ABSTRACT ALGEBRA (PMTH332)

SAMPLE SOLUTIONS FOR TUTORIAL 5

Question 1.

Let $\mathbb{R} := \operatorname{End}(A)$ be the set of all endomorphisms of the abelian group $A$, so that

$$\mathbb{R} = \{ \varphi : \mathbb{R} \rightarrow \mathbb{R} \mid \varphi \text{ is a homomorphism} \}$$

Define

$$+ : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad (\varphi, \psi) \mapsto \varphi + \psi$$

where

$$\varphi + \psi : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \varphi(x) + \psi(x)$$

$$\circ : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad (\varphi, \psi) \mapsto \varphi \circ \psi$$

where

$$\varphi \circ \psi : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \varphi(\psi(x))$$

Since the composition of homomorphisms yields a homomorphism, the composition of endomorphism must yield an endomorphism.

Thus, $\circ$ is well defined.

We define $0 : \mathbb{R} \rightarrow \mathbb{R}$ by

$$0(x) = 0$$

for every $x \in A$, where we have written 0 for the neutral element of $A$.

For $\varphi \in \mathbb{R}$, define $(-\varphi) : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(-\varphi)(x) = (-x)$$

for every $x \in A$.

We verify the ring axioms for $(\mathbb{R}, +, \circ)$.

Take $\varphi, \psi, \eta \in \mathbb{R}$, and $x \in A$. Then

$$((\varphi + \psi) + \eta)(x) = (\varphi + \psi)(x) + \eta(x)$$

$$= \varphi(x) + \psi(x) + \eta(x)$$

$$= \varphi(x)(\psi(x) + \eta(x))$$

$$= \varphi(x) + (\psi + \eta)(x)$$

$$= (\varphi + (\psi + \eta))(x)$$
Since this holds for all \( x \in A \),
\[
(\varphi + \psi) + \eta = \varphi + (\psi + \eta)
\]
This establishes the associativity of +.

Similar calculations, which are left to the reader to complete — and the reader is strongly encouraged to do so — verify that 0 is the neutral element for the addition in \( R \), that \( -\varphi \) is the additive inverse of \( \varphi \) in \( R \), that addition in \( R \) is commutative.

That multiplication in \( R \) is associative with neutral element \( \text{id}_A \) follows immediately from the properties of the composition of functions.

\[
((\varphi + \psi) \circ \eta)(x) = (\varphi + \psi)\eta(x)
= \varphi(\eta(x)) + \psi(\eta(x))
= (\varphi \circ \eta)(x) + (\psi \circ \eta)(x)
= ((\varphi \circ \eta) + (\psi \circ \eta))(x)
\]

\[
(\varphi \circ (\psi + \eta))(x) = \varphi((\psi + \eta)(x))
= \varphi(\psi(x) + \eta(x))
= \varphi(\psi(x)) + \varphi(\eta(x))
= (\varphi \circ \psi)(x) + (\varphi \circ \eta)(x)
= ((\varphi \circ \eta) + (\varphi + \eta))(x)
\]

Since these hold for all \( x \in A \)
\[
(\varphi + \psi) \circ \eta = (\varphi \circ \eta) + (\psi \circ \eta)
\varphi \circ (\psi + \eta) = (\varphi \circ \psi) + (\varphi \circ \eta)
\]
showing that the multiplication in \( R \) distributes over addition.

Hence \( \text{End}(A) \) is a unital ring.

**Question 2.**

For the ring \( R \) and group \( G \), put
\[
R[G] = \left\{ \sum_{g \in G} r_g g \mid r_g \in R \text{ with } r_g = 0 \text{ for all but finitely many } g \in G \right\}
\]
and define
\[
+: R[G] \times R[G] \to R[G], \quad \sum_{g \in G} r_g g + \sum_{g \in G} s_g g \mapsto \sum_{g \in G} (r_g + s_g) g
\]
\[
*: R[G] \times R[G] \to R[G], \quad \sum_{g \in G} r_g g \ast \sum_{g \in G} s_g g \mapsto \sum_{g, h \in G} r_g s_h g h
\]
Observe that
\[
\sum_{g, h \in G} r_g s_h g h = \sum_{k \in G} t_k k
\]
where

\[ t_k = \sum_{gh=k} r_g s_h \]

If \( r_g \neq 0 \) for at \( m \) elements of \( G \) and \( s_g \neq 0 \) for \( n \) elements of \( G \), then \((r_g + s_g) \neq 0\) for at most \( m + n \) elements of \( R \), and \( t_g \neq 0 \) for at most \( mn \) elements of \( G \), showing that the operations are well defined.

Take \( x = \sum r_g g \), \( y = \sum s_g g \) and \( z = \sum t_g \).

