Question 1.
Let \( f : X \rightarrow Y \) be a function.
Take \( A \subseteq X \) and \( B \subseteq Y \).

(i)
Given \( x \in A \), \( f(x) \in f(A) \), by the definition of \( f(A) \).
Hence, \( x \in f^{-1}(f(A)) \), by the definition of \( f^{-1}(f(A)) \).
Thus, \( A \subseteq f^{-1}(f(A)) \).

Given \( y \in f(f^{-1}(B)) \), \( y = f(x) \) for some \( x \in f^{-1}(B) \), by the definition of \( f(f^{-1}(B)) \).
Hence, \( y = f(x) \in B \), by the definition of \( f(f^{-1}(B)) \).
Thus, \( f(f^{-1}(B)) \subseteq B \).

(ii)
For \( X = Y := \mathbb{R}, A = B = [-1,1] \) and the function
\( f : X \rightarrow Y, \quad x \mapsto 0 \)
\[
A = [-1,1] \\
\neq \mathbb{R} \\
= f^{-1}([0]) \\
= f^{-1}(f(A)) \\
f(f^{-1}(B)) = \{0\} \\
\neq [-1,1] \\
= B.
\]

(iii)
Let \( f \) be injective.
Since, by (i), \( A \subseteq f^{-1}(f(A)) \), it is enough to show that \( f^{-1}(f(A)) \subseteq A \).
If \( x \in f^{-1}(f(A)) \), then \( f(x) \in f(A) \) by the definition of \( f^{-1}(f(A)) \).
Hence, by the definition of \( f(A) \), \( f(x) = f(a) \) for some \( a \in A \).
Since \( f \) is injective, \( x = a \), whence \( x \in A \).
Thus \( f^{-1}(f(A)) \subseteq A \).

For the converse, suppose that for all \( A \subseteq X \), \( f^{-1}(f(A)) = A \).
Take \( x, x' \in X \) with \( f(x') = f(x) \).
Putting \( A := \{x\} \)
\[
x' \in f^{-1}(f(\{x\})) \\
= \{x\}
\]
Thus, \( x' = x \), showing that \( f \) is injective.
(iv) Let $f$ be surjective.

Take $y \in B$.

Since $f$ is surjective, there is an $x \in X$ with $f(x) = y$.

Then, by definition, $x \in f^{-1}(B)$.

Thus, $y = f(x) \in f(f^{-1}(B))$.

Together with (i), this shows that $f(f^{-1}(B)) = B$.

Conversely, suppose that for every $B \subseteq Y$, we have $f(f^{-1}(B)) = B$.

Then $\text{im}(f) = f(X) = f(f^{-1}(Y)) = Y$, whence $f$ is surjective.

(v) Let $g : Y \rightarrow X$ be the inverse of $f$.

Then $y = f(x)$ if and only if $g(y) = x$ and so

$$f^{-1}(B) = \{x \mid f(x) \in B\} = \{g(y) \mid y \in B\}$$

as $x = g(y)$ if and only if $f(x) = y$.

Question 2.
Let $f : X \rightarrow Y$ be a function and take $A, B \subseteq X$, $C, D \subseteq Y$.

(i) $f(A \cup B) = f(A) \cup f(B)$, for

$$y \in f(A \cup B) \text{ if and only if } y = f(x), \text{ with } x \in A \cup B$$

if and only if $y = f(x)$, with $x \in A$ or $x \in B$.

if and only if $y \in f(A)$ or $y \in f(B)$.

if and only if $y \in f(A) \cup f(B)$.

(ii) $f(A \cap B) \subseteq f(A) \cap f(B)$, for

$$y \in f(A \cap B) \text{ if and only if } y = f(x) \text{ where } x \in A \cap B$$

if and only if $y = f(x)$ where $x \in A$ and $x \in B$.

if and only if $y \in f(A)$ and $y \in f(B)$.

if and only if $y \in f(A) \cap f(B)$.

To see that equality need not hold, take

$$X := Y := \mathbb{R}, \quad A := \mathbb{R}^{-} = \{x \in \mathbb{R} \mid x < 0\}, \quad B := \mathbb{R}^{+} = \{x \in \mathbb{R} \mid x > 0\}$$

and

$$f : X \rightarrow Y, \quad x \mapsto 0.$$  

Then $f(A \cap B) = \emptyset$ as $A \cap B = \emptyset$, but $f(A) \cap f(B) = \{0\} \neq \emptyset$.

