+y^2} = 1 \\
\frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \Rightarrow \sqrt{x^2+y^2} = 1
\end{aligned}$$

$\therefore f'(z_0)$ exists for all $z_0 \in \{z \in \mathbb{C} | \; |z| \; = 1 \; , \; \mathrm{A}rg(z) \neq \pi\}$ but **nowhere else**.

(2) $f(z) = c$ (constant $\in \mathbb{C}$) $\Rightarrow f'(z_0) = 0$ for all$z_0 \in \mathbb{C}$ (exercise)

$$\begin{aligned}
f(z) &= z^n \Rightarrow f'(z_0) = \lim_{\triangle z \to 0} \frac{1}{\triangle z} \left[(z_0 + \triangle z)^n - z_0^n \right] \\
&= \lim_{\triangle z \to 0} \frac{1}{\triangle z} \left[z_0^n + \binom{n}{1} z_0^{n-1} \triangle z + \cdots + \binom{n}{n-1} z_0 (\triangle z)^{n-1} + (\triangle z)^n - z_0^n \right] \\
&= n z_0^{n-1} \;\; \therefore \;\; f'(z_0) \text{ exists for all } z_0 \in \; \mathbb{C}.
\end{aligned}$$

Properties of the Derivative: Suppose $f'(z_0)$ and $g'(z_0)$ exist.

Theorem

(1) $(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0)$

(2) $(f \cdot g)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$ – Product Rule

(3) $(\frac{f}{g})'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{(g(z_0))^2}$ $\;(g(z_0) \neq 0)$ – Quotient Rule

(4) $(fog)'(z_0) = f'(g(z_0))g'(z_0)$ – Chain Rule.

Pf: 1-4 Same as for functions of a real variable.

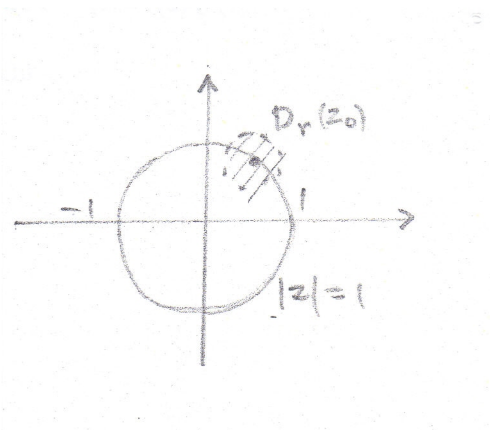
Definition 2.2. *$f(z)$ is* **analytic** *at $z_0 \in \mathbb{C}$ if $f'(z)$ exists for* **all** *$z \in D_r(z_0)$, some $r > 0$.*

Example:

(1) $f(z) = |z| \; + \mathbf{i}\mathrm{A}rg(z)$

$$f'(z_0) \text{ exists for } |z_0| \; = 1 \;, \; \mathrm{A}rg(z_0) \neq \pi \textbf{ only}$$

$$\therefore \textbf{ nowhere} \text{ analytic.}$$



(2) $f(z) = z^n \; (n \geq 0)$ $\quad .f'(z_0) = nz_0^{n-1}$ for all $z_0 \in \mathbb{C}$.

$$\therefore \text{analytic } \textbf{for all } z_0 \in \mathbb{C}$$

(Such f are said to be **entire** functions.)

if $f(z)$ is analytic for all $z_0 \in \Omega \subseteq \mathbb{C}$, we denote the **"derivative function"** $\frac{df}{dz}$, such that $\frac{df}{dz}(z_0) = f'(z_0)$ for all $z_0 \in \Omega$.

Example:

(1) $f(z) = c$ constant $\Rightarrow \frac{df}{dz} \equiv 0$
(2) $f(z) = z^n \Rightarrow \frac{df}{dz} = nz^{n-1}$
(3) $f(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n \Rightarrow$
$\frac{df}{dz} = a_1 + 2a_2 z + \cdots + (n-1)a_{n-1}z^{n-2} + na_n z^{n-1}$
(4) $f(z) = \frac{1}{z^n} \Rightarrow \frac{df}{dz} = -nz^{-n-1} = \frac{-n}{z^{n+1}}$

(5) Product, Quotient, Chain Rules for $\frac{d}{dz}$ (cf. previous Theorem)

Remark

(1) If $f(z)$ analytic on Ω, then $f(z)$ *continuous* on Ω (Proof easy).

(2) If $f(z)$ analytic on Ω, then $\frac{df}{dz}$ analytic on Ω (Proof discussed later).

L'Hôpital's Rule: If f , g analytic at $z_0 \in \mathbb{C}, f(z_0) = g(z_0) = 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \left(\frac{df}{dz} / \frac{dg}{dz} \right) = \frac{f'(z_0)}{g'(z_0)}$$

Pf:

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \frac{\lim\limits_{\triangle z \to 0} \dfrac{f(z_0 + \triangle z) - f(z_0)}{\triangle z}}{\lim\limits_{\triangle z \to 0} \dfrac{g(z_0 + \triangle z) - g(z_0)}{\triangle z}} = \frac{f'(z_0)}{g'(z_0)}$$

Now $\frac{df}{dz}, \frac{dg}{dz}$ **continuous** near $z_0 \Rightarrow \lim\limits_{z \to z_0} \dfrac{df}{dz} = f'(z_0)$

$$\lim_{z \to z_0} \frac{dg}{dz} = g'(z_0)$$

Example: $\lim\limits_{z \to \mathbf{i}} \dfrac{2z^7 + z + \mathbf{i}}{z^{24} - 1} = \dfrac{-13\mathbf{i}}{24}$

Theorem: If $f,\ g$ analytic at $z_0 \in \mathbb{C}$, then

(1) $f \pm g$ analytic at z_0

(2) $fg,\ f/g\ (g(z_0) \neq 0),\ f \circ g$ all analytic at z_0.

Example: $f(z) = z^3 + 2z - 1 \quad g(z) = z^2 + 1$

f, g are **entire** functions $\therefore \dfrac{f}{g}\ (z) = \dfrac{z^3 + 2z - 1}{z^2 + 1}$ is analytic for all $z_0 \in\ \mathbb{C}\ st\ z_0^2 + 1 \neq 0$, ie $z_0 \neq \pm\ \mathbf{i}$.

Definition 2.3. *If $f(z)$ not analytic at $z_0 \in \mathbb{C}$, but for all $r > 0$ there exists $\xi \in\ D_r(\hat{z}_0)$ such that f analytic at ξ then we say z_0 is a* **"singularity"** *of f.*

(**Ex.** $f(z) = \frac{z^3+2z-1}{z^2+1}$ has singularities at $z_0 = \pm\ \mathbf{i}$.)

3. Harmonic Functions

Consider $f(z) = u(x, y) + \mathbf{i}\ v(x, y)$ **analytic**. Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ;\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since $\frac{df}{dz} = f'(z)$ also analytic, it follows that u & v are **twice differentiable**, hence

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \ \& \ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}. \qquad (\dagger)$$

(**Theorem in** $\mathbb{R}^2$: If $\dfrac{\partial^2 v}{\partial x \partial y}, \dfrac{\partial^2 v}{\partial y \partial x}, \dfrac{\partial^2 v}{\partial x^2}, \dfrac{\partial^2 v}{\partial y^2}$ all **continuous** fns on $\Omega \subseteq \mathbb{R}^2$, then $\dfrac{\partial^2 v}{\partial x \partial y} = \dfrac{\partial^2 v}{\partial y \partial x}$ on $\Omega \subseteq \mathbb{R}^2$.)

$\therefore (\dagger) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Write $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta$, then f **analytic** on $\Omega \Rightarrow u, v$ satisfy **"Laplace's eqn":**

$$\Delta u = 0 \ , \ \Delta v = 0 \ , \text{ on } \Omega.$$

Definition 3.1. $\varphi(x, y) : \mathbb{R}^2 \to \mathbb{R}$ *is* **"harmonic"** *on* $\Omega \subseteq \mathbb{R}^2$ *if* $\Delta\varphi = 0$ *on* Ω. *When we say that* φ *is harmonic, it will also be assumed that all the second partial derivatives of* φ *are continuous on* Ω.

(Hence $f = u + \mathbf{i}\, v$ **analytic** $\Rightarrow u, v$ harmonic)

Example: $\varphi(x, y) = x^3 - 3y^2 x$

$$\frac{\partial^2 \varphi}{\partial x^2} = 6x \qquad \frac{\partial^2 \varphi}{\partial y^2} = -6x$$

$\therefore \ \Delta\varphi = 0$.

Theorem: If $\varphi(x, y)$ harmonic on a **simply connected** domain $\Omega \subseteq \mathbb{R}^2$, there exists an analytic fn $f(z)$ on $\Omega \subseteq \mathbb{C}$ such that $\varphi(x, y) = Re(f)$. Also there exists analytic $g(z)$ on $\Omega \subseteq \mathbb{C}$ such that $\varphi(x, y) = Im(g)$.

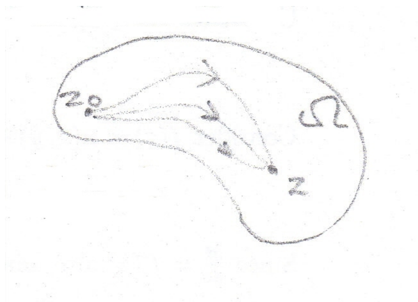
Proof: We have to use a **"Lemma"** which we'll prove later:

Lemma: If Ω simply connected, and $F(z)$ analytic on Ω, then there exists $f(z)$ analytic on Ω such that

$$\frac{df}{dz} = F(z).$$

(**Idea:** $f(z) = \displaystyle\int_{z_0}^{z} F(\xi) d\xi$, well defined if **all** paths from z_0 to z inside Ω are continuously "equivalent", i.e.,Ω is simply connected).

Now consider $F(z) = \dfrac{\partial\varphi}{\partial x} - \mathbf{i}\dfrac{\partial\varphi}{\partial y} := \mu + \mathbf{i}\nu$



$$F \textbf{ analytic} \text{ since the partial derivatives satisfy } \frac{\partial \mu}{\partial x} = \frac{\partial \nu}{\partial y} \ \& \ \frac{\partial \mu}{\partial y} = -\frac{\partial \nu}{\partial x}$$

and are all continuous, $\therefore$ there exists

$$f(z) = \vartheta(x,y) + \mathbf{i}\ \lambda(x,y) \text{ analytic}$$

such that

$$\frac{\partial \varphi}{\partial x} - \mathbf{i}\frac{\partial \varphi}{\partial y} = \frac{df}{dz} = \frac{\partial \vartheta}{\partial x} + \mathbf{i}\frac{\partial \lambda}{\partial x} = \frac{\partial \vartheta}{\partial x} - \mathbf{i}\frac{\partial \vartheta}{\partial y}$$

$$\therefore \ \frac{\partial \varphi}{\partial x} = \frac{\partial \vartheta}{\partial x}\ ,\ \frac{\partial \varphi}{\partial y} = \frac{\partial \vartheta}{\partial y} \Rightarrow \varphi(x,y) = \vartheta(x,y) + c$$

ie $\varphi = Re(f) - c$.

Alternatively: Consider $G(z) = \dfrac{\partial \varphi}{\partial y} + \mathbf{i}\ \dfrac{\partial \varphi}{\partial x} = \nu + \mathbf{i}\mu$

$$\left(G \text{ analytic, since } \frac{\partial \nu}{\partial x} = \frac{\partial \mu}{\partial y}\ ,\ \frac{\partial \nu}{\partial y} = -\frac{\partial \mu}{\partial x}\right)$$

$\therefore$ have $g(z) = u(x,y) + \mathbf{i}\ v(x,y)$ analytic such that

$$\frac{\partial \varphi}{\partial y} + \mathbf{i}\frac{\partial \varphi}{\partial x} = \frac{dg}{dz} = \frac{\partial u}{\partial x} + \mathbf{i}\ \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + \mathbf{i}\frac{\partial v}{\partial x}$$
$$\therefore\ \varphi(x,y) = v(x,y) + c' = \mathrm{Im}(g) + c'$$

Example: $\varphi(x,y) = x^3 - 3y^2x$. Find $f(z) st\ \varphi = \mathrm{Re}(f) \pm c$

$$\begin{aligned} f(z) = \vartheta(x,y) + \mathbf{i}\ \lambda(x,y)\ st\ \frac{\partial \vartheta}{\partial x} &= \frac{\partial \lambda}{\partial y} = \frac{\partial \varphi}{\partial x} = 3x^2 - 3y^2 \\ \frac{\partial \vartheta}{\partial y} &= -\frac{\partial \lambda}{\partial x} = \frac{\partial \varphi}{\partial y} = -6xy \end{aligned}$$

$$\begin{aligned}\therefore\ \vartheta &= \varphi \Rightarrow \lambda = \int 3(x^2-y^2)dy = \int 6\ xy\ dx\\ &= 3x^2y - y^3 + C(x) = 3x^2y + D(y)\end{aligned}$$

$$\therefore C(x) = c,\ \ D(y) = -y^3 + c \Rightarrow$$

$\lambda(x,y) = 3x^2y - y^3 + c \Rightarrow f(z) = x^3 - 3y^2x + \mathbf{i}(3x^2y - y^3) + \mathbf{i}c$
ie., $f(z) = (x+\mathbf{i}y)^3 = z^3 + \mathbf{i}c$

Definition 3.2. *Given* φ *st* $\Delta\varphi = 0$*, say* ψ *is the* **"harmonic conjugate"** *of* φ *if* $\varphi(x,y) + \mathbf{i}\psi(x,y) := f(z)$ **analytic**

Example: $\varphi = x^3 - 3y^2x \Rightarrow \psi = 3x^2 - y^3(+c)$.

Remark: If φ harmonic $= \mathrm{R}e(f)$ for some analytic $f(z)$, then $\varphi = \mathrm{I}m(\mathbf{i}f)$.

Example: $\varphi(x,y) = \dfrac{x}{x^2+y^2}, \dfrac{\partial^2\varphi}{\partial x^2} = \dfrac{2x(x-3y^2)}{(x^2+y^2)^3} = -\dfrac{\partial^2\varphi}{\partial y^2}$

$$\therefore\ \triangle\varphi = 0 \text{ on } \mathbb{R}^2\backslash\{(0,0)\}$$

$$F = \frac{\partial\varphi}{\partial x} - \mathbf{i}\frac{\partial\varphi}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} + \mathbf{i}\frac{2xy}{(x^2+y^2)^2} = \frac{-(x-\mathbf{i}\ y)^2}{(x^2+y^2)^2} = \frac{-(\overline{z})^2}{(z\overline{z})^2} = \frac{-1}{z^2}$$

$$\begin{aligned}= \frac{d}{dz}(\frac{1}{z})\ \therefore \varphi(x,y) &= \mathrm{Re}(\frac{1}{z}) = \mathrm{I}m(\frac{\mathbf{i}}{z})\\ \psi(x,y) &= \frac{-y}{x^2+y^2} = \mathrm{I}m(\frac{1}{z})\\ &= -\mathrm{Re}(\frac{\mathbf{i}}{z})\end{aligned}$$

CHAPTER 3

Transcendental Functions

1. The Exponential Function

Definition 1.1. $e^z = e^{x+\mathbf{i}y} = e^x(\cos y + \mathbf{i}\sin y)$

$\therefore e^{\mathbf{i}y} := \cos y + \mathbf{i}\sin y$ *for all* $y \in \mathbb{R}$

Properties:

(i) $e^x = e^{\mathrm{Re}(z)} = |e^z|,\ y = \mathrm{I}m(z) = \mathrm{A}rg(e^z) \pm 2\pi k \qquad k = 0, 1, 2, \cdots$

(ii) $e^0 = 1, e^{i\pi} = -1$

(iii)

$$\begin{aligned} u(x,y) &= e^x\cos(y), v(x,y) = e^x\sin(y) \\ \therefore \frac{\partial u}{\partial x} &= e^x\cos(y) = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -e^x\sin(y) = -\frac{\partial v}{\partial x} \end{aligned}$$

$\therefore$ e^z **analytic** on $\mathbb{C}$ (i.e. e^z **entire**).

(**Exercise:** show $\dfrac{d}{dz}[e^z] = e^z$ and $\dfrac{d}{dz}[e^{f(z)}] = f'(z)e^{f(z)}$.)

