Tutorial Questions

The tutorial exercises range from revision and routine practice, through filling in details in the notes, to applications of the theory.

While the tutorial problems are not compulsory, and are not formally assessed, it is difficult to imagine mastering the material without attempting the tutorial questions, or similar exercises.
Tutorial 1

Question 1.
Find all real numbers \(x, y, z\) satisfying the system of equations
\[
\begin{align*}
x + 7y + 4z &= 21 \\
3x - 6y + 5z &= 2 \\
5x + y - 3z &= 14
\end{align*}
\]

Question 2.
For each of the following matrices, \(A\), below, find \(A^n\) \((n \in \mathbb{N} \setminus \{0\})\).

(i) \(A := \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}\).

(ii) \(A := \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix}\).

(iii) \(A := \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix}\).

Question 3.
Prove that if \(A := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}\), and if \(n \in \mathbb{N}\), then
\[
A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}
\]

Question 4.
Let \(f, g: \mathbb{R} \rightarrow \mathbb{R}\) both satisfy the differential equation
\[
\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 3y = 0 \quad (\ast)
\]
Show that for all real numbers \(\lambda, \mu\), the function
\[
h: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \lambda f(x) + \mu g(x)
\]
also satisfies \((\ast)\).
Tutorial 2

Question 1.
Show that \( \mathbb{R} \) is a vector space over \( \mathbb{Q} \) with respect to the usual addition and multiplication of real numbers.

Question 2.
Take \( X = \{a, b\} \), with \( a \neq b \).
Define binary operations, + and \( \cdot \), on \( X \) by

\[
\begin{array}{cc}
+ & a & b \\
\hline
a & a & b \\
b & b & a \\
\end{array}
\quad \text{and} \quad
\begin{array}{cc}
\cdot & a & b \\
\hline
a & a & a \\
b & a & b \\
\end{array}
\]

Prove that \( X \) is a field with respect to + and \( \cdot \). This field is usually written as \( \mathbb{F}_2 \).

Question 3.
Decide whether the following are vector spaces.

(a) Take \( F := \mathbb{C} \) and \( V := \mathbb{C} \).
Let \( \oplus \) to be the usual addition of complex numbers, and define \( \Box \) by
\[
\alpha \Box z := \alpha^2 z \quad (\alpha, z \in \mathbb{C}).
\]

(b) Let \( F \) be any field and take \( V := \mathbb{F}_2 \).
Let \( \oplus \) be the usual (component-wise) addition of ordered pairs, and define \( \Box \) by
\[
\alpha \Box (\beta, \gamma) := (\alpha \beta, 0) \quad (\alpha, \beta, \gamma \in F).
\]

(c) Take \( F := \mathbb{F}_2 = \{a, b\} \) with operations + and \( \cdot \) defined as in Question 2.
Let \( V := (\mathbb{F}_2)^2 = \{(a, a), (a, 1), (1, a), (1, 1)\} \).
Define \( \Box : \mathbb{F}_2 \times V \rightarrow V \) by
\[
\alpha \Box (\beta, \gamma) := \begin{cases} 
(\alpha \beta, \alpha \gamma) & \text{if } \gamma \neq 0 \\
(\alpha^2, 0) & \text{if } \gamma = 0 
\end{cases}
\]
Define \( \oplus : V \times V \rightarrow V \) by
\[
\begin{array}{l}
| & (a, a) & (a, b) & (b, a) & (b, b) \\
(a, a) & (a, a) & (a, b) & (b, a) & (b, b) \\
(a, b) & (a, b) & (a, a) & (b, b) & (b, a) \\
(b, a) & (b, a) & (b, b) & (a, a) & (a, b) \\
(b, b) & (b, b) & (a, b) & (a, a) & (a, a) \\
\end{array}
\]
(d) Take $F := \mathbb{C}$ and $V := \mathbb{C}$.

Let $\oplus$ be the usual addition of complex numbers, and define $\odot$ by
\[ \alpha \odot z := \Re(\alpha)z \quad (\alpha, z \in \mathbb{C}), \]
where $\Re(\alpha)$ denotes the real part of the complex number $\alpha$.