(iii) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$, for

$$x \in f^{-1}(G \cup H) \text{ if and only if } f(x) \in G \cup H$$

if and only if $f(x) \in G$ or $f(x) \in H$.

if and only if $x \in f^{-1}(G)$ or $x \in f^{-1}(H)$.

if and only if $x \in f^{-1}(G) \cup f^{-1}(H)$.  

2
We can illustrate the difference diagrammatically as well.

Then to see that equality need not hold, take

\[ f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \{0\} \\ 0 & \text{otherwise} \end{cases} \]

In other words,

\[ f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H) \]

for

\[ x \in f^{-1}(G \cap H) \quad \text{if and only if} \quad f(x) \in G \cap H \]

\[ \text{if and only if} \quad f(x) \in G \quad \text{and} \quad f(x) \in H \]

\[ \text{if and only if} \quad x \in f^{-1}(G) \quad \text{and} \quad x \in f^{-1}(H) \]

\[ \text{if and only if} \quad x \in f^{-1}(G) \cap f^{-1}(H). \]

The following examples show there is no fixed relationship between \( f(X \setminus A) \) and \( Y \setminus f(A) \).

(a) For \( X = Y \), \( f := \text{id}_X \) and any \( A \subseteq X \),

\[ f(X \setminus A) = Y \setminus f(A) \]

(b) For \( X := \mathbb{R}, A := [0, 1], Y := [-1, 1] \) and \( f(x) := \sin(2\pi x) \),

\[ f(X \setminus A) = Y \]

\[ Y \setminus f(A) = \emptyset \]

(c) For \( X = Y = A = \mathbb{R} \) and \( f(x) = 0 \) for all \( x \in X \).

\[ f(X \setminus A) = \emptyset \quad \text{as} \quad X \setminus A = \emptyset \]

\[ Y \setminus f(A) \neq \emptyset \]

(vi) \( f^{-1}(Y \setminus G) = X \setminus f^{-1}(G) \), for

\[ x \in f^{-1}(Y \setminus G) \quad \text{if and only if} \quad f(x) \in Y \setminus G \]

\[ f(x) \notin G \]

\[ \text{if and only if} \quad x \notin f^{-1}(G) \]

\[ \text{if and only if} \quad x \in X \setminus f^{-1}(G). \]

**Question 3.**

Take sets \( A, B, C \) and \( D \).

(i) \( (A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D) \), for

\[ (x, y) \in (A \cup C) \times (B \cup D) \quad \text{if and only if} \quad x \in (A \cup C) \quad \text{and} \quad y \in (B \cup D) \]

\[ (x \in A \quad \text{or} \quad x \in C) \quad \text{and} \quad (y \in B \quad \text{or} \quad y \in D) \]

\[ (x \in A \quad \text{and} \quad y \in B) \quad \text{or} \quad (x \in C \quad \text{and} \quad y \in B) \]

\[ \text{or} \quad (x \in A \quad \text{and} \quad y \in D) \quad \text{or} \quad (x \in C \quad \text{and} \quad y \in D) \]

\[ (x, y) \in A \times B \quad \text{or} \quad (x, y) \in C \times B \quad \text{or} \quad (x, y) \in A \times D \]

\[ \text{or} \quad (x, y) \in C \times D \]

In other words,

\[ (A \cup C) \times (B \cup D) = (A \times B) \cup (C \times B) \cup (A \times D) \cup (C \times D) \]

To see that equality need not hold, take \( a \neq c \) and \( b \neq d \) and consider

\[ A := \{a\}, \quad B := \{b\}, \quad C := \{c\}, \quad D = \{d\}, \]

Then

\[ (A \times B) \cup (C \times D) = \{(a, b), (c, d)\}, \]

whereas

\[ (A \cup C) \times (B \cup D) = \{(a, b), (a, d), (c, b), (c, d)\}. \]

We can illustrate the difference diagrammatically as well.
Let $A$ and $C$ be disjoint (finite) intervals in $\mathbb{R}$.
Let $B$ and $D$ also be disjoint (finite) intervals in $\mathbb{R}$.
Then $(A \times B) \cup (C \times D)$ and $(A \cup C) \times (B \cup D)$ are the subsets of $\mathbb{R}^2$ illustrated in the following diagrams.

(ii) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. For

$(x, y) \in (A \times B) \cap (C \times D)$ if and only if $x \in A$ and $y \in B$, and $x \in C$ and $y \in D$
if and only if $x \in A$ and $x \in B$, and $y \in C$ and $y \in D$
if and only if $x \in A \cap B$ and $y \in C \cap D$.