(iv)

$$\begin{aligned} e^{z+w} &= e^{\mathrm{Re}(z+w)+\mathbf{i}\mathrm{I}m(z+w)} \\ &= e^{\mathrm{Re}(z+w)}(\cos(\mathrm{I}m(z+w)) + \mathbf{i}\sin(\mathrm{I}m(z+w))) \\ e^z e^w &= e^{\mathrm{Re}(z)}e^{\mathrm{Re}(w)}\,(\cos(\mathrm{I}m(z))) + \mathbf{i}\sin(\mathrm{I}m(z)))(\cos(\mathrm{I}m(w)) + \mathbf{i}\sin\mathrm{I}m(w)) \\ &= e^{\mathrm{Re}(z+w)}\,\{\cos(\mathrm{I}m(z))\cos(\mathrm{I}m(w)) - \sin(\mathrm{I}m(z))\sin(\mathrm{I}m(w)) \\ &\qquad +\mathbf{i}(\sin(\mathrm{I}m(w))\cos(\mathrm{I}m(z)) + \cos(\mathrm{I}m(w))\sin(\mathrm{I}m(x)))\} \\ &= e^{\mathrm{Re}(z+w)}(\cos(\mathrm{I}m(z) + \mathrm{I}m(w)) + \mathbf{i}\sin(\mathrm{I}m(z) + \mathrm{I}m(w))) \\ &= e^{z+w}(\therefore (e^z)^m = e^{mz}) \end{aligned}$$

(v) $e^{z-w} = e^z/e^w$ (Proof similar)

Special Case: $z = 0 \Rightarrow e^{-w} = \frac{1}{e^w}$

$x = 0 : (e^{iy})^m = e^{imy} = \cos(my) + \mathbf{i}\sin(my)$ (cf. De Moivre)

Example: $f(z) = e^{(z-\frac{1}{z})\pi}$. Find

(a) $f(1+\mathbf{i})$
(b) $f'(1+\mathbf{i})$
(c) $\{z \in \mathbf{C} | |f(z)| = 1\}$

(a) $z = 1 + \mathbf{i} \Rightarrow \frac{1}{z} = \frac{1-\mathbf{i}}{2} \therefore z - \frac{1}{z} = \frac{1}{2} + \frac{3\mathbf{i}}{2}$

$$\begin{aligned} \therefore\ f(1+\mathbf{i}) = e^{(\frac{1}{2}+\frac{3\mathbf{i}}{2})\pi} &= e^{\frac{\pi}{2}}\left(\cos\left(\frac{\pi}{2}\right) + \mathbf{i}\sin\left(\frac{3\pi}{2}\right)\right) \\ &= -\mathbf{i}e^{\pi/2} \end{aligned}$$

(b)

$$\begin{aligned} f'(z) &= e^{(z-\frac{1}{z})\pi} \cdot \frac{d}{dz}\left(z - \frac{1}{z}\right) \cdot \pi \\ &= \pi e^{\pi(z-\frac{1}{z})}\left(1 + \frac{1}{z^2}\right) = \pi\left(1 + \frac{1}{z^2}\right) f(z) \\ f'(1+\mathbf{i}) &= -\pi\mathbf{i}e^{\pi/2}\left(1 + \frac{1}{(1+\mathbf{i})^2}\right) \\ &= -\pi\mathbf{i}e^{\pi/2}\left(1 + \frac{(1-\mathbf{i})^2}{4}\right) = \frac{\pi\mathbf{i}e^{\pi/2}}{4}(4 - 2\mathbf{i}) \end{aligned}$$

(c)

$$\begin{aligned} |f(z)| &= \left|e^{(z-\frac{1}{z})\pi}\right| = e^{\mathrm{Re}(z-\frac{1}{z})\pi} = e^{\pi(x-\frac{x}{x^2+y^2})} \\ &= 1 \Rightarrow e^{\pi\left(x-\frac{x}{x^2+y^2}\right)} = 1 \end{aligned}$$

$$\therefore \left(x - \frac{x}{x^2+y^2}\right) = 0 \ \therefore\ (x^2 + y^2 - 1) = 0 \text{ or } x = 0.$$

2. Trigonometric and Hyperbolic Functions

Recall

$$\begin{aligned} e^{\mathbf{i}y} &= \cos y + \mathbf{i}\sin(y) \\ \therefore\ e^{-\mathbf{i}y} &= \cos(-y) + \mathbf{i}\sin(-y) = \cos(y) - \mathbf{i}\sin(y) \\ \therefore\ \cos(y) &= \frac{1}{2}\left(e^{\mathbf{i}y} + e^{-\mathbf{i}y}\right) \\ \sin(y) &= \frac{1}{2\mathbf{i}}\left(e^{\mathbf{i}y} - e^{-\mathbf{i}y}\right) = \frac{\mathbf{i}}{2}\left(e^{-\mathbf{i}y} - e^{\mathbf{i}y}\right) \end{aligned}$$

Definition 2.1.

$$\begin{aligned} \cos(z) &= \frac{1}{2}\left(e^{\mathbf{i}z} + e^{-\mathbf{i}z}\right) \\ \sin(z) &= \frac{1}{2\mathbf{i}}\left(e^{\mathbf{i}z} - e^{-\mathbf{i}z}\right) \end{aligned}$$

Properties:

(1) $e^{\mathbf{i}z}, e^{-\mathbf{i}z}$ entire $\therefore$ $\sin(z)$ and $\cos(z)$ entire.

(2)

$$\begin{aligned} \frac{d}{dz}\sin(z) &= \frac{d}{dz}\left[\frac{1}{2\mathbf{i}}\left(e^{\mathbf{i}z} - e^{-\mathbf{i}z}\right)\right] \\ &= \frac{1}{2\mathbf{i}}\left(\mathbf{i}e^{\mathbf{i}z} + \mathbf{i}e^{-\mathbf{i}z}\right) = \frac{1}{2}\left(e^{\mathbf{i}z} + e^{-\mathbf{i}z}\right) \\ &= \cos(z) \\ \frac{d}{dz}\cos(z) &= -\sin(z) \quad \text{(Exercise)} \end{aligned}$$

(3) $\sin^2(z) + \cos^2(z) = 1$ (Exercise)

(4)

$$\begin{aligned} \sin(z \pm w) &= \frac{1}{2\mathbf{i}}\left[e^{\mathbf{i}(z\pm w)} - e^{-\mathbf{i}(z\pm w)}\right] \\ &= \frac{1}{2\mathbf{i}}\left[e^{\mathbf{i}z}e^{\pm\mathbf{i}w} - e^{-\mathbf{i}z}e^{\pm\mathbf{i}w}\right] \end{aligned}$$

$\sin(z)\cos(w) \pm \cos(z)\sin(w) = \frac{1}{4\mathbf{i}}\left[\left(e^{\mathbf{i}z} - e^{-\mathbf{i}z}\right)\left(e^{\mathbf{i}w} + e^{-\mathbf{i}w}\right) \pm \left(e^{\mathbf{i}z} + e^{-\mathbf{i}z}\right)\left(e^{\mathbf{i}w} - e^{-\mathbf{i}w}\right)\right]$
$= \frac{1}{4\mathbf{i}}\left[e^{\mathbf{i}(z+w)} + e^{\mathbf{i}(z-w)} - e^{-\mathbf{i}(z-w)} - e^{-\mathbf{i}(z+w)} \pm e^{\mathbf{i}(z+w)} \pm e^{\mathbf{i}(z-w)} \pm e^{-\mathbf{i}(z-w)} \pm e^{-\mathbf{i}(z+w)}\right]$
$= \frac{1}{2\mathbf{i}}\left[e^{\mathbf{i}(z+w)} - e^{-\mathbf{i}(z+w)}\right]$ or $\frac{1}{2\mathbf{i}}\left[e^{\mathbf{i}(z-w)} - e^{-\mathbf{i}(z-w)}\right]$
$= \sin(z \pm w)$.

(5) **Similarly**, $\cos(z \pm w) = \cos(z)\cos(w) \pm \sin(z)\sin(w)$.

Example: Where does $\frac{1}{\cos(\mathbf{i}z)}$ **fail** to be analytic?

Note: $f(z) = \cos(\mathbf{i}z)$ is **entire** $\therefore$ $\frac{1}{f(z)}$ analytic if $f(z) \neq 0$.

$$\begin{aligned}
\text{But } \cos(\mathbf{i}z) &= \frac{1}{2}\left[e^{-z} + e^{z}\right] = \frac{1}{2}\left[e^{-x}\left(\cos y - \mathbf{i}\sin y\right) + e^{x}\left(\cos y + \mathbf{i}\sin y\right)\right] \\
&= \frac{1}{2}\cos(y)\left(e^{-x} + e^{x}\right) + \frac{1}{2}\mathbf{i}\sin(y)\left(e^{x} - e^{-x}\right) \\
&= 0 \Leftrightarrow \cos(y) = 0 \ ; \ e^{x} = e^{-x}, \text{ i.e.} \\
&\quad \Leftrightarrow y = \frac{\pi}{2} \pm \pi k \ ; \ x = 0
\end{aligned}$$

Now define $\tan(z) = \frac{\sin(z)}{\cos(z)}$, $\sec(z) = \frac{1}{\cos(z)}$, etc

Recall "Hyperbolic" sine and cosine:

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}; \ \cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

Now define $\boxed{\sinh(z) = \frac{e^{z} - e^{-z}}{2}; \ \cosh(z) = \frac{e^{z} + e^{-z}}{2}}$

Properties:

(1) $\sinh(z), \cosh(z)$ **entire**.

(2) $\frac{d}{dz}\sinh(z) = \cosh(z)$; $\frac{d}{dz}\cosh(z) = \sinh(z)$

(3)
$$\begin{aligned}
\sinh(\mathbf{i}z) &= \frac{e^{\mathbf{i}z} - e^{-\mathbf{i}z}}{2} = \mathbf{i}\sin(z) \\
\cosh(\mathbf{i}z) &= \frac{e^{\mathbf{i}z} + e^{-\mathbf{i}z}}{2} = \cos(z)
\end{aligned}$$

(4) $\cosh^2(z) - \sinh^2(z) = (\cosh(z) + \sinh(z))(\cosh(z) - \sinh(z)) = e^{z} \cdot e^{-z} = 1$

$$\begin{aligned}
\text{Now define } \tanh(z) &= \frac{\sinh(z)}{\cosh(z)}; \\
sech(z) &= \frac{1}{\cosh(z)}; \text{ etc.}
\end{aligned}$$

Example: Give the domain of $\mathbb{C}$ in which $sech(z)$ is **analytic**.

Note: $\cosh(z)$ is **entire** $\therefore$ $sech(z)$ fails to be analytic when

$$\begin{aligned}
\cosh(z) = 0, \text{ i.e., } e^{z} + e^{-z} &= e^{x}(\cos y + \mathbf{i}\sin y) + e^{-x}(\cos y - \mathbf{i}\sin y) \\
&= 0
\end{aligned}$$

i.e., $\cos y(e^x + e^{-x}) = 0$ and $\sin y(e^x - e^{-x}) = 0$
i.e., $y = \frac{\pi}{2} \pm \pi k$ and $x = 0$.
$\therefore$ $sech(z)$ analytic on $\Omega = \mathbb{C} \setminus \{0 + \mathbf{i}\left(\frac{\pi}{2} \pm \pi k\right) | k = 0, 1, 2, \cdots\}$

3. The Complex Logarithm

Recall $e^z = e^x(\cos(y) + \mathbf{i}\sin(y)) \Rightarrow |e^z| = e^x$
$\mathrm{A}rg(e^z) = y = \mathrm{I}m(z) \pm 2\pi k$
$\therefore e^{\log z} = z \Rightarrow e^{\mathrm{Re}(\log(z))} = |z| \Rightarrow \mathrm{R}e(\log(z)) = \ln|z|$

$$\mathrm{A}rg(e^{\log(z)}) = \mathrm{A}rg(z) = \mathrm{I}m(\log(z)) \pm 2\pi k$$

$\therefore$ $\log(z) = \ln|z| + \mathbf{i}\mathrm{A}rg(z) \pm 2\pi k\mathbf{i}$; $k = 0, 1, 2, \cdots$

Definition 3.1. $Log(z) = \ln|z| + \mathbf{i}Arg(z)$

Example: $z_0 = \frac{1}{2}(-1 + \sqrt{3}\mathbf{i})$; $|z_0| = 1$; $\mathrm{A}rg(z_0) = \frac{2\pi}{3}$

$$\left.\begin{array}{rcl} Log(z_0) & = & \ln(1) + \frac{2\pi\mathbf{i}}{3} = \frac{2\pi\mathbf{i}}{3} \\ Log(z_0^2) & = & \ln(1) + \mathbf{i}\mathrm{A}rg(z_0^2) = -\frac{2\pi i}{3} \end{array}\right\} Log(z_0^2) = 2Log(z_0) - 2\pi\mathbf{i}$$

In general,

$$\begin{aligned} \log(z_1 z_2) &= \ln|z_1 z_2| + \mathbf{i}\mathrm{A}rg\ (z_1 z_2) \pm 2\pi k\mathbf{i} \\ &= \ln|z_1| + \ln|z_2| + \mathbf{i}\mathrm{A}rg(z_1) + \mathbf{i}\mathrm{A}rg(z_2) \pm 2\pi k\mathbf{i} \end{aligned}$$

$$\begin{aligned} \therefore\ \log(z_1 z_2) &= \log(z_1) + \log(z_2) \pm 2\pi k\mathbf{i} \\ \log(z^n) &= n\log(z) \pm 2\pi k\mathbf{i}. \end{aligned}$$

Example: Solve $(e^z - 1)^2 = e^{2z}$
Now $e^{2z} - 2e^z + 1 = e^{2z}$ iff $2e^z = 1$
i.e. $z = \ln\left(\frac{1}{2}\right) \pm 2\pi k\mathbf{i}$

$$\begin{aligned}
\text{Note } \log(e^z) &= \ln|e^z| + \mathbf{i}\mathrm{A}rg\ (e^z) \pm 2\pi k\mathbf{i} \\
&= \log e^{\mathrm{Re}(z)} + \mathbf{i}\mathrm{I}m(z) \pm 2\pi k\mathbf{i} \\
&= \mathrm{R}e(z) + \mathbf{i}\mathrm{I}m(z) \pm 2\pi k\mathbf{i} \\
&= z \pm 2\pi k\mathbf{i}
\end{aligned}$$

$\therefore\ Log(e^z) = z$ if $\mathrm{I}m(z) \in (-\pi, \pi]$

Consider $Log(z) = \ln|z| + \mathbf{i}\mathrm{A}rg(z) = u + \mathbf{i}v$

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{(x^2+y^2)^{1/2}} \cdot \frac{1}{2(x^2+y^2)^{1/2}} \cdot 2x = \frac{x}{x^2+y^2} \\
\frac{\partial u}{\partial y} &= \frac{1}{(x^2+y^2)^{1/2}} \cdot \frac{1}{2(x^2+y^2)^{1/2}} \cdot 2y = \frac{y}{x^2+y^2}
\end{aligned}$$

$$v = \mathrm{A}rg(z) = \begin{cases} \arctan(\frac{y}{x}) & x \neq 0 \\ \frac{\pi}{2} & x = 0, y > 0 \\ -\frac{\pi}{2} & x = 0, y < 0 \end{cases} \quad \therefore \text{cts on } \mathbb{C}\backslash\{y = 0, x \leq 0\}$$

$$\begin{aligned}
\therefore\ \frac{\partial v}{\partial x} &= \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2};\ \frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} \\
\therefore\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ on } \mathbb{C}\backslash\{y = 0, x \leq 0\}
\end{aligned}$$

$$\begin{aligned}
\text{Now } \frac{d}{dz}Log(z) &= \frac{\partial}{\partial x}\log(z) = \frac{\partial u}{\partial x} + \mathbf{i}\frac{\partial v}{\partial x} \\
&= \frac{x}{x^2+y^2} - \mathbf{i}\frac{y}{x^2+y^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}
\end{aligned}$$

(cf. $\frac{d}{dx}\log x = \frac{1}{x}$) $\therefore$ $Log(z)$ **analytic** on $\Omega = \mathbb{C}\backslash\{y = 0; x \leq 0\}$

Definition 3.2. *: The analytic function $f(z) = Log(z)$ on the domain*

$$\Omega = \mathbb{C}\backslash\{Im(z) = 0; Re(z) \leq 0\}$$

*is called the "***Principal Branch***" of* $\log(z)$.

Remark: $f(z) = \log|z| + \mathbf{i}\text{Arg}(z) + 2\pi\mathbf{i}k$, for any **fixed** $k = \pm 1, \pm 2$, defines a secondary (or non–Principal) branch of the complex logarithm on Ω. (**Note:** Ω **simply connected**).

The half-line $\{\text{Im}(z) = 0; \text{Re}(z) \leq 0\}$ is called a "**branch cut**" for $\log(z)$.

Example: Consider $\phi(x,y) = \arctan\left(\frac{x}{y}\right)$ where $-\pi < \arctan\left(\frac{x}{y}\right) \leq \pi$

Find

(a) The **harmonic conjugate** $\psi(x,y)$

(b) The domain Ω on which $\phi + \mathbf{i}\psi$ is **analytic**.

(a)

$$F(x,y) = \frac{\partial \phi}{\partial x} - \mathbf{i}\frac{\partial \phi}{\partial y} \;\; ; \;\; \frac{\partial \phi}{\partial x} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{1+\left(\frac{x}{y}\right)^2} \cdot \frac{-x}{y^2} = \frac{-x}{x^2+y^x}$$

$$\therefore \;\; F(x,y) = \frac{y+\mathbf{i}x}{x^2+y^2} = \frac{\mathbf{i}\bar{z}}{z\bar{z}} = \frac{\mathbf{i}}{z} = \frac{df}{dz} \qquad (y + \mathbf{i}x = \mathbf{i}\bar{z})$$

$$\Rightarrow f(z) = \mathbf{i}\log(z) = -\text{Arg}(z) + C + \mathbf{i}\log|z| \;\; (\therefore \psi = \log(x^2+y^2)^{\frac{1}{2}})$$

(b) But $\phi(x,y) = \text{R}e(f) = \arctan\left(\frac{x}{y}\right) = \arg(\mathbf{i}\bar{z}) = \frac{\pi}{2} - \arg(z)$

$$\text{where} \quad -\pi < \arctan\left(\frac{x}{y}\right) \leq \pi \;\Leftrightarrow\; -\pi < \frac{\pi}{2} - \arg(z) \leq \pi$$

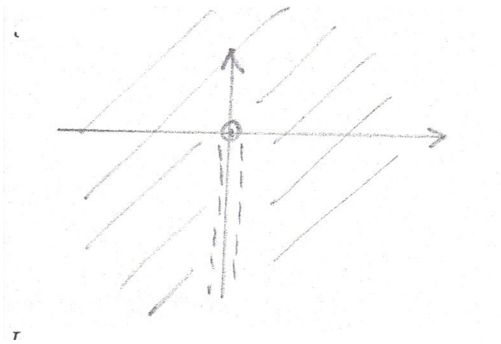
$$\Leftrightarrow\; -\frac{\pi}{2} \leq \arg(z) < \frac{3\pi}{2}$$

$$\therefore \text{ Define } \; f(z) = \frac{\pi}{2} - \arg(z) + \mathbf{i}\log|z|$$

$$= \arg(z) + \mathbf{i}\log|z| \text{ where } -\frac{\pi}{2} \leq \arg(z) < \frac{3\pi}{2}$$

$\therefore\;\; f = \phi + \mathbf{i}\psi$ has an **analytic branch** on

$$\Omega = \mathbb{C}\backslash\{\text{R}e(z) = 0; \text{I}m(z) \leq 0\}$$



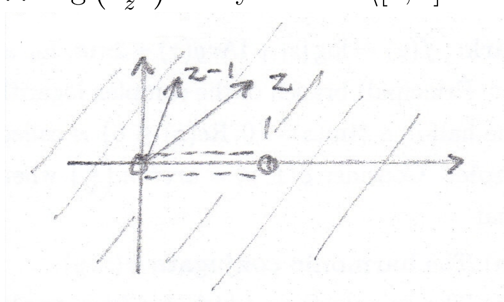
Example: Find the maximum domain $\Omega \subseteq \mathbb{C}$ on which $f(z) = \mathrm{Log}\left(\frac{z-1}{z}\right)$ is analytic.