Put \( 0 = \sum 0 g \) and \( (-x) = \sum (-r_g)g \). Then

\[
(x + y) + z = \sum_{g \in G} ((r_g + s_g) + t_g) g \\
= \sum_{g \in G} (r_g + (s_g + t_g)) g \\
= x + (y + z)
\]

\[
x + 0 = \sum_{g \in G} (r_g + 0) g \\
= \sum_{g \in G} r_g g \\
= x
\]

\[
x + (-x) = \sum_{g \in G} ((r_g + (-r_g)) g \\
= \sum_{g \in G} 0 g \\
= 0
\]

\[
x + y = \sum_{g \in G} (r_g + s_g) g \\
= \sum_{g \in G} (s_g + r_g) g \\
= y + x
\]

\[
(x \times y) \times Z = \sum_{g,h,k \in G} ((r_g s_h) t_k) ghk \\
= \sum_{g,h,k \in G} (r_g (s_h t_k)) ghk \\
= x \times (y \times z)
\]
\[ x \times (y + z) = \sum_{g,h \in G} (r_g(s_h + t_h))gh \]
\[ = \sum_{g,h \in G} (r_g(s_h + r_g t_h))gh \]
\[ = \sum_{g,h \in G} (r_g s_h)gh + \sum_{g,h \in G} (r_g t_h)gh \]
\[ = (x \times y) + (x \times z) \]
\[ (x + y) \times z = \sum_{g,h \in G} ((r_g + s_g) t_h) gh \]
\[ = \sum_{g,h \in G} (r_g(t_h + s_g t_h)) gh \]
\[ = \sum_{g,h \in G} (r_g t_h) gh + \sum_{g,h \in G} (s_g t_h) gh \]
\[ = (x \times z) + (y \times z) \]

Thus, \( R[G] \) is a ring.

**Question 3.**

Let \( \varphi: R \rightarrow S \) be a homomorphism of rings.

For \( x, y \in R \),
\[ \varphi(x) = \varphi(y) \quad \text{if and only if} \quad \varphi(x - y) = 0 \quad \text{as} \quad \varphi \text{ is a homomorphism} \]
\[ \text{if and only if} \quad x - y \in \ker \varphi \]

Hence, if \( \ker \varphi = \{0\} \), then \( \varphi(x) = \varphi(y) \) if and only if \( x = y \), showing that \( \varphi \) is injective.

Conversely, let \( \varphi \) be injective.

As \( \ker \varphi = \{x \in R \mid \varphi(x) = 0\} \) and \( \varphi(0) = 0 \), \( \ker \varphi = \{0\} \).

Thus, \( \varphi \) is injective if and only if \( \ker \varphi = \{0\} \).

**Question 4.**

Let \( F \) be a field and \( \varphi: F \rightarrow R \) a homomorphism of rings.

Then \( \ker \varphi \) is an ideal of \( F \).

Being a field, the only ideals \( F \) has are \( \{0\} \) and \( F \).

If \( \ker \varphi = \{0\} \), then \( \varphi \) is injective.

If \( \ker \varphi F \), then \( \varphi \) is the zero homomorphism.
Question 5.

For the integral domain $D$, put
\[ \tilde{D} := \{(b, a) \mid a, b \in D, a \neq 0\} \]
and define the binary relation $\sim$ on $\tilde{D}$ by
\[ (b, a) \sim (d, c) \text{ if and only if } bc = ad \]

(a) We show that $\sim$ is an equivalence relation on $\tilde{D}$.

**Reflexivity:** $(b, a) \sim (b, a)$, since $ba = ab$, as $D$ is commutative.

**Symmetry:** $(b, a) \sim (d, c)$ if and only if $bc = ad$ if and only if $da = cb$ if and only if $(d, c) \sim (b, a)$.

**Transitivity:** If $(b, a) \sim (d, c)$ and $(d, c) \sim (f, e)$, then $bc = ad$ and $de = cf$.

Thus, $bec = bce = ade = acf = afc$ whence, since $D$ has no zero-divisors, $be = af$, or, equivalently, $(b, a) \sim (f, e)$.

(b) Define
\[ + : \tilde{D} \times \tilde{D} \to \tilde{D}, \quad ((b, a), (d, c)) \mapsto (bc + ad, ac) \]
\[ \times : \tilde{D} \times \tilde{D} \to \tilde{D}, \quad ((b, a), (d, c)) \mapsto (bd, ac) \]

Since $D$ has no zero-divisors, $ac \neq 0$, so that the operations are well defined.

Suppose that $(b, a) \sim (f, e)$ and $(d, c) \sim (h, g)$, so that
\[ be = af \quad \text{and} \quad dg = ch \]

Then
\[ (b, a) + (d, c) = (bc + ad, ac) \]
\[ (f, e) + (h, g) = (fg + eh, eg) \]
\[ (b, a) \times (d, c) = (bd, ac) \]
\[ (f, e) \times (h, g) = (fh, eg) \]

and so
\[ (bc + ad)eg = bceg + adeg \]
\[ = afcg + aceh \]
\[ = (fg + eh)ac \]

showing that
\[ ( (b, a) + (d, c) ) \sim ( (f, e) + (h, g) ) \]

Similarly, from
\[ bdeg = afch \]

it follows that
\[ ( (b, a) \times (d, c) ) \sim ( (f, e) \times (h, g) ) \]