Assignment Problems

There are six sets of assignment problems, the marks for which comprise 30% of your total mark for the unit.

The assignments should be posted by the post-by date for marking.

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The assignments are intended to be the most challenging part of the course. View them as miniature research projects. They provide the opportunity to pursue independent work.

Do not hesitate to contact me for assistance with background information or guidance with the assignment problems.

By all means work together with colleagues. But make sure that the work you submit is your own. Anyone who provides you with more or less complete solutions before you have submitted your own attempts is doing you a gross disservice indeed.

Submitting Assignments

Mathematics assignments should be hand-written, single-sided, with space left for comments, and preferably submitted by post with an assignment cover sheet attached. These are available via the PMTH332 Moodle site by clicking on “Assignment X” and then “Download Coversheet” on the right. If you have received an exemption from the requirement to have computer access, you will receive your assignment cover sheets in the mail.

Assignments should be sent by post to:

Assignment Section
Learning Innovations Hub
University of New England
Armidale 2351
N.S.W.

DO NOT FAX YOUR ASSIGNMENTS. Faxed assignments will not be accepted.

Although we prefer assignments to be submitted by post, you can submit electronically. To do so, scan each assignment and submit the .pdf file via the PMTH332 Moodle site.

If you insist on typing your assignments, you must use some version of \TeX, with \LaTeX most convenient.

The dates by which you must send off your assignments are listed in the table above.
Question 1.
Show that every equivalence relation on the set $X$ determines a unique partition of $X$, and conversely.

Question 2.
Let $*: G \times G \rightarrow G$ be an associative binary operation on the set $G$ such that

GL2: there is an $e \in G$ such that for all $a \in G$,
$$e * a = a$$

GL3: for each $a \in G$ there is an $\pi \in G$ such that
$$\pi * a = e$$

Show that $(G, *)$ is a group.

This shows that axioms (G2L), (G3L) and (G1) together imply (G1), (G2) and (G3).

Question 3.
Given $m \in \mathbb{Z}, m \neq 0$, let $\mathbb{Z}_m$ be the set of residue classes modulo $m$. Show that

$$+ : \mathbb{Z}_m \times \mathbb{Z}_m \rightarrow \mathbb{Z}_m; \quad ([l], [k]) \mapsto [l + k]$$

is a well-defined binary operation, that is, $[l + k]$ does not depend on the choice of representatives $l$ and $k$ of the equivalence classes $[l]$ and $[k]$.

Prove that $(\mathbb{Z}_m, +)$ is a group.

Question 4.
Let $G$ be a group such that $(a * b)^2 = a^2 * b^2$ for all $a, b \in G$.

Show that $(G, *)$ is abelian.
Question 1.
Let $H = (\mathbb{R}^+, \times)$ be the group consisting of the set of all strictly positive real numbers with the binary operation given by multiplication.

Show that $H$ is isomorphic to $(\mathbb{R}, +)$, the group consisting of all real numbers with the binary operation given by addition.

Question 2.
Given isomorphisms $\varphi: G \rightarrow H$, $\psi: H \rightarrow K$, show that their composition $\psi \circ \varphi: G \rightarrow K$ is also an isomorphism.

Question 3.
Let $H$ be a subset of the set underlying the group $G$.

Show that $x \sim_H y$ if and only if $xy^{-1} \in H$ defines an equivalence on $G$ relation if and only if $H$ is a subgroup of $G$.

Question 4.
An automorphism of the group $G$ is isomorphism $\varphi: G \rightarrow G$.

(a) Show that $\text{Aut}(G)$, the set of all automorphisms of $G$, is a subgroup of $S(G)$, the group of invertible functions of the set underlying $G$ to itself, with the binary operation given by composition of functions.

(b) Given $g \in G$ define
$$\varphi_g: G \rightarrow G, \quad x \mapsto gxg^{-1}$$
Show that $\varphi_g$ is an automorphism for every $g \in G$.

(c) Show that $\text{Inn}: G \rightarrow \text{Aut}(G), \quad g \mapsto \varphi_g$ is a homomorphism.
Question 1.
Recall that
\[ \text{Inn}: G \rightarrow \text{Aut}(G), \quad g \mapsto \varphi_g \]
where, for \( x \in G \), \( \varphi_g(x) = gxg^{-1} \) is a homomorphism.
Show that the kernel of Inn is \( C(G) \), the centre of \( G \), that is
\[ \ker(\text{Inn}) = C(G) \]

Question 2.
Show that every subgroup \( H \) of the group \( G \) of index two is normal.