Note: $\mathrm{Log}\left(\frac{z-1}{z}\right) = \log\left|\frac{z-1}{z}\right| + \mathbf{i}\mathrm{Arg}\left(\frac{z-1}{z}\right)$

Clearly $z \neq 0, 1, \mathrm{Arg}\left(\frac{z-1}{z}\right) \neq \pm\pi \Rightarrow \arg(z-1) \neq \arg(z) \pm \pi$

But $\arg(z-1) = \arg(z) \pm \pi \Leftrightarrow z \in (0,1)$

$\therefore \mathrm{Log}\left(\frac{z-1}{z}\right)$ analytic on $\mathbb{C}\backslash[0,1] = \Omega$



4. Complex Exponents

Recall $x \neq 0 \in \mathbb{R} \Rightarrow x^c = e^{c\log(x)}$ for any $c \in \mathbb{R}$

$\therefore$ **Define** $z \neq 0 \Rightarrow z^c = e^{c\log(z)}$ for any $c \in \mathbb{C}$

Example:

(1) $z = \mathbf{i}, c = \frac{1}{3} \Rightarrow$

$$\begin{aligned} c\log(z) &= \frac{1}{3}(\log|\mathbf{i}| + \mathbf{i}\mathrm{A}rg(\mathbf{i}) \pm 2\pi\mathbf{i}k) \\ &= \frac{\pi\mathbf{i}}{6} \pm \frac{2\pi k\mathbf{i}}{3} \quad k = 0, 1, 2 \\ &= \frac{\pi\mathbf{i}}{6}; \frac{5\pi\mathbf{i}}{6}; \frac{9\pi\mathbf{i}}{6} = \frac{3\pi\mathbf{i}}{2} = \frac{-\pi\mathbf{i}}{2} \end{aligned}$$

$$\therefore\; i^{\frac{1}{3}} = \begin{cases} e^{\frac{\pi}{6}\mathbf{i}} &= \frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{i} \quad (k=0) \\ e^{\frac{5\pi}{6}\mathbf{i}} &= -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathbf{i} \quad (k=1) \\ e^{-\frac{\pi}{2}\mathbf{i}} &= -\frac{1}{2}\mathbf{i} \quad (k=2) \end{cases}$$

(2)

$$\begin{aligned} z = 1+\mathbf{i}, c = \mathbf{i} \Rightarrow c\log(z) &= \mathbf{i}(\log|1+\mathbf{i}| + \mathbf{i}\mathrm{A}rg(1+\mathbf{i}) \pm 2\pi\mathbf{i}k) \\ &= \mathbf{i}(\log\sqrt{2} + \mathbf{i}\frac{\pi}{4} \pm 2\pi k\mathbf{i}) \\ &= -\frac{\pi}{4} \pm 2\pi k + \frac{\mathbf{i}}{2}\log 2 \end{aligned}$$

$$\begin{aligned} \therefore\; (1+\mathbf{i})^{\mathbf{i}} &= e^{-\left(\frac{1\pm 8k}{4}\right)\pi} e^{\mathbf{i}\log\sqrt{2}} \\ &= e^{-\left(\frac{1\pm 8k}{4}\right)\pi}\left(\cos(\log\sqrt{2}) + \mathbf{i}\sin(\log\sqrt{2})\right) \end{aligned}$$

Now $f(z) = z^c = e^{cLog(z)} \Rightarrow f(z)$ is **analytic** on the same **domain**

$$\Omega = \mathbb{C}\backslash\{\mathrm{I}m(z) = 0; \mathrm{R}e(z) \le 0\}$$

as $Log(z)$ (note e^z is **entire**) $\therefore$ we may define the "**Principal Branch**" of z^c as $e^{cLog(z)}$ on Ω. Moreover

$$\begin{aligned} \frac{d}{dz}z^c = \frac{d}{dz}e^{cLog(z)} &= e^{cLog(z)} \cdot \frac{d}{dz}(cLog(z)) \\ &= \frac{c}{z}e^{cLog(z)} = \frac{c}{z}z^c = cz^{c-1} \end{aligned}$$

Example: Using the principal branch of $f(z) = z^{\cosh(z)}$, find $f'\left(\frac{\pi i}{2}\right)$. Where is $f(z)$ analytic?

$$\begin{aligned} \frac{d}{dz}z^{\cosh(z)} &= \cosh(z)z^{\cosh(z)-1} \cdot \frac{d}{dz}\cosh(z) \\ &= \cosh(z) \cdot z^{\cosh(z)-1} + \sinh(z)z^{\cosh(z)}Log(z) \end{aligned}$$

$$\begin{aligned} \text{Now } \cosh\left(\frac{\pi\mathbf{i}}{2}\right) &= \frac{e^{\pi\mathbf{i}/2} + e^{-\pi\mathbf{i}/2}}{2} = 0 \\ \sinh\left(\frac{\pi\mathbf{i}}{2}\right) &= \frac{e^{\pi\mathbf{i}/2} - e^{-\pi\mathbf{i}/2}}{2} = \mathbf{i} \\ \therefore f'\left(\frac{\pi\mathbf{i}}{2}\right) &= 0 \cdot \left(\frac{\pi\mathbf{i}}{2}\right)^{-1} + \mathbf{i}Log\left(\frac{\pi\mathbf{i}}{2}\right) \end{aligned}$$

Note $\cosh(z)$ is **entire** $\therefore$ $z^{\cosh(z)} = e^{\cosh(z)\log(z)}$ is analytic on

$$\Omega = \{\mathrm{I}m(z) = 0; \mathrm{R}e(z) \leq 0\}.$$

(**Exercise:** Do the same for $f(z) = z^{\tanh(z)}$)

CHAPTER 4

The Complex Integral

1. Complex Path Integration

Definition 1.1. *: A "***smooth path***" between two points z_0 and z_1 in $\mathbb{C}$ corresponds to a map $\gamma : [0,1] \longrightarrow \mathbb{C}$, $\gamma(t) = x(t) + \mathbf{i}y(t)$, such that*

(i) $\gamma(0) = z_0; \gamma(1) = z_1$

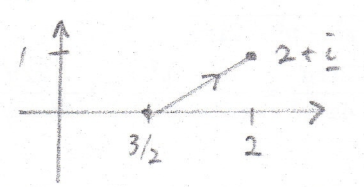
(ii) *$x(t)$ and $y(t)$ are differentiable functions for all $t \in (0,1)$*

(iii) *For each $z \in \gamma([0,1])$, there is* **at most one** *$t \in [0,1]$ such that $\gamma(t) = z$.*

Remark: γ may also be represented in "**implicit form**" as a graph $y = f(x)$ or $x = g(y)$ in $\mathbb{R}^2$.

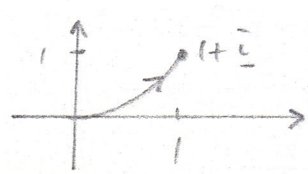
Examples:

(1) $z_0 = \frac{3}{2}, z_1 = 2 + \mathbf{i}, \gamma(t) = \frac{1}{2}(3+t) + \mathbf{i}t$; **implicit form:** $y = 2x - 3$ between $\frac{3}{2}$ and $2 + \mathbf{i}$



(2) $z_0 = \mathbf{i}, z_1 = 1, \gamma(t) = e^{\mathbf{i}(1-t)\frac{\pi}{2}}$; **implicit form:** $y = \sqrt{1-x^2}$ between $\mathbf{i}$ and 1.

(3) $z_0 = 0, z_1 = 1 + \mathbf{i}, \gamma(t) = t + \mathbf{i}t^2$; **implicit form:** $y = x^2$ between 0 and $1 + \mathbf{i}$



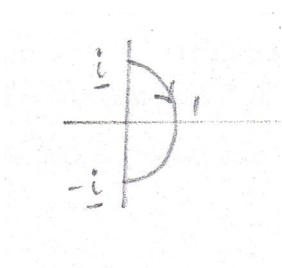
Definition 1.2. *:*

(i) "**Complex line element**" $\triangle z = \triangle x + \mathbf{i} \triangle y \Rightarrow$
"**Infinitesimal line element**" $dz = dx + \mathbf{i}dy$
Hence if $\gamma(t) = x(t) + \mathbf{i}y(t)$ smooth path in $\mathbb{C}$, can define "**path element**" $dz = \frac{dx}{dt}dt + \mathbf{i}\frac{dy}{dt}dt$. *Now*

(ii) *for any complex function $f(z) = u(x,y) + \mathbf{i}v(x,y)$, define the "Path integral"*

$$\begin{aligned}\int_{z_0}^{z_1} f(z)dz &:= \int_{z_0}^{z_1} (u(x,y) + \mathbf{i}v(x,y))(dx + \mathbf{i}dy) \\ &= \int_{x_0}^{x_1} udx - \int_{y_0}^{y_1} vdy + \mathbf{i}\left\{\int_{x_0}^{x_1} vdx + \int_{y_0}^{y_1} udy\right\} \\ &= \int_0^1 (ux' - vy')dt + \mathbf{i}\int_0^1 (vx' + uy')dt\end{aligned}$$

Example: $z_0 = \mathbf{i}, z_1 = -\mathbf{i}$, find $\int_{\mathbf{i}}^{-\mathbf{i}} \frac{1}{\bar{z}^2} dz$ along the **unit** semicircular arc lying in $\mathrm{Re}(z) \geq 0$.



Method (1):

$$\begin{aligned}f = u + \mathbf{i}v &= \frac{1}{(x - \mathbf{i}y)^2} = \frac{1}{x^2 - y^2 - 2\mathbf{i}xy} \\ &= \frac{(x^2 - y^2) + 2\mathbf{i}xy}{(x^2 - y^2)^2 + 4x^2y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} + \mathbf{i}\frac{2xy}{(x^2 + y^2)^2}\end{aligned}$$

$$\begin{aligned}\int_{\mathbf{i}}^{-\mathbf{i}} f(z)dz &= \int_0^0 udx - \int_1^{-1} vdy + \mathbf{i}\left\{\int_0^0 vdx + \int_1^{-1} udy\right\}\\ &= \int_{-1}^1 \frac{2xy}{(x^2+y^2)^2}dy - \mathbf{i}\int_{-1}^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy\end{aligned}$$

$x^2+y^2=1, \mathrm{Re}(z) = x \geq 0 \Rightarrow x = \sqrt{1-y^2}$

$$\begin{aligned}\therefore \int_{-1}^1 \frac{2xy}{(x^2+y^2)^2}dy &= \int_{-1}^1 2y\sqrt{1-y^2}dy = -\int_0^0 \sqrt{\mu}d\mu = 0 \ (\mu = 1-y^2)\\ \int_{-1}^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy &= \int_{-1}^1 (1-2y^2)dy = \frac{2}{3}\\ \therefore \int_{\mathbf{i}}^{-\mathbf{i}} f(z)dx = -\frac{2}{3}\mathbf{i}\end{aligned}$$

Method (2): $x(t) = \cos((1-2t)\frac{\pi}{2})$; $y(t) = \sin((1-2t)\frac{\pi}{2})$

$$\begin{aligned}\therefore ux'(t) - vy'(t) &= \left[\cos^2\left((1-2t)\frac{\pi}{2}\right) - \sin^2\left((1-2t)\frac{\pi}{2}\right)\right]\pi\sin\left((1-2t)\frac{\pi}{2}\right)\\ &\quad + \left[2\cos\left((1-2t)\frac{\pi}{2}\right)\sin\left((1-2t)\frac{\pi}{2}\right)\right]\pi\cos\left((1-2t)\frac{\pi}{2}\right)\end{aligned}$$

$$\begin{aligned}\therefore \int_0^1 ux'dt &= \int_0^0 2\mu^2 - 1d\mu = 0 \ (\mu = \cos((1-2t)\frac{\pi}{2}))\\ -\int_0^1 vy'dt &= \int_0^0 2\mu^2 d\mu = 0 \ (\mu = \cos((1-t)\frac{\pi}{2}))\end{aligned}$$

$$\therefore \int_0^1 ux' - vy'dt = 0$$

Exercise: Show $\int_0^1 vx' + uy'dt = -\frac{2}{3}$

Define

$$\begin{aligned}|dz| = |dx + \mathbf{i}dy| &= |x' + \mathbf{i}y'|dt\\ &= \sqrt{(x')^2 + (y')^2}dt\end{aligned}$$

$\therefore$ Given $\gamma : [0,1] \longrightarrow \mathbb{C}$ a smooth path st $\gamma(0) = z_0, \gamma(1) = z_1$

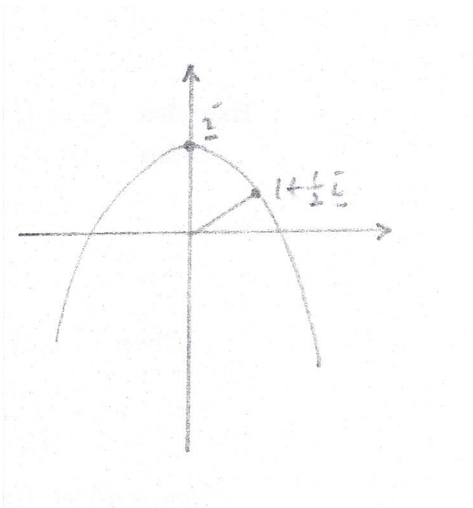
$$\int_{z_0}^{z_1} |dz| = \int_0^1 \sqrt{(x')^2+(y')^2}\ dt = L(\gamma) \ \textbf{“path length”}$$

Now consider $f(z)$ st $f \circ \gamma : [0,1] \longrightarrow \mathbb{C}$, and $|f \circ \gamma(t)| \leq M_\gamma$, i.e., M_γ is an upper bound for $|f|$ on the image of γ, then

$$\begin{aligned}
|\int_{z_0}^{z_1} f(z)dz| &\leq \int_{z_0}^{z_1} |f(z)||dz| \\
&= \int_0^1 |f \circ \gamma(t)| \sqrt{(x')^2 + (y')^2}\ dt \\
&\leq M_\gamma \int_0^1 \sqrt{(x')^2 + (y')^2}\ dt = M_\gamma L(\gamma)
\end{aligned}$$

Example: Consider $I = \displaystyle\int_{\mathbf{i}}^{1+\frac{1}{2}\mathbf{i}} e^{\mathbf{i}\,\mathrm{Log}(\bar{z})} dz$ along $y = 1 - \frac{1}{2}x^2$. Show that $|I| \leq (1.2)e^{\pi/2}$ without evaluating I precisely.

$$\begin{aligned}
y = 1 - \frac{x^2}{2} \Longrightarrow x &= t \\
y &= 1 - \frac{t^2}{2} \therefore \sqrt{(x')^2 + (y')^2} = \sqrt{1+t^2} \\
\Longrightarrow L(\gamma) &= \int_0^1 \sqrt{1+t^2}\ dt \qquad \begin{array}{l} t = \tan(\theta) \Longrightarrow \sqrt{1+t^2} = \sec(\theta) \\ dt = \sec^2(\theta)d\theta \end{array} \\
&= \int_0^{\pi/4} \sec^3\theta d\theta = \int_0^{\pi/4} \sec\theta + \tan^2\theta \sec\theta d\theta \\
\int_0^{\pi/4} \sec\theta d\theta &= \ln(\sec\theta + \tan\theta)\Big]_0^{\frac{\pi}{4}} = \ln(1+\sqrt{2}) \\
\int_0^{\pi/4} \tan^2\theta \sec\theta d\theta &= \tan\theta\sec\theta\Big]_0^{\pi/4} - \int_0^{\pi/4} \sec^3\theta d\theta
\end{aligned}$$



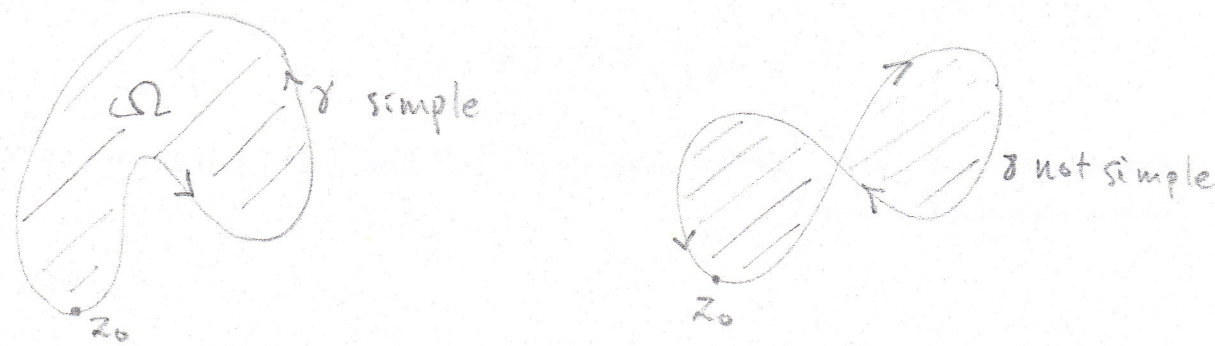
$$\therefore \int_0^{\pi/4} \sec^3 \ \theta \ d \ \theta = \frac{1}{2}(\ln(1+\sqrt{2})+\sqrt{2}) = L(\gamma)$$