(e) Take $F := \mathbb{R}$ and $V := \mathbb{R}^+ = \{ r \in \mathbb{R} \mid r > 0 \}$.

Define $\oplus$ and $\odot$ by
\[ x \oplus y := xy \quad (x, y \in \mathbb{R}^+) \]
\[ \alpha \odot x := x^\alpha \quad (\alpha \in \mathbb{R}, x \in \mathbb{R}^+) \]

**Question 4.**

Let $V$ be a vector space over the field $F$.

Take $u, v, w \in V$ and $\alpha \in F$.

Prove each of the following statements.

(i) If $u + v = u + w$, then $v = w$.

(ii) The equation $u + x = v$ has a unique solution, $x$.

(iii) $-(-u) = u$.

(iv) $0v = 0_V$.

(v) $-(\alpha u) = (-\alpha)u = \alpha(-u)$.

(vi) $(-\alpha)(-u) = \alpha u$.

(vii) If $\alpha u = \alpha v$, then either $\alpha = 0$ or $u = v$. 
Tutorial 3

Unless otherwise specified, we regard $\mathbb{R}^n$ as a vector space over $\mathbb{R}$ with the vector space operations defined coordinate-wise.

Question 1.
Determine $T \circ S$ and $S \circ T$ for

$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (u, v) \mapsto (u + 2v, 2u + 5v)$

$S : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto (x + y, x)$

Question 2.
Find all linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ which maps the line with equation $u = v$ onto the line with equations $x = y = 0$.

Question 3.
Find, if possible, linear transformations $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfying the conditions specified.

(i) $T(1, 0, 0) = (1, 0, 0)$, $T(1, 1, 0) = (0, 1, 0)$, $T(1, 1, 1) = (0, 0, 1)$

(ii) $T(1, 2, 3) = (1, 0, 0)$, $T(3, 1, 2) = (0, 0, 1)$, $T(2, 3, 1) = (0, 1, 0)$

(iii) $T(1, 2, 1) = (1, 0, 0)$, $T(1, 2, 2) = (1, 1, 0)$, $T(0, 0, 1) = (0, 0, 0)$

(iv) $T(1, 0, 0) = (1, 2, 3)$, $T(0, 2, 2) = (6, 1, 0)$, $T(1, 0, 1) = (2, 0, 1)$, $T(5, 2, 5) = (14, 5, 9)$

Where there is no solution, explain why not.

Question 4.
Take a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfying

$T(1, 0, 0) = (1, 0, 0)$

$T(0, 1, 0) = (0, 0, 1)$

$T(0, 0, 1) = (1, 0, 1)$

Find all solutions $(x, y, z) \in \mathbb{R}^3$, of

(i) $T(x, y, z) = (6, 0, 7)$

(ii) $T(x, y, z) = (6, 1, 7)$. 
Tutorial 4

Question 1.
Let $X$ be a non-empty set.

The set of all real valued functions defined on $X$, 
$$F(X) = \{ f: X \rightarrow \mathbb{R} \mid f \text{ is a function } \}$$
forms a real vector space with respect to “point-wise” operations.

More precisely, given $f, g \in F(X)$ and $\lambda \in \mathbb{R}$, $f + g$ and $\lambda f$ are the functions defined by
$$f + g: X \rightarrow \mathbb{R}, \quad x \mapsto f(x) + g(x)$$
$$\lambda f: X \rightarrow \mathbb{R}, \quad x \mapsto \lambda f(x)$$

Decide which of the following subsets of $F(\mathbb{R})$ are vector subspaces.

