Overview

Topology is one of the newer branches of mathematics, its origins being at about the close of the nineteenth century and beginning of the twentieth. The word topology only came to be commonly used later. The first tracts on topology included Poincaré’s *Analysis situs* (“Analysis of Place”) and Hausdorff’s *Mengenlehre* (“Set Theory”).

The twentieth century saw a rapid expansion in the study of topology. Several branches of topology emerged as disciplines in their own right, including point-set topology, algebraic topology, differential topology and geometric topology. Hardly any branch of mathematics remains untouched by topology.

Topology has also spawned other disciplines as well, notably category theory and homological algebra.

In the late twentieth century, and the twentyfirst century, topology, category theory and homological algebra have become indispensable to theoretical physics as well.

What Is Topology and Why Study It?

One way to regard topology is as the attempt to understand continuity in its broadest possible context. This is the approach adopted here.

More specifically, we seek to answer a concrete question.

**What additional structure must sets support in order to be able to speak of the continuity of functions between them?**

Moreover, this structure should characterise continuity in the sense that two such structures on the space are “essentially the same” if and only if they lead to the same collection of continuous functions.

Our investigations begin with the notion of continuity familiar from calculus. A close examination of what is essential leads to the notion of a *metric space*, which is a set with a way of measuring distance between its elements.

Many definitions and arguments from calculus can be translated into the language of metric spaces without requiring further modification, showing that much of calculus is actually valid in this, more general, setting. This is essential for the study of differential geometry and hence the theory of relativity and non-Euclidean geometry.

We shall see that the notion of a metric space is not the most general setting for studying continuity. That setting provided by the notion of a *topological space*.

Not only do topological spaces topological spaces provide the right context for studying continuing, but they also have more satisfactory features than metric spaces. Many important facts about metric spaces depend not on the metric, but on the fact that metric spaces are topological spaces. Many important results are easier to prove using properties of all topological spaces than by restricting to metric spaces.

Important features of topological spaces studied in this course are *connectedness*, *compactness* and *separation properties*. Many frequently used facts depend only on these.
(i) The Intermediate Value Theorem of Calculus depends only on general properties of
connected sets and the fact that a subset of the set of real numbers is connected if
and only if it is an interval. We prove this in this course.

(ii) The Extreme Value Theorem of Calculus depends only on general properties of com-
 pact sets and separation properties, together with the fact that closed finite intervals
are compact subsets of the set of real numbers. We prove this in this course.

(iii) Even more remarkable is the Fundamental Theorem of Algebra, whose statement of
the theorem is purely algebraic: a non-constant polynomial in one indeterminate with
complex coefficients can always be written as a product of as many linear factors as
the degree of the polynomial. Yet all known proofs depend crucially on topology. One
elementary proof is presented in this course, involving compactness.

(iv) The Banach Fixed Point Theorem states that under some mild conditions, given a
function $f: X \rightarrow X$, the equation $f(z) = z$ has one and only one solution. This
fact is at the core of common arguments in the theory of differential equations, and is
often used to show that an equation, or system of equations, has a (unique) solution.
This theorem requires $X$ to be a metric space.

(v) A appendix presents a construction of the set of real numbers from the rational num-
 bers, using Dedekind’s method. Some of you will be meeting such a construction and
seen proofs of the properties of the set of real numbers. Once again, you will meet
proofs of properties which most calculus textbooks simply assume.

These examples should suffice to indicate how widely topology is used, either explicitly or
implicitly, in much of mathematics.

**Course Notes**

There is a complete set of notes for the course. They are self-contained and cover all the
material in the course. They cannot be simply read, you will need to work through them.
These notes should be the focus of your study.

**Intensive School**

There is an optional four-day intensive school, commencing on Wednesday 17th February,
2016, held in Room 206 of the Mathematics and Computing Building (C26) of UNE’s campus.
External students will receive the notes at this school. If you cannot attend, contact me to
arrange for the notes to be sent to you.

Internal students are encouraged to attend as well.