Now

$$\begin{aligned} f(z) = e^{\mathbf{i}\,\mathrm{Log}(\bar{z})} \Longrightarrow |f(z)| &= \left|e^{\mathbf{i}\log|\bar{z}| - \mathrm{Arg}(\bar{z})}\right| \\ &= e^{-\mathrm{Arg}(\bar{z})} = e^{\mathrm{Arg}(z)} \leq e^{\pi/2} \\ \therefore M_\gamma L(\gamma) &= \frac{e^{\,\pi/2}}{2}\left\{\ln(1+\sqrt{2})+\sqrt{2}\right\} \\ &\approx 1.15 e^{\pi/2} \end{aligned}$$

2. Cauchy's Theorem

Definition 2.1. *A "***simple closed contour***" is a* **loop** γ $(\gamma(0)=\gamma(1)=z_0)$ *such that* $\gamma([0,1])$ *forms the boundary of a simply connected bounded domain* $\Omega \subseteq \mathbb{C}$



We say γ has "**positive**" orientation with respect to Ω if interior of Ω always left of $\gamma(t)$ as t moves from 0 to 1. For **positively** oriented γ, we define the "**contour integral**" $\displaystyle\int_\gamma f(z)\ dz$, and recall

Green's theorem: If $P(x,y), Q(x,y), \dfrac{\partial P}{\partial x}, \dfrac{\partial P}{\partial y}, \dfrac{\partial Q}{\partial x}, \dfrac{\partial Q}{\partial y}$ are **all** *cts fns* on $\Omega \cup \partial\Omega$, where $\partial\Omega = \gamma$ is a positively oriented simple closed contour, then

$$\int_\gamma P\ dx + Q\ dy = \iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dx\ dy.$$

Now $f(z) = u + \mathbf{i}v$ **analytic** $\Longrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\begin{aligned} \therefore \int_\gamma f(z)\ dz &= \int_\gamma u\ dx - v\ dy + \mathbf{i} \int_\gamma v\ dx + u\ dy \\ &= -\iint_\Omega \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)\ dx\ dy + \ \mathbf{i} \iint_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)\ dx\ dy \\ &= 0, \end{aligned}$$

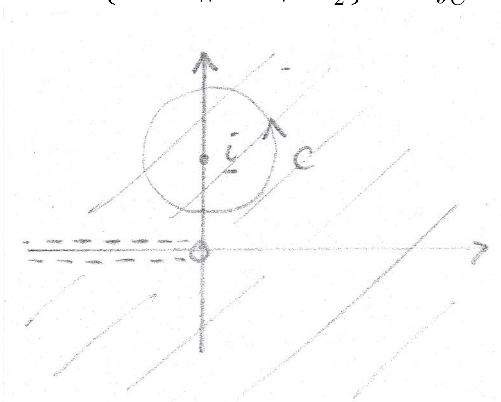
and we have proved

Cauchy's theorem: Let C be a closed simple contour, and let $f(z)$ be a complex function which is analytic on some domain containing C. Then $\int_C\ f(z)\ dz = 0$.

Example:

(1) $f(z) = \mathrm{Log}(z)$, analytic on $\Omega = \mathbb{C} \setminus \{\mathrm{Im}(z) = 0; \mathrm{Re}(z) \le 0\}$

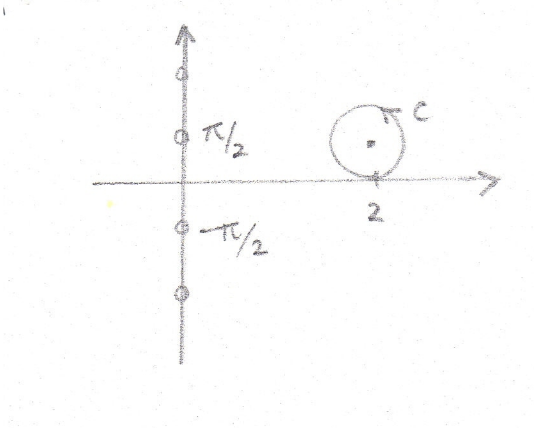
$\therefore C = \left\{z \in \mathbb{C} \middle| |z - \mathbf{i}| = \frac{1}{2}\right\} \Longrightarrow \int_C Log(z)\ dz = 0$



(2) $f(z) = \tanh(z) = \dfrac{\sinh(z)}{\cosh(z)}$

$\therefore$ analytic on $\Omega = \mathbb{C} \setminus \left\{z = \mathbf{i}\left(\frac{\pi}{2} \pm \pi k\right), k = 0, 1, 2, \cdots\right\}$

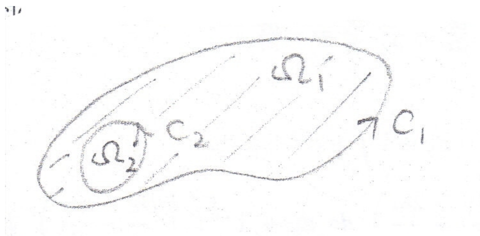
$\Longrightarrow \displaystyle\int_{|z-2-\mathbf{i}|=1} \tanh(z)\ dz = 0$



(3) $f(z) = \frac{1}{z}$ analytic on $\mathbb{C} \setminus \{0\}$ $\therefore$ **cannot** apply Cauchy theorem to $\displaystyle\int_{|z|=1} \frac{dz}{z}$. But $|z| = 1 \Longrightarrow z(t) = e^{\mathbf{i}2\pi t} \Longrightarrow$

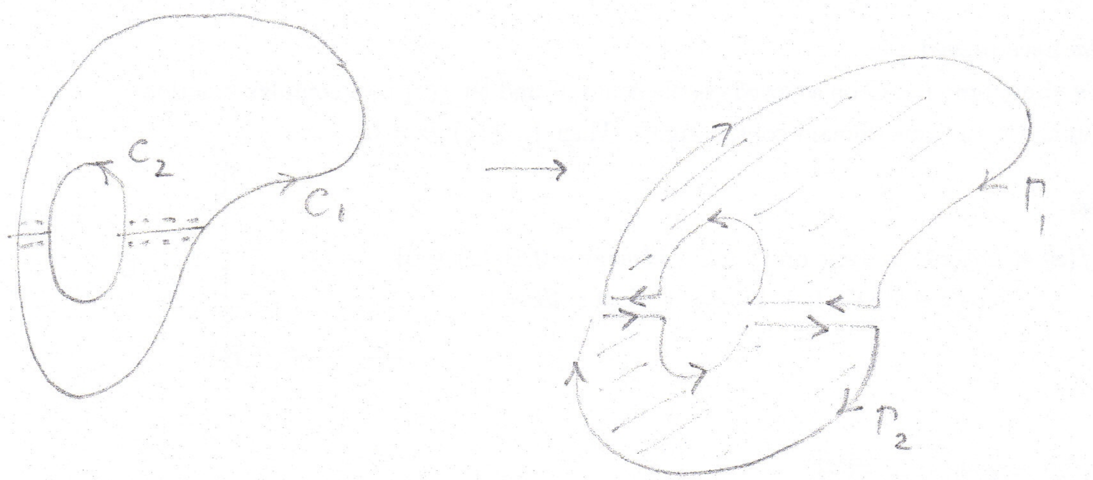
$$\int_{|z|=1} \frac{dz}{z} = \int_0^1 \frac{2\pi\mathbf{i}e^{\mathbf{i}2\pi t}}{e^{2\pi\mathbf{i}t}}\ dt = 2\pi\mathbf{i} \int_0^1\ dt = 2\pi\mathbf{i}$$

Exercise: Show $\displaystyle\int_{|z|=1} \frac{dz}{z^n} = 0$ for all integer $n \geq 2$.



Deformation of Contours:

Let C_1, C_2 be closed simple contours such that $C_j = \partial\Omega_j, j = 1, 2,\ C_2 \subset \Omega_1$



Divide $\Omega_1 \setminus \Omega_2$ into **two** simply connected domains with boundary Γ_1, Γ_2 as follows

Now consider $f(z)$ **analytic** on $\Omega_1 \setminus \Omega_2 \Longrightarrow$

$$\int_{\Gamma_1} f(z)\ dz \ = \ \int_{\Gamma_2} f(z)\ dz \ = \ 0 \text{ by Cauchy's theorem.}$$

Note

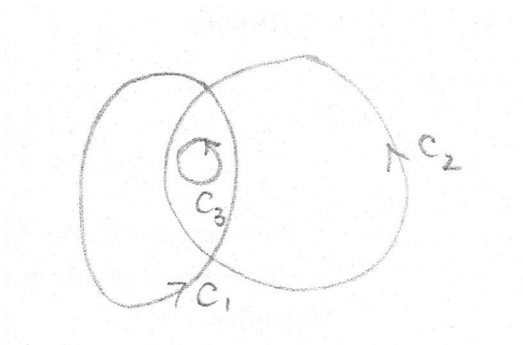
$$\Gamma_1 \cup \Gamma_2 = C_2 \cup (-C_1) \therefore 0 = \int_{\Gamma_1 \cup \Gamma_2} f(z)\ dz \ = \ \int_{C_2 \cup (-C_1)} f(z)\ dz$$
$$= \int_{C_2} f(z)\ dz - \int_{C_1} f(z)\ dz$$

$\therefore \int_{C_1} f(z)\ dz \ = \ \int_{C_2} f(z)\ dz.$

More generally, consider C_1, C_2 as follows:

Then $\exists\, C_3$ such that

$$\begin{aligned} \int_{C_1} f(z)\, dz &= \int_{C_3} f(z)\, dz \\ \int_{C_2} f(z)\, dz &= \int_{C_3} f(z)\, dz \\ \therefore \int_{C_1} f(z)\, dz &= \int_{C_2} f(z)\, dz \end{aligned}$$

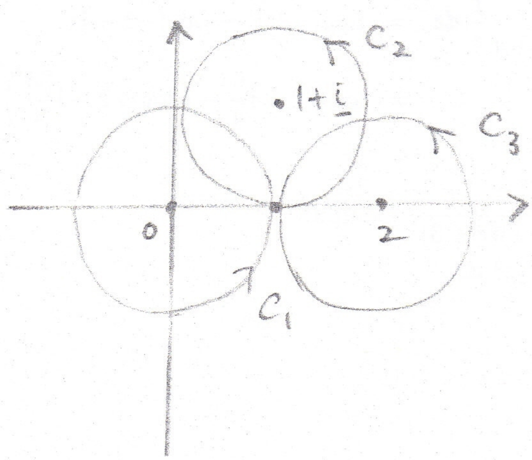


Theorem (deformation of contours)

If C_1, C_2 are closed simple contours such that C_1 can be continuously deformed into C_2 without passing through a singularity of some analytic function $f(z)$, then

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$$

Example: $f(z) = \frac{1}{z}$; $\quad C_1 = \{|z| = 1\}$; $\quad C_2 = \{|z - 1 - \mathbf{i}| = 1\}$; $\quad C_3 = \{|z - 2| = 1\}$



$$\int_{C_1} f(z)\ dz\ =\ 2\pi\mathbf{i}\ ;\quad \int_{C_2} f\ dz\ =\ \int_{C_3} f\ dz\ =\ 0$$

3. The Fundamental Theorem of Calculus

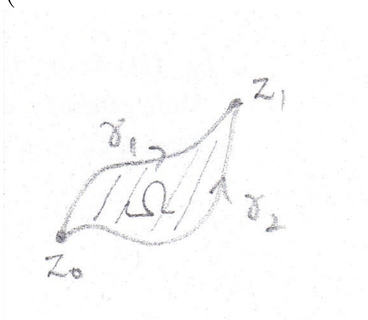
Principal of Path Independence:

If $\gamma_1, \gamma_2 : [0,1] \longrightarrow \mathbb{C}$ are two paths such that $\gamma_j(0) = z_0, \gamma_j(1) = z_1, j = 1,2$ and the region Ω bounded by $\gamma_1 \cup \gamma_2$ is **simply connected**, then any function $f(z)$, analytic on a domain containing Ω, satisfies

$$\int_{\gamma_1} f(z)\ dz\ =\ \int_{\gamma_2} f(z)\ dz,$$

$$\text{ie} \int_{z_0}^{z_1} f(z)\ dz \quad \frac{\textbf{does not depend on } \gamma}{st\ \gamma(0) = z_0, \gamma(1) = z_1}$$

(**Proof:**



$$\int_{\gamma_2 - \gamma_1} f(z)\ dz\ =\ 0\ = \int_{\gamma_1} f(z)\ dz\ -\ \int_{\gamma_2} f(z)\ dz.)$$

Now suppose $f(z) = \frac{dF}{dz}$ on some simply connected domain $\Omega \subseteq \mathbb{C}$, then for any $\gamma : [0,1] \longrightarrow \Omega\ st\gamma(0) = z_0, \gamma(1) = z_1 \in \Omega$ we have the **Fundamental Theorem of Calculus:**

$$\begin{aligned} \int_{z_0}^{z_1} f(z)\ dz &= \int_{z_0}^{z_1} \tfrac{d}{dz} F(z)\ dz &&= \int_0^1 \tfrac{dF}{dz}\ \tfrac{dz}{dt}\ dt \\ &= \int_0^1 \tfrac{d(F\circ\gamma)}{dt}\ dt &&= F(\gamma(1)) - F(\gamma(0)) \\ & &&= F(z_1) - F(z_0). \end{aligned}$$

Example:

(1)

$$\begin{aligned} f(z) = \frac{1}{z} &= \frac{d}{dz}\operatorname{Log}(z) \text{ on } \mathbb{C} \setminus \{\operatorname{Im}(z) = 0, \operatorname{Re}(z) \le 0\} \\ \Longrightarrow \int_{-1-\mathbf{i}}^{1+\mathbf{i}} \frac{1}{z}\, dz &= \operatorname{Log}(1+\mathbf{i}) - \operatorname{Log}(-1-\mathbf{i}) \\ &= \log\sqrt{2} + \mathbf{i}\frac{\pi}{4} - \left(\log\sqrt{2} - \mathbf{i}\frac{3\pi}{4}\right) \\ &= \mathbf{i}\pi \end{aligned}$$

(2) $f(z) = \dfrac{1}{(z-1)(z+3)}$. Compute

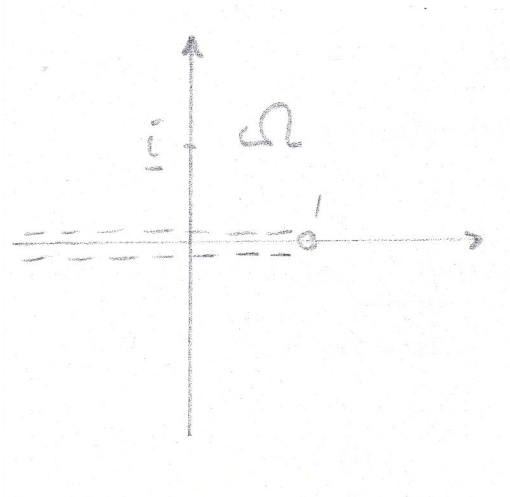
(a) $\int_i^{2+2\mathbf{i}} f(z)\, dz$

(b) $\int_{|z+3|=1} f(z)\, dz$

Solution: (a) $\dfrac{1}{(z-1)(z+3)} = \dfrac{A}{(z-1)} + \dfrac{B}{(z+3)}$

$$\Longrightarrow (A+B)z + 3A - B = 1, \text{ ie } \left.\begin{array}{rcl} A+B &=& 0 \\ 3A-B &=& 1 \end{array}\right] A = \frac{1}{4}\ B = \frac{-1}{4}$$

$$\begin{aligned} \therefore \int_{\mathbf{i}}^{2+2\mathbf{i}} f(z)\, dz &= \frac{1}{4}\int_{\mathbf{i}}^{2+2\mathbf{i}} \frac{1}{z-1}\, dz - \frac{1}{4}\int_{\mathbf{i}}^{2+2\mathbf{i}} \frac{1}{z+3}\, dz \\ &= \frac{1}{4}\operatorname{Log}(z-1)\Big]_{\mathbf{i}}^{2+2\mathbf{i}} - \frac{1}{4}\operatorname{Log}(z+3)\Big]_{\mathbf{i}}^{2+2\mathbf{i}} \\ &= \frac{1}{4}\{\log|1+2i| + \mathbf{i}\operatorname{Arg}(1+2\mathbf{i}) - \log|-1+\mathbf{i}| - \mathbf{i}\operatorname{Arg}(-1+\mathbf{i}) \\ &\qquad - \log|5+2\mathbf{i}| - \mathbf{i}\operatorname{Arg}(5+2\mathbf{i}) + \log|3+\mathbf{i}| + \mathbf{i}\operatorname{Arg}(3+\mathbf{i})\} \\ &= \frac{1}{4}\left\{\log\left(\frac{5}{\sqrt{29}}\right) + \mathbf{i}\operatorname{Arg}\left(\frac{(1+2\mathbf{i})(3+\mathbf{i})}{(-1+\mathbf{i})(5+2\mathbf{i})}\right)\right\} \\ &= \frac{1}{4}\left\{\log\left(\frac{5}{\sqrt{29}}\right) + \mathbf{i}\operatorname{Arg}\left(\frac{14}{58} - \mathbf{i}\frac{52}{58}\right)\right\} \\ &= \frac{1}{4}\log\left(\frac{5}{\sqrt{29}}\right) + \frac{\mathbf{i}}{4}\arctan(-\frac{52}{14}) \end{aligned}$$



$$(b) \int_{|z+3|=1} f(z)\, dz = \frac{1}{4}\int_{|z+3|=1} \frac{1}{z-1}\, dz \;-\; \frac{1}{4}\int_{|z+3|=1} \frac{1}{z+3}\, dz$$

Now $\displaystyle\int_{|z+3|=1} \frac{1}{z-1}\, dz \;=\; 0$ by Cauchy's theorem, while

$$w = z+3 \Longrightarrow \qquad \int_{|z+3|=1} \frac{1}{z+3}\, dz \;=\; \int_{|w|=1} \frac{1}{w}\, dw \;=\; 2\pi\mathbf{i}$$

$$\therefore \int_{|z+3|=1} \frac{dz}{(z-1)(z+3)} \;=\; \frac{-2\pi\mathbf{i}}{4} \;=\; \frac{-\pi\mathbf{i}}{2}$$

4. The Cauchy Integral Formula

Theorem: Let $f(z)$ be analytic on a domain $\Omega \subseteq \mathbb{C}$. Let C be a closed simple contour in Ω, and $z_o \in \Omega$ lying in the interior of C, then $\boxed{f(z_0) = \frac{1}{2\pi\mathbf{i}} \int \frac{f(z)dz}{z-z_0}}$.

