ABSTRACT ALGEBRA (PMTH332)

SAMPLE SOLUTIONS FOR TUTORIAL 4

Question 1.
If \( n \geq 3 \), then \((12), (13) \in S_n\).
Since 
\[(12)(13) = (132), \text{ but } (13)(12) = (123) \neq (132), \]
\( S_n \) is not commutative for \( n \geq 3 \).

Question 2.
Let \( G \) be a non-commutative group.
Then there are \( a, b \in G \) with \( ab \neq ba \).
Hence \( c = aba^{-1}b^{-1} \neq e \), where \( e \) is the neutral element of \( G \).
Put \( H = \langle c \rangle = \{ c^n \mid n \in \mathbb{Z} \} \), the subgroup of \( G \) generated by \( c \).
Since \( c \neq e \), \( H \) is not the trivial subgroup.
Since \( H \) is a cyclic group, it is commutative, whence \( H \) cannot be all of \( G \).

Question 3.
Let \( G_1, G_2 \) be groups.

(a) Consider the binary operation
\[
*: (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2, \quad ((g_1, g_2), (g'_1, g'_2)) \mapsto (g_1g'_1, g_2g'_2)
\]
Take \((g_1, g_2), (g'_1, g'_2), (g''_1, g''_2) \in G_1 \times G_2\).
\[
((g_1, g_2) * (g'_1, g'_2)) * (g''_1, g''_2) = (g_1g'_1, g_2g'_2) * (g''_1, g''_2)
= ((g_1g'_1)g''_1, (g_2g'_2)g''_2)
= (g_1(g'_1g''_1), g_2(g_2g''_2)) \quad \text{as } G_1, G_2 \text{ are groups}
= (g_1, g_2) * (g'_1g''_1, g'_2g''_2)
= (g_1, g_2) * ((g'_1, g'_2) * (g''_1, g''_2))
\]
showing that * is associative.

Plainly,
\[
e_{G_1 \times G_2} = (e_{G_1}, e_{G_2})
\]
and
\[
(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1})
\]
Hence \( G_1 \times G_2 \) is a group with respect to *.

(b) For \( i = 1, 2 \), consider the function
\[
\pi_i: G_1 \times G_2 \to G_i, \quad (g_1, g_2) \mapsto g_i
\]
Given \((g_1, g_2), (g'_1, g'_2) \in G_1 \times G_2\),
\[
\pi_i((g_1, g_2) * (g'_1, g'_2)) = \pi_i(g_1g'_1, g_2g'_2)
= g_i g'_i
= \pi_i(g_1, g_2) \pi_i(g'_1, g'_2)
\]
showing that \( \pi_i \) is a homomorphism.
Question 4.
Given groups \( G_1, G_2 \), with neutral elements \( e_1 \) and \( e_2 \) respectively, put
\[
\tilde{G}_1 := \{(e_1, y) \mid y \in G_2\} \subseteq G_1 \times G_2
\]
\[
\tilde{G}_2 := \{(x, e_2) \mid x \in G_1\} \subseteq G_1 \times G_2
\]
Since \((e_1, e_2) \in \tilde{G}_1 \cap \tilde{G}_2, \tilde{G}_1, \tilde{G}_2 \neq \emptyset\).
Take \((x, e_2), (u, e_2) \in \tilde{G}_2\). Then
\[
(x, e_2) \ast (u, e_2)^{-1} = (xu^{-1}, e_2 e_2^{-1})
\]
\[
= (xu^{-1}, e_2)
\]
\[
\in \tilde{G}_2
\]
as \(xu^{-1} \in G_1\)
Thus, \(\tilde{G}_2\) is a subgroup of \(G_1 \times G_2\).

A completely analogous argument shows that \(\tilde{G}_1\) is also a subgroup of \(G_1 \times G_2\).

For \(i = 1, 2\), consider the function
\[
\pi_i : G_1 \times G_2 \rightarrow G_i, \quad (g_1, g_2) \mapsto g_i
\]
As shown in Question 3(b), \(\pi_i\) is a homomorphism.

Being the natural projection from the Cartesian product of two sets onto one of the factors, it is surjective. Hence, by the Noether Isomorphism Theorem,
\[
G_i = \text{im}(\pi_i) \cong (G_1 \times G_2)/\ker(\pi_i)
\]
It remains to determine \(\ker(\pi_i)\).
\[
(g_1, g_2) \in \ker(\pi_i) \text{ if and only if } \pi_i(g_1, g_2) = e_i
\]
if and only if \(g_i = e_i\)
if and only if \((g_1, g_2) \in \tilde{G}_i\)
Hence, \(\ker(\pi_i) = \tilde{G}_i\).

Being the kernel of \(\varphi\), \(\tilde{G}_i\) is a normal subgroup of \(G_1 \times G_2\) and
\[
G_i \cong (G_1 \times G_2)/\tilde{G}_i
\]

Question 5.
Given a homomorphism of groups, \(\varphi : G \rightarrow H\), and \(K \subseteq G\),
\[
x \in \varphi^{-1}(\varphi(K)) \text{ if and only if } \varphi(x) \in \varphi(K)
\]
if and only if \(\varphi(x) = \varphi(k)\) for some \(k \in K\)
if and only if \(\varphi(k)^{-1} \varphi(x) = e_H\) for some \(k \in K\)
if and only if \(k^{-1} x \in \ker \varphi\) for some \(k \in K\)
if and only if \(x \in k \ker \varphi\) for some \(k \in K\)
if and only if \(x \in K \ker \varphi\)
Thus,
\[
\varphi^{-1}(\varphi(K)) = K \ker \varphi
\]