(a) $\{ f \in F(\mathbb{R}) \mid f(x) \leq 0 \text{ for all } x \in \mathbb{R} \}$
(b) $\{ f \in F(\mathbb{R}) \mid f(7) = 0 \}$
(c) $\{ f \in F(\mathbb{R}) \mid f(1) = 2 \}$
(d) $\{ f \in F(\mathbb{R}) \mid \text{ there are } a, b \in \mathbb{R} \text{ with } f(x) = a + b \sin x \text{ for all } x \in \mathbb{R} \}$
(e) $D^n(\mathbb{R}) := \{ f \in F(\mathbb{R}) \mid f \text{ is } n \text{ times differentiable} \} \quad (n \in \{1, 2, \ldots\})$
(f) $C^n(\mathbb{R}) := \{ f \in F(\mathbb{R}) \mid f \text{ is } n \text{ times continuously differentiable} \} \quad (n \in \{1, 2, \ldots\})$

Question 2.

Let $\mathbb{R}[t]$ denote the set of all polynomials with real coefficients, so that
$$\mathbb{R}[t] := \{ a_0 + a_1 t + \cdots + a_m t^m \mid m \in \mathbb{N} \text{ and } a_j \in \mathbb{R} \text{ for } 0 \leq j \leq m \}$$
This forms a real vector space with respect to the usual addition of polynomials and multiplication of a polynomial by a fixed real number, so that for $p = a_0 + a_1 t + \cdots + a_m t^m, q = b_0 + b_1 t + \cdots + b_n t^n \in \mathbb{R}[t]$ with $m \leq n$ and $\lambda \in \mathbb{R}$,
$$\lambda p = \lambda a_0 + \lambda a_1 t + \cdots + \lambda a_m t^m$$
$$p + q = c_0 + c_1 t + \cdots + c_n t^n \quad \text{ with } c_j = \begin{cases} a_j + b_j & \text{for } j \leq m \\ b_j & \text{for } j > m \end{cases}$$

Show that for each $n \in \mathbb{N}$, the set of all real polynomials of degree at most $n$,
$$\mathcal{P}_n := \{ a_0 + a_1 t + \cdots + a_n t^n \mid a_0, \ldots, a_n \in \mathbb{R} \},$$
forms a vector subspace of $\mathbb{R}[t]$.

Question 3.

$\mathbf{M}(2; \mathbb{R})$, the set of all $2 \times 2$ matrices with real coefficients, is a real vector space with respect to the usual operations on real matrices.

Determine which of the following subsets of $\mathbf{M}(2; \mathbb{R})$ form vector subspaces.
(a) \( \mathbf{M}(2; \mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z} \right\} \)

(b) \( \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{M}(2; \mathbb{R}) \mid a + b + c + d = 0 \right\} \)

(c) \( \{ \mathbf{A} \in \mathbf{M}(2; \mathbb{R}) \mid \det(\mathbf{A}) = 0 \} \)

(d) \( \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \)
Tutorial 5

Question 1.
Determine whether \((2, 2, 2)\) and \((3, 1, 5)\) \(\in \mathbb{R}^3\) are linear combinations of \((0, -1, 1)\) and \((1, -1, 3)\).

Question 2.
Using the notation in Question 2 of Tutorial 4, decide which of the following sets of elements of \(P_2\) are linearly independent.

(a) \(\{4t^2 - t + 2, 2t^2 + 6t + 3, -4t^2 + 10t + 2\}\)
(b) \(\{4t^2 - t + 2, 2t^2 + 6t + 3, 6t^2 + 5t + 5\}\)
(c) \(\{t^2 + t + 23, 5t^2 - t + 2\}\)
(d) \(\{3t^2 + 3t + 1, t^2 + 6t + 3, 5t^2 + t + 2, -t^2 + 2t + 7\}\)

Question 3.
Recall that \(\mathcal{F}(\mathbb{R})\), the set of all real valued functions defined on \(\mathbb{R}\), is a real vector space for point-wise defined addition of functions and multiplication of a function by a real constant.

Decide whether \(\{f, g, h\}\) is a linearly independent set of elements of \(\mathcal{F}(\mathbb{R})\) when \(f, g\) and \(h\) are defined by

\[ \begin{align*}
(a) & \quad f(x) := \cos 2x, \quad g(x) := \sin x, \quad h(x) := 7 \\
(b) & \quad f(x) := \ln(x^2 + 1), \quad g(x) := \sin x, \quad h(x) := e^x
\end{align*} \]
Tutorial 6

Question 1.
This question investigates finding the matrix representation of a linear transformation $T: V \rightarrow W$. To do so, we specify bases $\{e_j\}$ for $V$ and $\{f_i\}$ for $W$.