Proof: For r sufficiently small, let $|z-z_0| = r$ lie in the interior of C, then deformation of contours $\Rightarrow$

$$\int_C \frac{f(z)dz}{z-z_0} = \int_{|z-z_0|=r} \frac{f(z)dz}{z-z_0},$$

since $\frac{f(z)}{z-z_0}$ **analytic** in the region between $|z-z_0| = r$ and C.

Similarly $\displaystyle\int_C \frac{dz}{z-z_0} = \int_{|z-z_0|=r} \frac{dz}{z-z_0} = 2\pi\mathbf{i}.$

Now consider

$$\int_{|z-z_0|=r} \frac{f(z)}{z-z_0} dz - 2\pi \mathbf{i} f(z_0) = \int_{|z-z_0|=r} \frac{f(z)-f(z_0)}{z-z_0} dz.$$

Note that $f(z)$ **cts** at $z_0 \Rightarrow \forall \varepsilon > 0,\ \exists \delta > 0$ such that $|z-z_0| < \delta \Rightarrow |f(z)-f(z_0)| < \varepsilon$.

$\therefore$ choose $r < \delta$ so that $\left.\frac{|f(z)-f(z_0)|}{|z-z_0|}\right|_{|z-z_0|=r} \le \sup_{|z-z_0|=r} \frac{|f(z)-f(z_0)|}{r} < \frac{\varepsilon}{r}$.

$$\text{Hence } M_\gamma L(\gamma) \text{ inequality, cf. p. 35 } \Rightarrow \int_{|z-z_0|=r} \frac{|f(z)-f(z_0)|}{|z-z_0|} |dz| \le M \int_{|z-z_0|=r} |dz| < \frac{\varepsilon}{r} \cdot 2\pi r = 2\pi\varepsilon$$

$$\text{Now } \left| \int_C \frac{f(z)dz}{z-z_0} - 2\pi \mathbf{i} f(z_0) \right| = \left| \int_{|z-z_0|=r} \frac{f(z)-f(z_0)dz}{z-z_0} \right| \qquad (*)$$

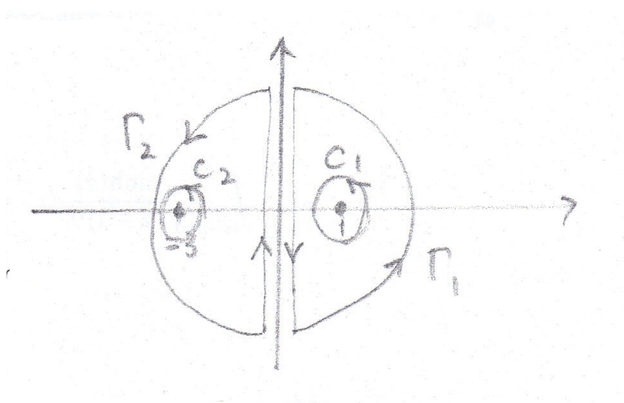
$$\le \int_{|z-z_0|=r} \frac{|f(z)-f(z_0)|}{|z-z_0|} |dz| < 2\pi\varepsilon. \qquad (\dagger)$$

But (*) **does not depend** on $r > 0, \therefore$ for any $\varepsilon > 0$, choose $r < \delta$, and conclude $(\dagger) \Rightarrow \int_C \frac{f(z)}{z-z_0} dz = 2\pi \mathbf{i} f(z_0)$.

Example:

(1)

$$\begin{aligned} g(z) &= \frac{z^2+3z-2}{z^2+2z-3} \\ \int_{|z+3|=1} g(z)dz &= \int_{|z+3|=1} \frac{z^2+3z-2}{(z-1)(z+3)} dz = \frac{1}{4} \int_{|z+3|=1} \frac{z^2+3z-2}{z-1} dz - \frac{1}{4} \int_{|z+3|=1} \frac{z^2+3z-2}{z+3} dz \\ &= \frac{1}{4} \cdot 0 - \frac{1}{4} \cdot 2\pi \mathbf{i} f(-3) \quad \text{where} \quad f(z) = z^2+3z-2 \\ &= \frac{-\pi \mathbf{i}}{2} \cdot (-2) = \pi \mathbf{i}. \end{aligned}$$



(2)

$$
\begin{aligned}
g(z) &= \frac{z^2+3z-2}{z^2+2z-3} \\
\int_{|z|=4} g(z)dz &= \int_{\Gamma_1\cup\Gamma_2} g(z)dz \\
&= \int_{\Gamma_1} g(z)dz + \int_{\Gamma_2} g(z)dz \\
&= \int_{C_1} g(z)dz + \int_{C_2} g(z)dz \\
&= \frac{1}{4}\int_{C_1} \frac{z^2+3z-2}{z-1}dz - \frac{1}{4}\int_{C_2} \frac{z^2+3z-2}{z+3}dz \\
&= \frac{\pi\mathbf{i}}{2}\{f(1)-f(-3)\} = 2\pi\mathbf{i}.
\end{aligned}
$$

Rewrite Cauchy Formula: $f(z) = \frac{1}{2\pi\mathbf{i}}\int_C \frac{f(w)}{w-z}dw$

$\therefore$ differentiating under $\int$ sign, e.g., when $\frac{df}{dz}$ **cts** on C,

$$\frac{df}{dz} = \frac{1}{2\pi\mathbf{i}}\int_C f(w)\frac{d}{dz}\left(\frac{1}{w-z}\right)dw = \frac{1}{2\pi\mathbf{i}}\int_C \frac{f(w)}{(w-z)^2}dw$$

Moreover $\boxed{\frac{d^n}{dz^n}f(z) = \frac{n!}{2\pi\mathbf{i}}\int_C \frac{f(w)}{(w-z)^{n+1}}dw}$

Example:

(1) $$\int_{|z-1|=1} \frac{z^2+1}{(z-1)^4(z+2)} = \int_{|z-1|=1} \frac{g(z)dz}{(z-1)^4} = \frac{2\pi\mathbf{i}}{3!}g'''(1)$$

$$\begin{aligned}
g(z) &= \frac{z^2+1}{z+2}\ ;\ g'(z) = \frac{2z+1}{z+2} - \frac{z^2+1}{(z+2)^2} = \frac{z^2+5z+1}{(z+2)^2} \\
g''(z) &= \frac{2z+5}{(z+2)^2} - \frac{2(z^2+52+1)}{(z+2)^3} = \frac{-z+8}{(z+2)^3} \\
g'''(z) &= \frac{-1}{(z+2)^3} + \frac{3(z-8)}{(z+2)^4} = \frac{z-26}{(z+2)^4}\ \therefore \int_{|z-1|=1} \frac{g(z)dz}{(z-1)^4} = \frac{2\pi\mathbf{i}}{6}\cdot\frac{-25}{81} = \frac{-25\pi\mathbf{i}}{3^5}
\end{aligned}$$

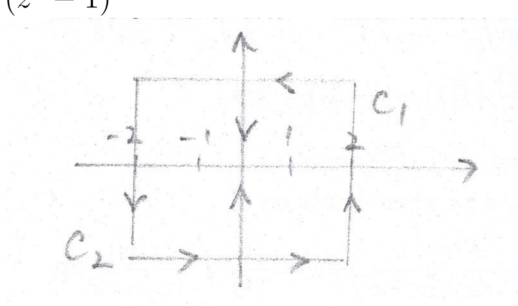
(2) $$\int_{|z-\mathbf{i}|=1} \frac{\sinh(z)dz}{(z-\mathbf{i})^{17}}.$$

$$\text{Note } f(z) = \sinh(z) \Rightarrow \frac{d^n}{dz^n}f(z) = \begin{cases} \sinh(z) \text{ if } n \text{ even} \\ \cosh(z) \text{ if } n \text{ odd} \end{cases}$$

$$\text{And } \frac{\sinh(z)}{(z-\mathbf{i})^{17}} = \frac{d^{16}}{dw^{16}}\left(\frac{\sinh(z)}{(z-w)}\right)\Bigg|_{w=\mathbf{i}}$$

$$\begin{aligned}
\therefore \int_{|z-\mathbf{i}|=1} \frac{\sinh(z)}{(z-\mathbf{i})^{17}}dz &= \frac{2\pi\mathbf{i}}{16!}\frac{d^{16}}{dw^{16}}(sinh(w))\Bigg|_{w=\mathbf{i}} \\
&= \frac{2\pi\mathbf{i}}{16!}\sinh(\mathbf{i}) = \frac{\pi\mathbf{i}}{16!}(e^{\mathbf{i}} - e^{-\mathbf{i}})
\end{aligned}$$

(3) $\int_C \frac{e^z dz}{(z^2-1)^2}$, where C is the rectangle with corners $z = 2 \pm \mathbf{i}, -2 \pm \mathbf{i}$.



$(z^2-1)^2 = (z-1)^2(z+1)^2 \therefore$ let $C = C_1 \cup C_2$, where

C_1 has corners $2 \pm \mathbf{i}, \pm\mathbf{i}$

C_2 has corners $-2 \pm \mathbf{i}, \pm\mathbf{i}$

$$\begin{aligned}
\text{On } C_1, \text{ let } f(z) &= \frac{e^z}{(z+1)^2} \Rightarrow \\
\int_{C_1} \frac{e^z}{(z^2-1)^2}dz = \int_{C_1} \frac{f(z)}{(z-1)^2}dz &= 2\pi\mathbf{i}\frac{d}{dw}f(w)\Big|_{w=1} \\
&= 2\pi\mathbf{i}\left(\frac{e^w}{(w+1)^2} - \frac{2e^w}{(w+1)^3}\right)\Big|_{w=1} \\
&= 2\pi\mathbf{i}\left(\frac{e}{4} - \frac{2e}{8}\right) = 0
\end{aligned}$$

$$\begin{aligned}
\text{On } C_2, \text{ let } f(z) = \frac{e^z}{(z-1)^2} &\Rightarrow \\
\int_{C_2} \frac{e^z dz}{(z^2-1)^2} = \int_{C_2} \frac{f(z)}{(z+1)^2}dz &= 2\pi\mathbf{i}\frac{d}{dw}f(w)\Big|_{w=-1} \\
&= 2\pi\mathbf{i}\left(\frac{e^2}{(w-1)^2} - \frac{2e^w}{(w-1)^3}\right)\Big|_{w=-1} = 2\pi\mathbf{i}\left(\frac{e}{4} + \frac{2e}{8}\right) \\
\therefore \int_C \frac{e^z dz}{(z^2-1)^2} &= \mathbf{i}\pi e + 0
\end{aligned}$$

Remark: Cauchy's formula implies that if $f(z)$ is **analytic**, then $\frac{d^n}{dz^n}f(z)$ exists for **all** n.

5. Applications of the Cauchy Formula

Let $f(z)$ be **analytic** on a **simply connected** domain $\Omega \subseteq \mathbb{C}$. Consider $z_0 \in \Omega$, and a circle $|z-z_0| = r$ contained in Ω. Then

$$\begin{aligned}
f(z_0) = \frac{1}{2\pi\mathbf{i}} \int_{|z-z_0|=r} \frac{f(z)}{z-z_0}dz &= \frac{1}{2\pi\mathbf{i}} \int_0^1 \frac{f(z_0 + re^{2\pi\mathbf{i}t})}{re^{2\pi\mathbf{i}t}} \cdot 2\pi\mathbf{i}re^{2\pi\mathbf{i}t}dt \\
&= \int_0^1 f(z_0 + re^{2\pi\mathbf{i}t})dt \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{\mathbf{i}\theta})d\theta
\end{aligned}$$

$\therefore f(z_0) =$ **average value** of $f(z)$ on the circle $|z-z_0| = r$. (Gauss' Mean Value Theorem)

Can use this to prove:

"**Maximum Modulus Principle:**" If f is analytic on a **closed, bounded region** $R \subseteq \mathbb{C}$, and f is **not constant** then

$$M_f = \sup_{z\in R} |f(z)| = f(\zeta) \;\text{ for some }\; \zeta \in \partial R, \;\text{ and}$$
$$|f(\zeta)| > |f(z)| \;\text{ for all }\; z \in R \backslash \partial R.$$

Also have

Liouville's Theorem: If $f(z)$ is **entire**, and $|f(z)| \leq M$ for some **finite** $M \in \mathbb{R}$ and for all $z \in \mathbb{C}$, then $f(z)$ must be **constant**.

Proof: Consider $C : |z - z_0| = r$, some fixed $z_0 \in \mathbb{C}$, then

$$\begin{aligned} f'(z_0) &= \frac{1}{2\pi\mathbf{i}} \int_C \frac{f(z)}{(z-z_0)^2} dz \Rightarrow \\ |f'(z_0)| &\leq \frac{1}{2\pi} \int_C \left| \frac{f(z)}{(z-z_0)^2} \right| |dz| \leq \frac{1}{2\pi} M_r \cdot 2\pi r \\ \text{where }\; M_r &= \sup_{t\in[0,1]} \left| \frac{f(re^{2\pi\mathbf{i}t})}{(re^{2\pi\mathbf{i}t})^2} \right| = \sup_{t\in[0,1]} \frac{|f(re^{2\pi\mathbf{i}t})|}{r^2} \end{aligned}$$

Now, we assumed that $|f(z)| \leq M$ for all $z \in \mathbb{C}$

$$\therefore \sup_{t\in[0,1]} |f(re^{2\pi\mathbf{i}t})| \leq M \Rightarrow M_r \leq \frac{M}{r^2}$$

$\therefore$ for **any** $z_0 \in \mathbb{C}$, we have $|f'(z_0)| \leq \frac{M}{r^2} \cdot r = \frac{M}{r}$ – **independently** of $r > 0 \therefore |f'(z_0)| = 0$.

Corollary (Fundamental Theorem of Algebra):

The polynomial equation $P(z) = a_0 + a_1 z + \cdots + a_n z^n = 0, a_n \neq 0$, always has n **solutions** $z_0 \in \mathbb{C}$.

Proof: Suppose $P(z) = 0$ has **no solutions** for some non–constant polynomial$P$. Then $P(z) \neq 0$ for all $z \in \mathbb{C}$. Now P **entire** and **nowhere** zero $\Rightarrow f(z) = \frac{1}{P(z)}$ also **entire**. Moreover,

$$\begin{aligned} &\Rightarrow \;\; |P(z)| = |z|^n |a_n + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}| \\ &\Rightarrow \;\; \lim_{|z|\to\infty} |P(z)| = \lim_{|z|\to\infty} |z|^n |a_n| = \infty \\ &\qquad \therefore \lim_{|z|\to\infty} \frac{1}{|P(z)|} = 0 \Rightarrow |f(z)| \leq M \end{aligned}$$

$\therefore \frac{1}{P(z)}$ **constant** $\Rightarrow P(z)$ **constant**.

CHAPTER 5

Complex Power Series

1. Convergence of Complex Series

A "**complex series**" is a summation $S(z) = \sum_{n=0}^{\infty} a_n z^n$ where $a_n \in \mathbb{C}$ for all n. We denote by $S_N(z)$ the "**N-th Partial sum**", i.e., $S_N(z) = \sum_{n=1}^{N} a_n z^n$, and hence define the **sequence** $\{S_N(z)\}_{N=1}^{\infty}$ of **partial sums**.

Definition 1.1. (1) *Given $z_0 \in \mathbb{C}$, we say $S(z_0)$ "**converges**" at z_0 if $\sigma = \lim_{N\to\infty} S_N(z_0)$* **exists**, *i.e., for all $\varepsilon > 0$, there exists N_0 such that $|\sigma - S_N(z_0)| < \varepsilon$ when $N > N_0$.*

(2) *The "**Domain of Convergence**" of $S(z)$ is the set $\Omega \subseteq \mathbb{C}$ such that $z_0 \in \Omega \Leftrightarrow S(z_0)$ converges.*

(3) *Conversely, $S(z_0)$ "**diverges**" if $\lim_{N\to\infty} S_N(z_0)$ does not exist.*

Example: Find Domain of convergence of $S(z) = \sum_{n=0}^{\infty} z^n$.