Question 4.
Take $A := \mathbb{R}^*: = \mathbb{R} \setminus \{0\}$.

(a) The function

$$f: A \to \mathbb{R}, \quad x \mapsto \frac{x}{|x|} = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

clearly satisfies

$$\frac{df}{dx} = 0$$

everywhere on $A$, but is not constant, since $f(-1) \neq f(1)$.
(b) The function
\[ f: A \rightarrow \mathbb{R}, \quad x \mapsto -\frac{1}{x} \]
clearly satisfies
\[ \frac{df}{dx} = \frac{1}{x^2} > 0 \]
everywhere on A, but is not monotonically increasing, since \( f(-1) = 1 > -1 = f(1) \).

(c) For the function
\[ f: A \rightarrow \mathbb{R}, \quad x \mapsto x - \frac{1}{x} \]
\[ \frac{df}{dx} = 1 + \frac{1}{x^2} > 0 \]
for all \( x \in A \).
But \( f \) is not injective (1–1), since \( f(-1) = f(1) = 0 \).

**Question 5.**

Let \( \sim \) be an equivalence relation on the set \( X \)

Let \( [x] \) denote the equivalence class of \( x \in X \), so that
\[ [x] := \{ t \in X \mid x \sim t \}. \]

(a) Take \( x, x' \in X \) with \( x \sim x' \).

If \( z \in [x] \), then, by the definition of \([x]\), \( z \sim x \).

Since \( \sim \) is transitive, \( z \sim x' \).

Hence, by the definition of \([x']\), \( z \in [x'] \).

Thus, \([x]\) \subseteq \([x']\)\).

Since \( \sim \) is symmetric, we may interchange \( x \) and \( x' \), and, arguing as above, obtain \([x']\) \subseteq \([x]\)\).

Hence, combining the two inclusions, we deduce that \([x] = [x']\).

If \( z \in [x] \cap [x'] \), then, by definition, \( z \sim x \) and \( z \sim x' \).

By the symmetry and transitivity of \( \sim \), \( x \sim x' \), in which case, by the above, \([x] = [x']\).

It follows that \([x] \cap [x'] = \emptyset \) whenever \( x \not\sim x' \)

(b) Since \( \sim \) is reflexive, \( x \in [x] \) for every \( x \in X \). Thus
\[
X = \bigcup_{x \in X} \{x\} \\
\subseteq \bigcup_{x \in X} [x] \\
\subseteq \bigcup_{x \in X} X \\
= X.
\]

(c) Take the natural surjection
\[ \eta: X \rightarrow X/\sim, \quad x \mapsto [x] \]

Take a function \( f: X \rightarrow Y \) with \( f(x) = f(x') \) whenever \( x \sim x' \).
In order for a function
\[ \tilde{f}: X/\sim \rightarrow Y, \quad [x] \mapsto f(x) \]
to satisfy
\[ f = \tilde{f} \circ \eta \quad \text{(\ast)} \]
we must have, for \([x] \in X/\sim\),
\[
\tilde{f}([x]) = \tilde{f}(\eta(x)) \\
= (\tilde{f} \circ \eta)(x) \\
= f(x)
\]
by the definition of \(\eta\)
in order to satisfy (\ast)

Hence, the only possible function \(\tilde{f}\) is
\[ \tilde{f}: X/\sim \rightarrow Y, \quad [x] \mapsto f(x) \]
It (only) remains to verify that \(\tilde{f}\) is, in fact, a function.
Both the domain and co-domain are sets.
It is immediate from the definition of \(\tilde{f}\) that it assigns to each \([x] \in X/\sim\) an element of \(Y\).
Thus, the only way \(\tilde{f}\) could fail to be a function is that it assigns to some element of \(X/\sim\) more than one element of \(Y\).
For this to happen, we would have to have \([x] = [x']\) with \(\tilde{f}([x]) \neq \tilde{f}([x'])\).
But \([x] = [x']\) if and only if \(x \sim x'\), in which case, by hypothesis, \(f(x) = f(x')\).
Thus \(\tilde{f}\) is a function.

Comment. The requirement that \(\tilde{f} \circ \eta = f\) forced the definition of \(\tilde{f}\), ensuring that there cannot be more than one function satisfying the requirement.
This illustrates one of the ways in which a theoretical approach can simplify solving a problem, for we see that there is only one possibility, for which the theoretical considerations give a concrete formula.
All that remained was to check whether this sole possibility is actually a function.
In turn, this was the case if and only if \(f(x) = f(x')\) whenever \(x \sim x'\), which is the one and only condition imposed on \(f\).