(a) Take $V = W = \mathbb{R}^2$ and $T = id_{\mathbb{R}^2}$, so that $T(x, y) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$.

Find the matrix $A_T$ in each of the following cases.

(i) $e_1 := (1, 0)$, $e_2 := (0, 1)$ and $f_1 := (1, 0)$, $f_2 := (0, 1)$
(ii) $e_1 := (1, 0)$, $e_2 := (0, 1)$ and $f_1 := (0, 1)$, $f_2 := (1, 0)$
(iii) $e_1 := (1, 2)$, $e_2 := (3, 4)$ and $f_1 := (1, 0)$, $f_2 := (0, 1)$
(iv) $e_1 := (1, 0)$, $e_2 := (0, 1)$ and $f_1 := (1, 2)$, $f_2 := (3, 4)$
(v) $e_1 := (3, 4)$, $e_2 := (1, 2)$ and $f_1 := (1, 2)$, $f_2 := (3, 4)$

(b) Let $P_n$ be the set of all real polynomials in the indeterminate $t$ of degree at most $n$.

The polynomial $p \in \mathbb{R}[t]$ induces the function $f_p: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto p(x)$

This allows us to define the derivative of $p$, $D(p)$, to be the polynomial $q$ such that the functions $f_q$ and $f_p'$ agree. In other words, for all $x \in \mathbb{R}$,

$$f_q(x) = f_p'(x)$$

so that $f_{fof} p = a_0 + a_1 t + \cdots + a_n t^n$, we get $q = a_1 + 2a_2 t + \cdots + ma_m t^{m-1} + 0t^m$.

Prove that the function $D: P_3 \rightarrow P_2, \quad p \mapsto D(p)$
is a linear transformation and find its matrix with respect to each of the following bases.

(i) $e_1 := 1$, $e_2 := t$, $e_3 := t^2$, $e_4 := t^3$ and $f_1 := 1$, $f_2 := t$, $f_3 := t^2$
(ii) $e_1 := 1$, $e_2 := t$, $e_3 := t^2$, $e_4 := t^3$ and $f_1 := 6$, $f_2 := 6t$, $f_3 := 3t^2$
(iii) $e_1 := 1$, $e_2 := 1+t$, $e_3 := 1+t^2$, $e_4 := 1+t+t^2+t^3$ and $f_1 := 1$, $f_2 := 1+t$, $f_3 := 1+t+t^2$

Question 2.

Prove that a linear transformation between finitely generated vector spaces is an isomorphism if and only if every matrix representing it is invertible.
Tutorial 7

Question 1.
Consider the system of linear equations
\[
\begin{align*}
 a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
 \vdots & \quad \vdots \\
 a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]
Prove that there is a solution \((x_1, \ldots, x_n) \in \mathbb{F}^n\) if and only if \((b_1, \ldots, b_m) \in \mathbb{F}^m\) is an element of the vector subspace of \(\mathbb{F}^m\) generated by \(\{(a_{11}, \ldots, a_{m1}), \ldots, (a_{1n}, \ldots, a_{mn})\}\).
When is the solution unique?

Question 2.
Let \(T: V \rightarrow W\) be a linear transformation.
Prove that if the matrix of \(T\) with respect to some choice of bases is
\[
\begin{bmatrix}
 1 & 0 & 0 \\
 0 & 0 & 0
\end{bmatrix},
\]
then \(T\) is neither injective nor surjective.

Question 3.
Show that \(B := \{(1,2), (3,4)\}\) and \(B' := \{(2,1), (4,3)\}\) are bases for \(\mathbb{R}^2\).
Suppose that the matrix with respect to \(B\) of the linear transformation \(T: \mathbb{R}^2 \rightarrow \mathbb{R}^2\) is
\[
\begin{bmatrix}
 1 & 2 \\
 3 & 4
\end{bmatrix}
\]
What is the matrix of \(T\) with respect to \(B'\)?
Tutorial 8

Question 1.
Evaluate the determinant of each of the following matrices.