Note $1 - z^N = (1-z)(1 + z + \cdots + z^{N-1})$

$\therefore S_{N-1}(z) = \frac{1-z^N}{1-z}$

$$\begin{aligned} \text{Now } z = re^{\mathbf{i}\theta} \quad &\Rightarrow \quad z^N = r^N e^{\mathbf{i}N\theta} \\ &\Rightarrow \quad \lim_{N\to\infty} z^N = 0 \text{ if } r = |z| < 1. \end{aligned}$$

$$\therefore \text{ Guess } \lim_{N\to\infty} S_N(z) = \lim_{N\to\infty} \frac{1 - z^{N+1}}{1-z} \quad = \quad \frac{1}{1-z} \text{ if } |z| < 1.$$
$$(\quad = \quad \sigma(z))$$

$$\text{But } |\sigma(z) - S_N(z)| = \left| \frac{1}{1-z} - \left(\frac{1 - z^{N+1}}{1-z} \right) \right| = \frac{|z|^{N+1}}{|1-z|} < \varepsilon$$

$$\begin{aligned} \Leftrightarrow |z|^{N+1} < \varepsilon|1-z| \quad &\Leftrightarrow \quad (N+1)\log|z| < \log(\varepsilon|1-z|) \\ &\Leftrightarrow \quad N > \frac{\log(\varepsilon|1-z|)}{\log|z|} - 1 \end{aligned}$$

–noting $\log|z| < 0$ if $|z| < 1$.

$\therefore \lim_{N\to\infty} S_N(z) = \dfrac{1}{1-z}$ for all z *st* $|z| < 1$.

Now, by analogy with real series there is the

Divergence Test: If $\lim_{n\to\infty} |a_n z_0^n| \neq 0$, then $\sum_{n=0}^{\infty} a_n z_0^n$ **diverges**.

Example: Note $|z| \geq 1 \Rightarrow \lim_{n\to\infty} |z^n| = \lim_{n\to\infty} |z|^n = \lim_{n\to\infty} r^n \neq 0$

$\therefore S(z) = \sum_{n=0}^{\infty} z^n$ **diverges** if $|z| \geq 1$.

Definition 1.2. $S(z_0) = \sum_{n=0}^{\infty} a_n z_0^n$ *converges "***absolutely***" if the real series* $\sum_{n=0}^{\infty} |a_n z_0^n|$ *converges.* $S(z_0)$ *converges "***conditionally***" otherwise.*

Recall:

(1) Absolute convergence $\Rightarrow$ ordinary convergence.
(2) If $S(z_0)$ absolutely convergent, then order of summation not important.
(3) $S_1(z_0)$ and $S_2(z_0)$ abs convergent $\Rightarrow S_1 \cdot S_2(z_0)$ abs.cgt. where

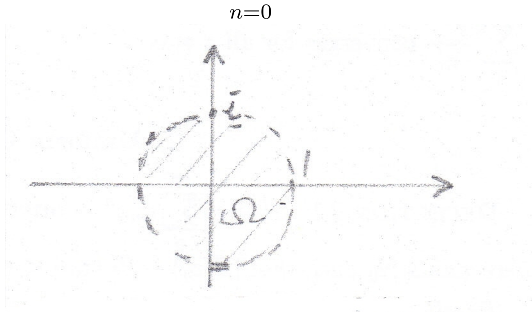
$$\begin{aligned} S_1(z) &= \sum_0^{\infty} a_n z^n, \ S_2(z) = \sum_0^{\infty} b_n z^n \Rightarrow \\ S_1 \cdot S_2(z) &= \sum_0^{\infty} c_n z^n \quad \text{where} \\ c_n &= \sum_{j=0}^{n} a_j b_{n-j} \end{aligned}$$

Example:

$$\begin{aligned} \frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n \text{ for } |z| < 1 \\ \therefore \frac{1}{(1-z)^2} &= \left(\sum_0^{\infty} z^n\right)^2 \\ &= \sum_{k=0}^{\infty} \left(\sum_{m+n=k} z^n z^m\right) = \sum_{k=0}^{\infty} (k+1) z^k \end{aligned}$$

Change of Variable: If $S(w) = \sum_{n=0}^{\infty} a_n w^n$ converges for all $w \in \Omega$ and $f : \mathbb{C} \to \mathbb{C}$ is an analytic function such that $w = f(z)$, then $S(f(z)) = \sum_{n=0}^{\infty} a_n f(z)^n$ converges for all $z \in f^{-1}(\Omega) = \{z \in \mathbb{C} | f(z) \in \Omega\}$.

Example: $S(w) = \sum_{n=0}^{\infty} w^n$ converges for $|w| < 1$:



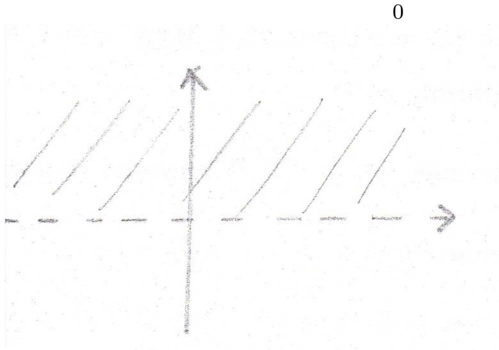
$w = e^{\mathbf{i}z} \Rightarrow S(e^{\mathbf{i}z}) = \sum_{n=0}^{\infty} e^{\mathbf{i}nz}$ converges for $|e^{\mathbf{i}z}| < 1$, where $|e^{\mathbf{i}z}| = e^{\mathrm{Re}(\mathbf{i}z)} = e^{-\mathrm{I}m(z)}$.

$\therefore e^{-\mathrm{I}m(z)} < 1 \Leftrightarrow -\mathrm{I}m(z) < 0 \Leftrightarrow \mathrm{I}m(z) > 0$

Conversely, $\mathrm{I}m(z) \leq 0 \Rightarrow \lim_{n\to\infty} |e^{\mathbf{i}z}| = \lim_{n\to\infty} e^{-\mathtt{Im}(z)} \neq 0$

$\therefore \sum_{n=0}^{\infty} e^{\mathbf{i}nz}$ diverges for $\mathtt{Im}(z) \leq 0$.

Domain of Convergence of $\sum_{0}^{\infty} e^{\mathbf{i}nz} (= \frac{1}{1 - e^{\mathbf{i}z}})$:



Ratio Test: Consider $\rho(z) = \lim_{n\to\infty} \left| \frac{a_{n+1} z^{n+1}}{a_n z^n} \right| = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} |z|$

Recall

(i) If $\rho(z_0) < 1$ then $S(z_0)$ **converges absolutely** at $z_0 \in \mathbb{C}$.
(ii) If $\rho(z_0) > 1$ then $S(z_0)$ **diverges** at z_0.

(iii) If $\rho(z_0) = 1$ or $\rho(z_0)$ **does not exist**, then **don't know**.

Example:

$$S(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow \rho(z) = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| |z| = \lim_{n\to\infty} \left|\frac{1}{n+1}\right| |z| = 0 \text{ for all } z \in \mathbb{C}.$$

$$\text{Now consider } \sum_{n=0}^{\infty} \frac{z^{-n}}{n!} = S\left(\frac{1}{z}\right) \therefore \rho\left(\frac{1}{z}\right) = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{1}{|z|} = 0 \text{ for all } z \neq 0$$

$\therefore \sum_{0}^{\infty} \frac{z^{-n}}{n!}$ converges for all $z \neq 0$.

2. Uniform Convergence

Definition 2.1. $S(z) = \sum_{0}^{\infty} a_n z^n$ "**converges uniformly**" *on* $R \subseteq \mathbb{C}$ *if for all* $\varepsilon > 0$ *there exists* N_0 *such that* $|S(z_0) - S_N(z_0)| < \varepsilon$ *for all* $N \geq N_0$ **independently** *of* $z_0 \in R$.

Recall "**Weierstrass' M-test**"*:*

Consider $S(z) = \sum_{0}^{\infty} a_n z^n$*. If there exists a sequence* $\{M_n\}_{n=0}^{\infty}$ *of* **positive real numbers** M_n *such that* $|a_n z^n| \leq M_n$ *for all* $z \in R \subseteq \mathbb{C}$*, and* $\sum_{0}^{\infty} M_n < \infty$*, then* $S(z)$ *converges* **uniformly** *on* R.

Example: $S(z) = \sum_{n=0}^{\infty} z^n$ converges **absolutely** on $|z| < 1$ $\therefore$, *e.g.*, $S\left(\frac{1}{2} + \frac{1}{2}\mathbf{i}\right)$ converges **absolutely** since $\left|\frac{1}{2} + \frac{1}{2}\mathbf{i}\right| = \frac{1}{\sqrt{2}} < 1 \Rightarrow \sum_{0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n < \infty$.

Now $|z| \leq \frac{1}{\sqrt{2}} \Rightarrow |z^n| = |z|^n \leq \left(\frac{1}{\sqrt{2}}\right)^n = M_n$.

$\therefore \sum_{0}^{\infty} z^n$ converges **uniformly** on $R = \{|z| \leq \frac{1}{\sqrt{2}}\}$.

In fact, $\sum_{0}^{\infty} z^n$ converges **uniformly** on $\{|z| \leq r\}$ for $0 \leq r < 1$, but **not** on $\{|z| < 1\}$.

Theorem: Suppose $S(w) = \sum_0^\infty a_n w^n$ converges **uniformly** on $R \subseteq \mathbb{C}$, and $f : \Omega \to \mathbb{C}$ is **analytic**, $w = f(z)$, such that $R \subseteq f(\Omega)$.

Then

(1) If γ is a **smooth path** in $f^{-1}(R) \subseteq \Omega$

$$\int_\gamma S(f(z))dz = \sum_0^\infty \int_\gamma a_n f(z)^n dz$$

(2) $\dfrac{d}{dz}S(f(z)) = \sum_0^\infty a_n \dfrac{d}{dz}(f(z)^n) = \sum_0^\infty n a_n f(z)^{n-1}\dfrac{df}{dz}$ on $f^{-1}(R)$.

(3) $S(f(z)) = \sum_0^\infty a_n f(z)^n$ is **analytic** on $f^{-1}(R) \subseteq \Omega$. In particular, $S(z)$ **analytic** on R (cf. (1) plus "Morera's theorem").

Example

(1) $S(w) = \sum_0^\infty w^n$ converges **uniformly** on $|w| \le r < 1$

$\therefore F(z) = S(e^{\mathbf{i}z}) = \sum_0^\infty e^{\mathbf{i}nz}$ **analytic** on $\mathrm{I}m(z) \ge -\log r > 0$

$\therefore$ Domain of **analyticity** of $F(z) = \{z \in \mathbb{C} | \mathrm{I}m(z) > 0\}$

(2) $\dfrac{1}{1-w} = \sum_0^\infty w^n$ on $|w| < 1 \therefore F(z) = \frac{1}{1-e^{\mathbf{i}z}}$ on $\{\mathrm{I}m(z) > 0\}$.

Hence if $\gamma \subset \{\mathrm{I}m(z) > 0\}$ st $\gamma(0) = z_0, \gamma(1) = z$

$$\begin{aligned}\int_\gamma \frac{d\zeta}{1-e^{\mathbf{i}\zeta}} = \int_{z_0}^z \sum_0^\infty e^{\mathbf{i}n\zeta} d\zeta &= \sum_0^\infty \int_{z_0}^z e^{\mathbf{i}n\zeta} d\zeta \\ &= \sum_1^\infty \left[\frac{-\mathbf{i}}{n} e^{\mathbf{i}n\zeta}\right]_{z_0}^z + (z - z_0) \\ &= \sum_1^\infty -\frac{\mathbf{i}}{n}\left(e^{\mathbf{i}nz} - e^{\mathbf{i}nz_0}\right) + (z - z_0)\end{aligned}$$

$z_0 = \mathbf{i} \Rightarrow \sum_1^\infty \dfrac{\mathbf{i}}{n} e^{\mathbf{i}nz_0} = \mathbf{i}\sum_1^\infty \dfrac{1}{ne^n}$

$S(x) = \sum_1^\infty \dfrac{x^n}{n} \Rightarrow \dfrac{d}{dx}S(x) = \sum_1^\infty x^{n-1} = \dfrac{1}{1-x}$ if $|z| < 1$

$$\therefore S(x) = \int_0^x \frac{1}{1-t}dt = \ln\left(\frac{1}{1-x}\right) \Rightarrow S\left(\frac{1}{e}\right) = \ln\left(\frac{1}{1-\frac{1}{e}}\right) = 1 - \ln(e-1)$$

$$\therefore \sum_1^\infty \frac{e^{\mathbf{i}nz}}{n} = \mathbf{i}\left(\int_{\mathbf{i}}^z \frac{d\zeta}{1-e^{\mathbf{i}\zeta}} - \mathbf{i}\left[1-\ln(e-1)\right] - (z-\mathbf{i})\right)$$

$$\begin{aligned}\int_{\mathbf{i}}^z \frac{d\zeta}{1-e^{\mathbf{i}\zeta}} = -\mathbf{i}\int_{\frac{1}{e}}^{e^{iz}} \frac{dw}{w(1-w)} &= -\mathbf{i}\left\{\int_{\frac{1}{e}}^{e^{iz}} \frac{dw}{w} + \int_{\frac{1}{e}}^{e^{iz}} \frac{dw}{1-w}\right\} \\ &= -\mathbf{i}\left\{\mathrm{Log}(e^{\mathbf{i}z}) + 1 - \mathrm{Log}(1-e^{\mathbf{i}z}) + \log(1-\frac{1}{e})\right\}\end{aligned}$$

$$\begin{aligned}\therefore \sum_1^\infty \frac{e^{\mathbf{i}nz}}{n} &= \mathbf{i}\left(\left[1+\mathbf{i}z - \mathrm{Log}(1-e^{\mathbf{i}z})\right](-\mathbf{i}) - z + \mathbf{i}\right) \\ &= -\mathbf{i}\,\mathrm{Log}(1-e^{\mathbf{i}z}) \\ &\quad \text{on } \{\mathrm{Im}(z) > 0\} \cap \{0 < \mathrm{Re}(z) < 2\pi\}\end{aligned}$$

3. Taylor Series

Theorem 1: If $S(z) = \sum_0^\infty a_n z^n$ converges at $z_0 \neq 0$, then $S(z)$ converges **uniformly** on the set of all z such that $|z| \leq r$, where $r < |z_0|$. Hence $S(z)$ is **analytic** on $\{|z| < |z_0|\}$.

Proof: $S(z_0)$ converges $\Rightarrow \lim_{n\to\infty} |a_n||z_0|^n = 0$ $\therefore \exists\, C > 0$ such that $|a_n||z_0|^n \leq C$ for all n. Now consider $z \in \mathbb{C}$ *st* $|z| \leq r < |z_0|$, write

$$|a_n||z|^n = |a_n||z_0|^n \left(\frac{|z|}{|z_0|}\right)^n \leq C\left(\frac{r}{|z_0|}\right)^n$$

for all n.

Let $M_n = C\left(\frac{r}{|z_0|}\right)^n$, then $\sum_0^\infty M_n = C\sum_0^\infty \left(\frac{r}{|z_0|}\right)^n = C\left(\frac{1}{1-\frac{r}{|z_0|}}\right)$.

Since $\frac{r}{|z_0|} < 1$. Hence **Weierstrass** $\Rightarrow \sum_0^\infty a_n z^n$ converges **uniformly** on $\{|z| \leq r\}$.

Change of Variable: If $S(z-a) = \sum_0^\infty a_n(z-a)^n$ converges at $z_0 \neq a \in \mathbb{C}$ then $S(z-a)$ converges uniformly on $\{|z-a| \leq r\}$, where $r < |z_0 - a|$. Hence $S(z-a)$ **analytic** on $\{|z-a| < |z_0 - a_0|\}$.

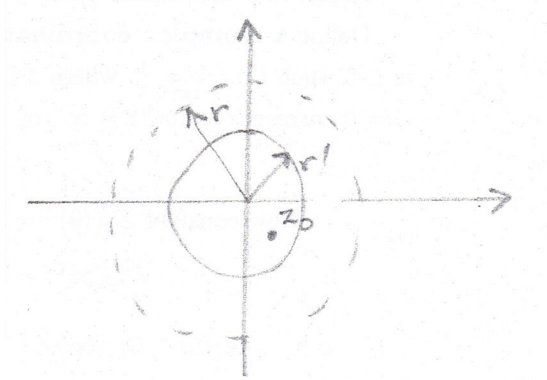
Definition 3.1. *: The* **largest** *r such that $\sum_0^\infty a_n(z-a)^n$ converges for all $z \in \{|z - a|| < r\}$, given $a \in \mathbb{C}$, is called the "***radius of convergence***" of $S(z-a)$. The set*

$\{|z-a| < r_{\max}\}$, *where* $r_{\max}$ *is the radius of convergence, is called the "disc of convergence" of* $S(z-a)$.

Theorem 2: Suppose $f(z)$ is **analytic** on a disc of radius r about $a \in \mathbb{C}$ then there exists a power series $S(z-a) = \sum_0^\infty a_n(z-a)^n$ which converges to $f(z)$ for all $z \in D_r(a)$. Moreover, $a_n = \frac{f^{(n)}(a)}{n!}$ for each n.