Question 3. (Cayley’s Theorem)
Show that every group \( G \) of order \( n \) is isomorphic to a subgroup of \( S_n \).

Question 4.
Let \( G \) be a group and let \( H \) be a subgroup and \( N \) a normal subgroup of \( G \). Define the subset \( HN \) of \( G \) by
\[ HN := \{hn \mid h \in H, \ n \in N\} \]
Show that
(i) \( HN \) is a subgroup of \( G \).
(ii) \( HN \) is the smallest subgroup of \( G \) containing both \( H \) and \( N \).
(iii) \( HN \) is normal in \( G \) whenever \( H \) is normal in \( G \).
ASSIGNMENT 4

Question 1.
A maximal normal subgroup of the group $G$ is a proper normal subgroup $M$ of $G$ such that any other normal subgroup $N$ of $G$ containing $M$ must be equal to $M$ or to $G$.

The group $G$ is simple if and only if $G$ has no non-trivial subgroups.

Show that $M$ is a maximal normal subgroup of $G$ if and only if $G/M$ is simple.

Question 2.
Let \( \{G_i \mid i = 1, \ldots, n\} \). Put
\[
G = \prod_{i=1}^{n} G_i
\]

Show that if for each $i \in \{1, \ldots, n\}$ the element $a_i$ of $G_i$ has finite order, then the element $(a_1, a_2, \ldots, a_n)$ of $G$ has finite order given by
\[
\text{ord}(a_1, a_2, \ldots, a_n) = \text{lcm}(\text{ord} a_1, \text{ord} a_2, \ldots, \text{ord} a_n)
\]

Question 3.
Show that for $m, n$ non-zero positive integers
\[
\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/mn\mathbb{Z}
\]
if and only if $m$ and $n$ are relatively prime, that is, $\gcd(m, n) = 1$.

Question 4.
Let $G$ be a group of order 324.

Show that $G$ has subgroups of order 2, 3, 4, 9, 27 and 81, but none of order 10.
PMTH 332  ABSTRACT ALGEBRA  (Post by: 15th September, 2014)

ASSIGNMENT 5

Question 1.
Let $M(2; R)$ denote the set of $(2 \times 2)$–matrices with coefficients in the commutative ring $R$, so that

$$M(2; R) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in R \right\}$$

Then $M(2; R)$ is a ring with respect to addition and multiplication of matrices.

Show that $M(2; R)$ has zero divisors and provide an example where $M(2; R)$ is not commutative.

Question 2.
Show that a finite integral domain is a field.

Question 3.
Show that the characteristic of a non–trivial integral domain with 1 must be either zero or a prime.

Question 4.
Given the integral domain, $D$, put

$$\tilde{D} := \{(b, a) \mid a, b \in D, a \neq 0\}.$$  

$$(b, a) \sim (d, c) \text{ if and only if } bc = ad$$
defines an equivalence relation on $\tilde{D}$ Let $F$ be the set of equivalence classes of this equivalence relation.

Show that $(F, +, \times)$ is a field, for

$$+ : F \times F \longrightarrow F, \quad ([b, a], [(d, c)] \longmapsto [(ad + bc, ac)]$$

$$\times : F \times F \longrightarrow F, \quad ([b, a], [(d, c)] \longmapsto [(bd, ac)]$$

and that

$$\iota : D \longrightarrow F, \quad a \longmapsto [(a, 1)]$$

is an injective ring homomorphism.

$F$ is the field of quotients of $D$. 
Question 1.
Show that for every integer \( n \) the number \( n^{33} - n \) is divisible by 15.

Question 2.
Given a field \( F \) show that every proper non–trivial prime ideal of \( F[x] \) is maximal.

Question 3.
Let \( F \) be a field and let \( f(x), g(x) \in F[x] \) be relatively prime.
Show that for every \( h(x) \in F[x], f(x)|h(x) \) and \( g(x)|h(x) \) only \( f(x)g(x)|h(x) \).

Question 4.
(a) Find all prime ideals of \( \mathbb{Z}_{12} \).
(b) Find a prime ideal of \( \mathbb{Z} \times \mathbb{Z} \) which is not maximal.

Question 5.
Write \( p(x) = x^3 + 2x + 3 \) as a product of irreducible polynomials over \( \mathbb{Z}_5 \).