SAMPLE SOLUTIONS FOR TUTORIAL 6

Question 1. Let \((X, T)\) be a topological space. Suppose that \(X = A \cup B\), with \(A\) and \(B\) closed subsets of \(X\).
Given continuous functions \(f: A \to Y\) and \(g: B \to Y\) with \(f(x) = g(x)\) whenever \(x \in A \cap B\), define
\[
h: X \to Y, \quad x \mapsto \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}
\]
Then \(h\) is a well-defined function, since \(X = A \cup B\) and \(f(x) = g(x)\) whenever \(x \in A \cap B\).
Let \(F\) be a closed subset of \(Y\). Then
\[
h^{-1}(F) = (h^{-1}(F) \cap A) \cup (h^{-1}(F) \cap B) = f^{-1}(F) \cap g^{-1}(F).
\]
By the continuity of \(f\), \(f^{-1}(F)\) is closed in \(A\) and so closed in \(X\), since \(A\) is closed in \(F\).
Similarly \(g^{-1}(F)\) is also closed in \(X\).
Thus, being the intersection of closed subsets of \(X\), \(h^{-1}(F)\) is a closed subset of \(X\), showing that \(h\) is continuous.

Question 2. Let \(X \neq \emptyset\) be connected and \(f: X \to Y\) continuous. Then the image of \(f\) is a connected subset of \(Y\). But \(Y\) is discrete, so that its only connected subsets are singletons. Hence \(f\) must be constant.

Question 3. We offer two proofs, one applying compactness, the other connectedness.

Compactness Argument: Since \(S^1\) is a closed and bounded subset of \(\mathbb{R}^2\), it is compact by the Heine-Borel Theorem.
Hence any continuous image of \(S^1\) must also be compact.
But by the Heine-Borel Theorem, \([0, 1]\) is not compact, since it is not a closed subset of \(\mathbb{R}\).
Thus there is no continuous surjection \(f: S^1 \to [0, 1]\).

Connectedness Argument: Since \(S^1\) is connected, any continuous image of \(S^1\) must also be connected.
So if \(f: S^1 \to [0, 1]\) is a continuous function, its image must be a subinterval of \([0, 1]\), as the connected subsets of \(\mathbb{R}\) are precisely the intervals.
Now, take \(y\) from the interior of \(\text{im}(f)\) (as subset of \(\mathbb{R}\)).
If \(f^{-1}(y)\) consists of a singleton, \(\{x\}\), then \(S^1 \setminus \{x\}\) is connected, but \(\text{im}(f) \setminus \{y\}\) is not.
Hence the interior of $\text{im}(f)$ is empty. Thus $\text{im}(f)$ must be a singleton, since it is connected. Thus there is no continuous injection $f : \mathbb{S}^1 \rightarrow [01, [.$

**Question 4.** Let $\{K_\lambda \mid \lambda \in \Lambda\}$ be a collection of compact subsets of the Hausdorff space $(X, \mathcal{T})$. Then each $K_\lambda$ is closed. Hence $K := \bigcap K_\lambda$ is also closed. Take $\mu \in \Lambda$. Then $K \subseteq K_\mu$, so that $K$ is a closed subset of the compact set $K_\mu$, whence $K$ is itself compact.

**Question 5.** Take $A \subseteq \mathbb{R}, A \neq \emptyset$. If $A$ is totally bounded, take a finite subset $\{a_1, \ldots, a_n\}$ with $a_1 < a_2 < \cdots < a_n$, such that

$$A \subseteq \bigcup_{j=1}^{n} [a_j - 1, a_j + 1[.$$

Then, plainly, $A \subseteq ]a_1 - 1, a_n + 1[$. Thus, $A$ is bounded.

For the converse, suppose that $A$ is bounded. Then $A \subseteq [-K, K]$ for some $K \in \mathbb{R}$. Since $A \neq \emptyset$, it has an infimum, $\alpha_0$.

Take $\varepsilon > 0$. By the definition of an infimum, we may choose an $a_0 \in A \cap [\alpha_0, \alpha_0 + \frac{\varepsilon}{2}[$.

Clearly $\alpha_0 \in B(a_0; \frac{\varepsilon}{2})$.

If $A \setminus B(a_0; \frac{\varepsilon}{2}) = \emptyset$, then $\{a_0\}$ is an $\varepsilon$-net for $A$.

Otherwise, suppose we have chosen $a_0 < \cdots < a_n$ with $a_{j+1} \in A \setminus \bigcup_{i=0}^{j-1} B(a_i; \frac{\varepsilon}{2})$ ($0 \leq j < n$).

If $A \setminus \bigcup_{i=0}^{n} B(a_i; \frac{\varepsilon}{2}) = \emptyset$, then $\{a_0, \ldots, a_n\}$ is an $\varepsilon$-net for $A$.

Otherwise, put $\alpha_{n+1} := \inf\{x \in A \setminus \bigcup_{i=0}^{n} B(a_i; \frac{\varepsilon}{2})\}$.

Take $a_{n+1} \in [\alpha_{n+1}, \alpha_{n+1} + \frac{\varepsilon}{2}[$.

Since $\alpha_{j+1} \geq a_j + \frac{\varepsilon}{2}$ ($j = 0, \ldots, n$), $a_{n+1} \geq a_0 + n \frac{\varepsilon}{2} > \alpha_0 + \frac{\varepsilon}{2}$.

Hence, if $n > \frac{4K}{\varepsilon}$, then $\alpha_0 + 2K > K$.

Thus

$$A \subseteq \bigcup_{i=0}^{n} B(a_i; \frac{\varepsilon}{2}),$$

so that $\{a_0, \ldots, a_n\}$ is an $\varepsilon$-net for $A$.

Thus $A$ is totally bounded.