Proof: First consider $a = 0 \in \mathbb{C}$, and $z_0 \in \mathbb{C}$ such that $|z_0| < r$. Now choose r' such that $|z_0| < r' < r$, and define $C' = \{|z| = r'\}$ in $\mathbb{C}$. Cauchy's formula $\Rightarrow$

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi\mathbf{i}} \int_{C'} \frac{f(z)}{z - z_0} dz \\ &= \frac{1}{2\pi\mathbf{i}} \int_{C'} \frac{f(z)dz}{z(1 - \frac{z_0}{z})}. \end{aligned}$$



Note $|z| = r' \Rightarrow \left|\frac{z_0}{z}\right| < 1 \therefore \frac{1}{1 - \left(\frac{z_0}{z}\right)} = \sum_{n=0}^\infty \left(\frac{z_0}{z}\right)^n$

$$\begin{aligned} \therefore f(z_0) &= \frac{1}{2\pi\mathbf{i}} \int_{C'} \frac{f(z)}{z} \sum_{n=0}^\infty \left(\frac{z_0}{z}\right)^n dz \\ &= \frac{1}{2\pi\mathbf{i}} \sum_0^\infty \left(z_0^n \int_{C'} \frac{f(z)dz}{z^{n+1}} \right) \end{aligned}$$

since f **analytic** and $\sum_0^\infty \left(\frac{z_0}{z}\right)^n$ converges **uniformly** on $|z_0| < r'' \le |z| \le r' < r$.

$$\therefore f(z_0) = \frac{1}{2\pi\mathbf{i}}\sum_0^\infty c_n z_0^n \text{ for all } z_0 \in D_r(0), \text{ where } c_n \begin{aligned}[t] &= \int_{C'} \frac{f(z)dz}{z^{n+1}} \\ &= \frac{2\pi i}{n!} f^{(n)}(0) \ , \end{aligned}$$

$$\therefore f(z) = \sum_0^\infty \frac{f^{(n)}(0)}{n!} z^n \text{ on } D_r(0).$$

Change of variable: $z = w - a, \tilde{f}(w) = f(w-a) = f(z)$

$$\therefore \tilde{f}(w) = f(w-a) = \sum_0^\infty \frac{\tilde{f}^{(n)}(a)}{n!}(w-a)^n.$$

Remark: If r is the largest radius for which $\sum_0^\infty \frac{f^{(n)}(a)}{n!}(z-a)^n$ converges to $f(z)$ on $D_r(a)$, then $f(z)$ must have a **singularity** at some pt z_0 such that $|z_0 - a| = r$.

Example: $f(z) = \dfrac{1}{1-z}$

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \therefore \frac{f^{(n)}(0)}{n!} = 1$$

Hence $\sum_0^\infty \frac{f^{(n)}(0)}{n!} z^n = \sum_0^\infty z^n$ converges to $f(z)$ on $D_1(0)$, with **singularity** at $z_0 = 1$.

4. Laurent series

Recall the "**Riemann sphere**" $\mathbb{S} = \mathbb{C} \cup \{\infty\}$ (cf., e.g., Churchill and Brown, p.38).

Define a "**complex coordinate near infinity**" to be the set of all complex numbers $w \in \mathbb{C}$ such that $w = \frac{1}{z}$, where $z \in \mathbb{C}$ is the usual complex coordinate near $0 \in \mathbb{C}$. Hence $w = 0$ corresponds to "$z = \infty$" on $\mathbb{S}$.

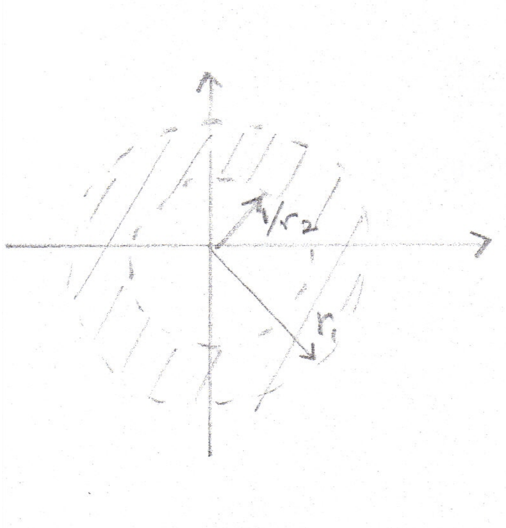
$$\begin{aligned} \text{Now consider } D_{r_1}(0) &= \{z \in \mathbb{C} \mid |z| < r_1\} \text{ and} \\ D_{r_2}(\infty) &= \{w \in \mathbb{C} \mid |w| < r_2\} \text{ such that } r_1 > \frac{1}{r_2}. \text{ Then} \\ D_{r_1}(0) \cap D_{r_2}(\infty) &= \{z \in \mathbb{C} \mid \frac{1}{r_2} < |z| < r_1\}. \end{aligned}$$

Suppose $S_1(z) = \sum_0^\infty a_n z^n$ converges on $D_{r_1}(0)$ and $S_2(w) = \sum_0^\infty b_n w^n$ converges on $D_{r_2}(\infty)$. Then on $D_{r_1}(0) \cap D_{r_2}(\infty)$ we can write

$$\begin{aligned} S_2(w) &= S_2\left(\frac{1}{z}\right) \\ &= \sum_0^\infty b_n z^{-n}. \end{aligned}$$

$$\text{Write } \mathbb{L}(z) = S_1(z) + S_2\left(\frac{1}{z}\right) = \sum_{-\infty}^{\infty} c_k z^k$$

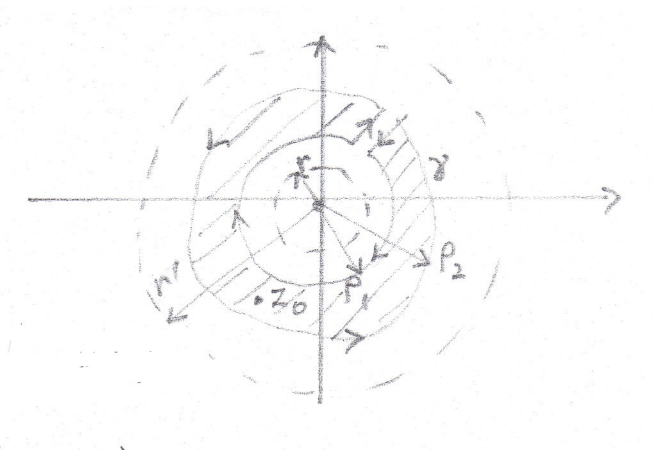
$$\text{where } c_k = \begin{cases} a_k \text{ if } k > 0 \\ b_k \text{ if } k < 0 \\ a_0 + b_0 \text{ if } k = 0. \end{cases}$$



Note $\mathbb{L}(z)$ is convergent on the annulus $D_{r_1}(0) \cap D_{r_2}(\infty) = D_{r_1}(0) \cap \tilde{D}_{\frac{1}{r_2}}(0)$

Laurent's theorem: Suppose $f(z)$ is **analytic** on an **annular region** $A = \{r < |z| < r'\}$, then for all $z_0 \in A$ there exists a power series $\sum_{-\infty}^{\infty} c_k z^k = \mathbb{L}(z)$ such that $\mathbb{L}(z_0)$ converges to $f(z_0)$. Moreover $c_k = \frac{1}{2\pi\mathbf{i}} \int_C \frac{f(z)dz}{z^{k+1}}$ for **all** k, where $C = \{|z| = s\}$ for some $s \in (r, r')$.

Proof:



Cauchy formula ⇒

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi\mathbf{i}}\int_\gamma \frac{f(z)}{z-z_0}dz \\ &= \frac{1}{2\pi\mathbf{i}}\int_{|z|=\rho_2} \frac{f(z)}{z-z_0}dz - \frac{1}{2\pi\mathbf{i}}\int_{|z|=\rho_1} \frac{f(z)}{z-z_0}dz \end{aligned}$$

$$\begin{aligned} \text{Now } \int_{|z|=\rho_2} \frac{f(z)}{z-z_0}dz &= \int_{|z|=\rho_2} \frac{f(z)}{z}\left(\sum_0^\infty \left(\frac{z_0}{z}\right)^n\right) dz \\ &= \sum_{n=0}^\infty z_0^n \left(\int_{|z|=\rho_2} \frac{f(z)dz}{z^{n+1}}\right), \text{ while} \end{aligned}$$

$$\begin{aligned} -\int_{|z|=\rho_1} \frac{f(z)dz}{z-z_0} = \int_{|z|=\rho_1} \frac{f(z)dz}{z_0\left(1-\frac{z}{z_0}\right)} &= \int_{|z|=\rho_1} \frac{f(z)}{z_0}\left(\sum_0^\infty \left(\frac{z}{z_0}\right)^n\right) dz \\ &= \sum_{n=0}^\infty z_0^{-(n+1)} \left(\int_{|z|=\rho_1} z^n f(z) dz\right) \\ (m=-(n+1)) &= \sum_{m=-\infty}^{-1} z_0^m \int_{|z|=\rho_1} \frac{f(z)dz}{z^{m+1}} \end{aligned}$$

$$\text{Hence } f(z_0) = \sum_{-\infty}^\infty c_k z_0^k, \quad where\ c_k = \begin{cases} \frac{1}{2\pi\mathbf{i}}\int_{|z|=\rho_2} \frac{f(z)dz}{z^{k+1}} \text{ if } k \geq 0 \\ \frac{1}{2\pi\mathbf{i}}\int_{|z|=\rho_1} \frac{f(z)}{z^{k+1}}dz \text{ if } k \leq -1 \end{cases}$$

$\therefore$ deformation of contours $\Rightarrow \displaystyle\int_{|z|=\rho_1} = \int_{|z|=\rho_2} = \int_{|z|=s}, s \in (r, r').$

Moreover $\sum_{-\infty}^{\infty} c_k z_0^k$ converges **uniformly** inside A. $\square$

Change of Variable: $f(z)$ analytic on $A = \{r < |z-a| < r'\} \Rightarrow f(z_0) = \sum_{-\infty}^{\infty} c_k (z-a)^k$ such that $c_k = \dfrac{1}{2\pi \mathbf{i}} \displaystyle\int_C \frac{f(z)dz}{(z-a)^{k+1}}$.

CHAPTER 6

Residue Theory and Applications

1. Complex Residues

Definition 1.1. *: Let $f(z)$ be analytic on the punctured disc*

$$D_r(z_0)\backslash\{z_0\} = \{z|0 < |z - z_0| < r\}.$$

The "Residue" of f at z_0 is given by $\boxed{\mathtt{Res}(f, z_0) = \frac{1}{2\pi\mathbf{i}}\int_{|z-z_0|=r'} f(z)dz}$

for any $0 < r' < r$.

Consider $\mathbb{L}_f(z, z_0) = \sum_{-\infty}^{\infty} c_k(z - z_0)^k, c_k = \frac{1}{2\pi\mathbf{i}}\int_{|z-z_0|=r} \frac{f(z)dz}{(z - z_0)^{k+1}}.$

$$\begin{aligned}
\text{Then } \frac{1}{2\pi\mathbf{i}}\int_{|z-z_0|=r'} f(z)dz &= \frac{1}{2\pi\mathbf{i}}\int \sum_{-\infty}^{\infty} c_k(z - z_0)^k dz \\
&= \frac{1}{2\pi\mathbf{i}} \sum_{-\infty}^{\infty} c_k \int_{|z-z_0|=r'} (z - z_0)^k dz. \\
\text{But } \int_{|z-z_0|=r'} (z - z_0)^k dz &= \begin{cases} 2\pi\mathbf{i} \text{ if } k = -1 \\ 0 \qquad k \neq -1 \end{cases}
\end{aligned}$$

$$\boxed{\therefore \frac{1}{2\pi\mathbf{i}}\int_{|z-z_0|=r'} f(z)dz = c_{-1}}$$

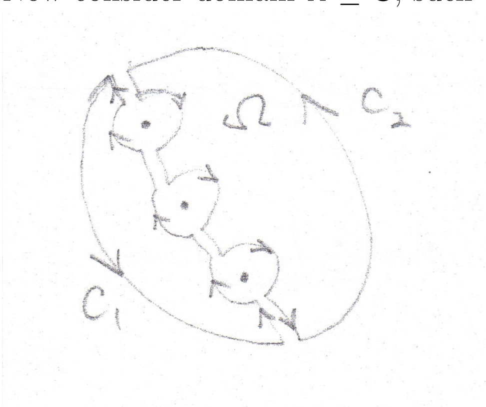
$\therefore$ **Theorem:** $\mathrm{R}es(f, z_0) = c_{-1}$.

Example: $f(z) = \dfrac{1}{z^2+1} = \dfrac{\mathbf{i}}{2}\dfrac{1}{z+\mathbf{i}} - \dfrac{\mathbf{i}}{2}\dfrac{1}{z-\mathbf{i}}$. $Res(f, \mathbf{i}) =?$

$$
\begin{aligned}
\text{Note } \frac{1}{z+\mathbf{i}} &= \frac{1}{(z-\mathbf{i})+2\mathbf{i}} = \frac{1}{2\mathbf{i}(1-\frac{\mathbf{i}}{2}(z-\mathbf{i}))} \\
\therefore \frac{\mathbf{i}}{2}\frac{1}{z+\mathbf{i}} &= \frac{1}{4}\sum_0^\infty \left(\frac{\mathbf{i}}{2}(z-\mathbf{i})\right)^k \\
\therefore \mathbb{L}_f(z, \mathbf{i}) &= \frac{1}{4}\sum_0^\infty \left(\frac{\mathbf{i}}{2}\right)^k (z-\mathbf{i})^k - \frac{\mathbf{i}}{2}(z-\mathbf{i})^{-1}
\end{aligned}
$$

$$\therefore c_{-1} = -\frac{\mathbf{i}}{2} = \mathrm{R}es(f, \mathbf{i})$$

Now consider domain $\Omega \subseteq \mathbb{C}$, such that $f(z)$ **analytic** on $\Omega \backslash \{z_1, \cdots, z_k\}, z_i \in \Omega$.



$$\int_{\partial\Omega} f(z)dz =$$

$$\int_{C_1 \cup C_2} f(z)dz + \sum_{i=1}^k \int_{|z-z_i|=\varepsilon} f(z)dz$$

Cauchy Theorem: $\displaystyle\int_{C_1 \cup C_2} f(z)dz = 0$

$\therefore$ **Residue Theorem:**

If $\partial\Omega$ is a closed simple contour, and $f(z)$ is analytic on $\Omega \backslash \{z_1, \cdots, z_k\}$ for $z_i \in \Omega$, then

$$\frac{1}{2\pi\mathbf{i}} \int_{\partial\Omega} f(z)dz = \sum_{i=1}^k \mathrm{R}es(f, z_i)$$

Example: Compute $\displaystyle\int_{|z|=3} \frac{dz}{z^2-1}$ using residues

$$\begin{aligned}
\int_{|z|=3} \frac{dz}{z^2-1} &= \mathrm{R}es\left(\frac{1}{z^2-1}, 1\right) + \mathrm{R}es\left(\frac{1}{z^2-1}, -1\right) \\
\text{Now } \frac{1}{z^2-1} &= \frac{\frac{1}{2}}{z-1} - \frac{\frac{1}{2}}{z+1} \Rightarrow \\
\mathrm{R}es\left(\frac{1}{z^2-1}, 1\right) &= \mathrm{R}es\left(\frac{\frac{1}{2}}{z-1}, 1\right) = \frac{1}{4\pi i}\int_{|z-1|=\varepsilon} \frac{dz}{z-1} = \frac{1}{2} \\
\mathrm{R}es\left(\frac{1}{z^2-1}, -1\right) &= \mathrm{R}es\left(\frac{-\frac{1}{2}}{z+1}, -1\right) = -\frac{1}{4\pi\mathbf{i}}\int_{|z+1|=\varepsilon} \frac{dz}{z+1} = -\frac{1}{2} \\
\therefore \int_{|z|=3} \frac{dz}{z^2-1} &= 0.
\end{aligned}$$

Example: $f(z) = \dfrac{\sin(z)}{z\sinh(z)}$. Compute $\mathrm{R}es(f, 0)$.

$$\begin{aligned}
e^z &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \\
e^{\mathbf{i}z} &= 1 + \mathbf{i}z + \frac{(\mathbf{i}z)^2}{2!} + \frac{(\mathbf{i}z)^3}{3!} + \cdots \\
\sin(z) &= \frac{e^{\mathbf{i}z} - e^{-\mathbf{i}z}}{2\mathbf{i}} = \frac{1}{2\mathbf{i}}\left\{2\mathbf{i}z + \frac{2(\mathbf{i}z)^3}{3!} + \frac{2(\mathbf{i}z)^5}{5!} + \cdots\right\} \\
&\qquad = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = z\left\{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right\} \\
\sinh(z) &= \frac{e^z - e^{-z}}{2} = \frac{1}{2}\left\{2z + 2\frac{z^3}{3!} + \frac{2z^5}{5!} + \cdots\right\} \\
&\qquad = \left\{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right\} = z\left\{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right\} \\
\therefore \frac{\sin(z)}{z\sin h(z)} &= \frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}{z\{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\}} \\
&= \frac{1}{z}\{1 - \frac{2z^2}{3!} + \cdots\} = \sum_{-\infty}^{\infty} c_k z^k \\
\therefore c_{-1} &= 1 = \mathrm{R}es(f, 0).
\end{aligned}$$

2. Isolated Singularities

Definition 2.1. *: Given* $\mathbb{L}_f(z, z_0) = \sum_{-\infty}^{\infty} c_k (z - z_0)^k$, *define the* "**Principal Part**"

$$[\mathbb{L}_f(z, z_0)] = \sum_{-\infty}^{-1} c_k (z - z_0)^k.$$

(1) $f(z)$ has a **"pole" of order** N at $z_0 \in \mathbb{C}$ if

$$[\mathbb{L}_f(z, z_0)] = \sum_{-N}^{-1} c_k (z - z_0)^k, \text{ i.e. } c_k = 0 \ \forall k < -N.$$

Example:
$$\begin{aligned} f(z) &= \frac{1}{(z^2-1)^3} \text{ has poles of order 3 at } z = \pm 1 \\ f(z) &= \frac{\sin(z)}{z \sinh(z)} \text{ has pole of order 1 at } z = 0. \end{aligned}$$

(2) $f(z)$ has an "**isolated essential singularity**" at $z_0 \in \mathbb{C}$ if $[\mathbb{L}_f(z, z_0))]$ has **infinitely** many terms.

Example:
$$\begin{aligned} e^{\frac{1}{z}} &= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \\ c_{-k} &= \frac{1}{k!} \ \forall \ k > 0 \end{aligned}$$

(3) $f(z)$ has a "**Removable singularity**" at $z_0 \in \mathbb{C}$ if $f(z_0)$ may be **redefined** to make f analytic at z_0.

Example: $f(z) = \frac{\sin(z)}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$

- **not defined** at $z_0 = 0$, but **analytic** if we define

$$f(z) = \begin{cases} \frac{\sin(z)}{z} & z \neq 0 \\ 1 & z = 0. \end{cases}$$

Remarks:

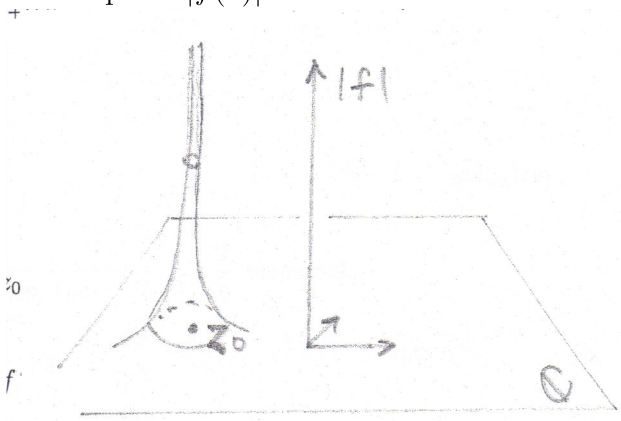
(1) $f(z)$ has a **pole or order** N at $z_0 \Rightarrow$

$0 \neq \lim_{z \to z_0} |(z - z_0)^N f(z)| < \infty$ and

$\lim_{z \to z_0} |f(z)| = \infty$

$$\text{since } \mathbb{L}_f(z, z_0) = c_{-N}(z-z_0)^{-N} + \cdots + c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0) + \cdots$$
$$= \frac{1}{(z-z_0)^N}\left(c_{-N} + c_{-N+1}(z-z_0) + \cdots c_0(z-z_0)^N + \cdots\right)$$

cf. Graph of $|f(z)|$:



(2) $f(z)$ has a **removable** singularity at $z_0 \Rightarrow$

$$\lim_{z \to z_0}(z-z_0)f(z) = 0.$$

(3) $f(z)$ has **essential** singularity at $z_0 \Rightarrow \lim_{z\to z_0}|f(z)|$ does not exist ($\neq \infty$).
(cf. "Cassorati-Weierstrass" thm)

Example:

(1) $f(z) = \dfrac{1}{(z^2-1)^3} \Rightarrow \lim_{z\to 1}(z-1)^3 f(z) = \lim_{z\to 1}\dfrac{1}{(z+1)^3} = \dfrac{1}{8}$

$\therefore \mathbb{L}_f(z,1) = \sum_{-3}^{\infty} c_k(z-1)^k \Rightarrow c_{-3} = \dfrac{1}{8}.$

(2) $f(z) = \dfrac{\sin(z)}{z} \Rightarrow \lim_{z\to 0} zf(z) = \lim_{z\to 0}\sin(z) = 0$

Definition 2.2. *: "**Order**" of $f(z)$ at z_0,*

$$ord_f(z_0) = \begin{cases} N \text{ if } f(z) \sim (z-z_0)^N \sum_0^\infty a_n(z-z_0)^n \quad a_0 \neq 0 (\text{"zero"}) \\ -N \text{ if } f(z) \sim \frac{1}{(z-z_0)^N}\sum_0^\infty a_n(z-z_0)^n \; a_0 \neq 0(\text{"pole"}) \\ -\infty \text{ if } z_0 \text{ essential.} \\ 0 \quad \text{if } 0 \neq |f(z_0)| < \infty \end{cases}$$

Suppose $ord_f(z_0) = k, ord_g(z_0) = \ell,$ *then*

$$\begin{aligned} f \cdot g &= \left[(z-z_0)^k \sum_0^\infty a_n (z-z_0)^n\right]\left[(z-z_0)^\ell \sum_0^\infty b_m (z-z_0)^m\right] \\ &= (z-z_0)^{k+\ell}\sum_0^\infty c_p (z-z_0)^p \quad \textit{where} \;\; c_0 = a_0 b_0 \neq 0. \end{aligned}$$

$$\begin{aligned} c_1 &= a_1 b_0 + a_0 b_1 \\ \vdots \;\; & \;\; \vdots \end{aligned}$$

$\therefore\, ord_{f\cdot g}(z_0) = k + \ell$

$$\begin{aligned} \textit{Moreover}\ \frac{1}{g} &= \frac{1}{(z-z_0)^\ell [b_0 + b_1(z-z_0) + \cdots]}\ (b_0 \neq 0) \\ \textit{Define}\ 1 - \varphi(z-z_0) &= \sum_{-}^{\infty} \frac{b_m}{b_0}(z-z_0)^m \Rightarrow \frac{1}{g} = \frac{1}{(z-z_0)^\ell b_0 (1-\varphi)} \\ \textit{Now}\ \ \frac{1}{1-\varphi} &= \sum_0^\infty \varphi^n \ \textit{if}\ |\varphi| < 1\ (\textit{i.e.}, |z - z_0) < \varepsilon) \\ \therefore\ \ \frac{1}{g} &= (z-z_0)^{-\ell}\frac{1}{b_0}\left[1 + \varphi + \varphi^2 + \cdots\right] \Rightarrow \\ ord_{\frac{1}{g}}(z_0) &= -\ell, \\ \therefore\ \ ord_{f/g}(z_0) &= k - \ell. \end{aligned}$$

Example: Find $ord_h(0)$ for $h(z) = \dfrac{z^2+3z}{\sin^3(z)}$.

$f(z) = 3z + z^2 \Rightarrow ord_f(0) = 1.$

$$\begin{aligned} \sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\ &= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right) \end{aligned}$$

$$\begin{aligned} \therefore g(z) &= \sin^3(z) = z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)^3 \Rightarrow ord_g(0) = 3 \\ &\therefore\ \ ord_h(0) = 1 - 3 = -2 \end{aligned}$$

In general

$f(z) = c_{-N}(z-z_0)^{-N} + \cdots + c_{-1}(z-z_0)^{-1} + c_0 + \cdots$

$$\Rightarrow (z-z_0)^N f(z) = c_{-N} + c_{-N+1}(z-z_0) + \cdots + c_{-1}(z-z_0)^{N-1} + c_0(z-z_0)^N + \cdots$$

$$\frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}[(z-z_0)^N f(z)] = c_{-1} + c_0(z-z_0) + \cdots$$

$$\therefore\ \mathrm{R}es(f,z_0) = \lim_{z\to z_0}\left\{\frac{1}{(N-1)!}\frac{d^{N-1}}{dz^{N-1}}\left[(z-z_0)^N f(z)\right]\right\}$$

Example: Compute $\int_{|z|=1} f(z)dz$ for $f(z) = \frac{1}{z\sin(z)}$.

Now $\displaystyle\int_{|z|=1}\frac{dz}{z\sin(z)} = 2\pi\mathbf{i}\mathrm{R}es\left(\frac{1}{z\sin(z)}, 0\right)$

$$\begin{aligned}
\mathrm{o}rd_f(0) &= -2 \Rightarrow f(z) = c_{-2}z^{-2} + c_{-1}z^{-1} + c_0 + c_1 z + \cdots \\
\therefore\ \mathrm{R}es(f,0) &= \lim_{z\to 0}\frac{1}{1!}\frac{d}{dz}\left[z^2 f(z)\right] \\
&= \lim_{z\to 0}\frac{d}{dz}\left[\frac{z}{\sin(z)}\right] \\
&= \lim_{z\to 0}\left\{\frac{1}{\sin(z)} - \frac{z}{\sin^2(z)}\cdot\cos(z)\right\} \\
&= \lim_{z\to 0}\left\{\frac{\sin(z) - z\cos(z)}{\sin^2(z)}\right\} \\
\text{(L'Hôpital)} \quad &= \lim_{z\to 0}\left\{\frac{\cos(z) - \cos(z) + z\sin(z)}{2\sin(z)\cos(z)}\right\} \\
&= \lim_{z\to 0}\left\{\frac{z}{2\cos(z)}\right\} = 0
\end{aligned}$$

Example: $\displaystyle\int_{|z|=1}\frac{\sin\left(\frac{1}{z}\right)dz}{z} = \mathrm{R}es\left(\frac{\sin\left(\frac{1}{z}\right)}{z}, 0\right)\cdot 2\pi\mathbf{i}$

$$\begin{aligned}
\sin\left(\frac{1}{z}\right) &= \frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^5 5!} - \cdots \\
\frac{1}{z}\sin\left(\frac{1}{z}\right) &= \frac{1}{z^2} - \frac{1}{z^4 3!} + \frac{1}{z^6 5!} - \cdots \\
&= \mathbb{L}_{\frac{1}{z}\sin\left(\frac{1}{z}\right)}(z,0)
\end{aligned}$$

$\therefore$ $\mathrm{R}es\left(\frac{1}{z}\sin\left(\frac{1}{z}\right), 0\right) = c_{-1} = 0.$

3. Applying Residues to Real Integration

1. Trigonometric Integrals:

Consider integrals of the form: $\displaystyle\int_0^{2\pi} \frac{P(\sin\theta,\cos\theta)}{Q(\sin\theta,\cos\theta)} d\theta$; P, Q polynomials.

$$\begin{aligned}
\text{Recall } sin\theta &= \frac{z-\bar{z}}{2\mathbf{i}} \text{ if } z = \cos\theta + \mathbf{i}\sin\theta = e^{\mathbf{i}\theta} \\
&= \frac{z-z^{-1}}{2\mathbf{i}} \text{ since } \bar{z} = z^{-1} \text{ when } |z| = 1. \\
\text{Similarly } \cos\theta &= \frac{z+z^{-1}}{2}. \quad \text{Now } z = e^{\mathbf{i}\theta} \Rightarrow dz = \mathbf{i}e^{\mathbf{i}\theta} d\theta \\
&\qquad\qquad\qquad\qquad\qquad = \mathbf{i}z d\theta \\
\therefore\ d\theta &= \frac{1}{\mathbf{i}z} dz \Rightarrow \\
\int_0^{2\pi} \frac{P}{Q}(\sin\theta,\cos\theta) d\theta &= \int_{|z|=1} \frac{P}{Q}\left(\frac{1}{2\mathbf{i}}(z-z^{-1}), \frac{1}{2}(z+z^{-1})\right) \frac{dz}{\mathbf{i}z}
\end{aligned}$$

Example:

$$\begin{aligned}
(a \in \mathbb{R}) \qquad \int_0^{2\pi} \frac{d\theta}{a+\sin\theta} &= \int_{|z|=1} \frac{dz}{\mathbf{i}z\left(a + \left(\frac{z-z^{-1}}{2\mathbf{i}}\right)\right)} \\
&= 2\int_{|z|=1} \frac{dz}{2\mathbf{i}az + z^2 - 1}
\end{aligned}$$

$$\begin{aligned}
\text{Note } z^2 + 2\mathbf{i}az - 1 &= (z+\mathbf{i}a)^2 - 1 + a^2 \\
&= (z+\mathbf{i}a)^2 - (\mathbf{i}\sqrt{a^2-1})^2 \\
&= (z + \mathbf{i}(a+\sqrt{a^2-1}))(z + \mathbf{i}(a-\sqrt{a^2-1}))
\end{aligned}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = \int_{|z|=1} \frac{2dz}{(z+\mathbf{i}(a+\sqrt{a^2-1}))(z+\mathbf{i}(a-\sqrt{a^2-1}))}.$$

$$\begin{aligned} \text{Now } \ a+\sqrt{a^2-1} = \frac{1}{(a-\sqrt{a^2-1})} \therefore |a+\sqrt{a^2-1}| &\geq 1 \Rightarrow |a-\sqrt{a^2-1}| \leq 1 \\ |a+\sqrt{a^2-1}| \leq 1 \Rightarrow |a-\sqrt{a^2-1}| &\geq 1 \end{aligned}$$

But $|a+\sqrt{a^2-1}| = 1 \Leftrightarrow |a| \leq 1 \therefore$ assume w.l.o.g.

$$\left.\begin{array}{r} |a| > 1 \\ |a+\sqrt{a^2-1}| > 1 \end{array}\right\} \Rightarrow \int_0^{2\pi} \frac{d\theta}{a+\sin\theta} = \mathrm{R}es(f(z), -(a-\sqrt{a^2-1})\mathbf{i}) \cdot 2\pi\mathbf{i}$$

$$(f(z) := 2\{[z+\mathbf{i}(a+\sqrt{a^2-1})][z+\mathbf{i}(a-\sqrt{a^2-1})]\}^{-1})$$

Let $g(z) = \dfrac{2}{z+\mathbf{i}(a+\sqrt{a^2-1})}$, then $f(z) = \dfrac{g(z)}{z+\mathbf{i}(a-\sqrt{a^2-1})}$
$\therefore$ Cauchy formula $\Rightarrow$

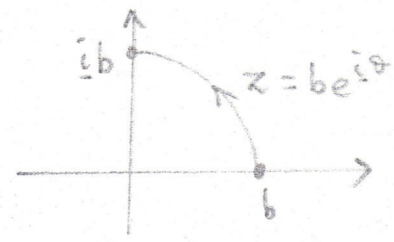
$$\begin{aligned} 2\pi\mathbf{i}\ \mathrm{R}es(f(z), -(a-\sqrt{a^2-1})\mathbf{i}) &= \int_{|z|=1} \frac{g(z)dz}{z+\mathbf{i}(a-\sqrt{a^2-1})} \\ &= 2\pi\mathbf{i} g(-\mathbf{i}(a-\sqrt{a^2-1})) \\ &= \frac{4\pi\mathbf{i}}{2\mathbf{i}\sqrt{a^2-1}} = \frac{2\pi}{\sqrt{a^2-1}} \end{aligned}$$

2. Improper Integrals:

Example: $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}dx$. Note $z^4+1=0$ when $z = e^{\mathbf{i}\left(\frac{\pi}{4}+\frac{2\pi k}{4}\right)}$ $(k=0,1,2,3)$

Write $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}dx = \lim_{a\to-\infty}\int_a^0 \frac{x^2dx}{x^4+1} + \lim_{b\to\infty}\int_0^b \frac{x^2dx}{x^4+1}$

$\therefore$ Consider contour $C_1 := [0,b] \cup [0,\mathbf{i}b] \cup \gamma$ such that $\gamma = \{be^{i\theta} | 0 \leq \vartheta \leq \frac{\pi}{2}\}$ and $b > 1$.



$$
\begin{aligned}
\text{Note } \int_{C_1} \frac{z^2dz}{z^4+1} &= 2\pi\mathbf{i}\mathrm{R}es\left(\frac{z^2}{z^4+1}, e^{\mathbf{i}\pi/4}\right) \\
&= 2\pi\mathbf{i} \lim_{z\to e^{\mathbf{i}\pi/4}} \left[(z - e^{\mathbf{i}\pi/4})\frac{z^2}{z^4+1}\right] \\
&\qquad (\textbf{simple} \text{ pole: } N = 1) \\
&= (2\pi\mathbf{i}) \lim_{z\to e^{\mathbf{i}\pi/4}} \left[\frac{z^2}{4z^3}\right] = \lim_{z\to e^{\mathbf{i}\pi/4}} \left[\frac{\mathbf{i}\pi}{2z}\right] \\
&\qquad (\text{L'Hôpital's Rule}) \\
&= \frac{\mathbf{i}\pi e^{-\mathbf{i}\pi/4}}{2} = \frac{\mathbf{i}\pi\sqrt{2}}{4}(1-\mathbf{i}) = \frac{\pi}{2\sqrt{2}}(1+\mathbf{i})
\end{aligned}
$$

$$
\left.\begin{array}{l}
\text{Now } z \in [0,b] \Rightarrow z = tb \ (t \in [0,1]) \\
z \in [0,\mathbf{i}b] \Rightarrow z = tb\mathbf{i} \ (t \in [0,1]) \\
z = be^{i\theta}(dz = \mathbf{i}zd\theta)
\end{array}\right\} \Rightarrow
$$

$$
\int_{C_1} \frac{z^2dz}{z^4+1} = b\int_0^1 \frac{b^2t^2dt}{b^4t^4+1} - \mathbf{i}b\int_0^1 \frac{-b^2t^2dt}{b^4t^4+1} + \mathbf{i}\int_\gamma \frac{z^2dz}{z^4+1}.
$$

Recall $M = \max\limits_{0\le\theta\le\pi/2} \left|\frac{z^2}{z^4+1}\right| \Rightarrow \left|\mathbf{i}\int_\gamma \frac{z^2dz}{z^4+1}\right| \le ML = M\frac{\pi b}{2}$

Moreover $|z| = b \Rightarrow \left|\frac{z^2}{z^4+1}\right| \le \frac{|z|^2}{|z|^4-1} = \frac{b^2}{b^4-1}$

$\therefore \lim\limits_{b\to\infty} ML = \lim\limits_{b\to\infty} \frac{\pi b^3}{2(b^4-1)} = 0$

$\therefore \frac{\pi}{2\sqrt{2}}(1+\mathbf{i}) = (1+\mathbf{i}) \lim\limits_{b\to\infty} \int_0^b \frac{x^2dx}{x^4+1} \quad (x = bt)$

$$\therefore \int_0^\infty \frac{x^2 dx}{x^4+1} = \frac{\pi}{2\sqrt{2}} \frac{1+\mathbf{i}}{1+\mathbf{i}} = \frac{\pi}{2\sqrt{2}}.$$

Finally $\dfrac{x^2}{x^4+1}$ **even** $\therefore \displaystyle\int_0^\infty \frac{x^2}{x^4+1} dx = \int_{-\infty}^0 \frac{x^2}{x^4+1} dx$

$$\Rightarrow \int_{-\infty}^\infty \frac{x^2}{x^4+1} dx = \frac{\pi}{\sqrt{2}}$$