(i)
\[
\begin{bmatrix}
1 & 6 & 4 & 7 \\
4 & 5 & 0 & 8 \\
6 & 2 & 1 & 9 \\
7 & 3 & 5 & 6 \\
\end{bmatrix}
\]

(ii)
\[
\begin{bmatrix}
1 \\
2 \\
3 & 9 & 15 \\
\end{bmatrix}
\]

(iii)
\[
\begin{bmatrix}
a & a^2 & a^3 \\
b & b^2 & b^3 \\
c & c^2 & c^3 \\
\end{bmatrix}
\]

(iv)
\[
\begin{bmatrix}
a & b & c \\
b & c & a \\
c & a & b \\
\end{bmatrix}
\]

Question 2.
(a) Find the determinant and the trace of each of the following matrices.

(i)
\[
\begin{bmatrix}
4 & -3 \\
1 & 0 \\
\end{bmatrix}
\]

(ii)
\[
\begin{bmatrix}
4 & -4 \\
1 & 0 \\
\end{bmatrix}
\]

(iii)
\[
\begin{bmatrix}
4 & -5 \\
1 & 0 \\
\end{bmatrix}
\]
(b) Find the determinant of each of the following matrices.

(i) \[
\begin{bmatrix}
4 - \lambda & -3 \\
1 & -\lambda \\
\end{bmatrix}
\]

(ii) \[
\begin{bmatrix}
4 - \lambda & -4 \\
1 & -\lambda \\
\end{bmatrix}
\]

(iii) \[
\begin{bmatrix}
4 - \lambda & -5 \\
1 & -\lambda \\
\end{bmatrix}
\]

(c) Compare the corresponding results in parts (a) and (b).

**Question 3.**

Decide which of the following sets of vectors form a basis for \( \mathbb{R}^3 \).

(i) \{ (1, 2, 3), (6, 5, 4), (31, 20, 9) \}

(ii) \{ (1, 2, 3), (1, 4, 9), (1, 8, 27) \}

(iii) \{ (1, 2, 3), (2, 3, 1), (3, 1, 2) \}
Tutorial 9

Question 1.
Show that eigenvectors for different eigenvalues of an endomorphism are linearly independent.

Question 2.
Consider the linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, where

(a) $T(x, y, z) = (-2x + 4y + 4z, -2x + 4y + 2z, -x + y + 3z)$

(b) $T(x, y, z) = (-2x + 8y, -2x + 6y, -x + 2y + 2z)$

(c) $T(x, y, z) = (-3x + 9y + 2z, -3x + 7y + 2z, -x + 2y + 2z)$

For each of the above

(i) find the matrix of $T$ with respect to the standard basis for $\mathbb{R}^3$;

(ii) find the eigenvalues of $T$;

(iii) find the eigenvectors of $T$ for each eigenvalue;

(iv) find, if possible, a basis for $\mathbb{R}^3$ consisting of eigenvectors of $T$;

(v) find the matrix of $T$ with respect to this new basis for $\mathbb{R}^3$.

Question 3.
Let $C^\infty(\mathbb{R})$ be the set of all infinitely differentiable real valued functions of a real variable, so that

$$C^\infty(\mathbb{R}) = \{ f: \mathbb{R} \rightarrow \mathbb{R} \mid \frac{d^n f}{dx^n} \text{ is continuous for every } n \in \mathbb{N} \}$$

$C^\infty(\mathbb{R})$ is a real vector space with respect to point-wise addition of functions and point-wise multiplication of functions by real numbers.

Show that each of the following mappings, $T: C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$, is a linear transformation and find all eigenvalues of each $T$, as well as the corresponding eigenvectors.

(i) $T(f) := \frac{df}{dx}$

(ii) $T(f) := \frac{d^2 f}{dx^2}$

(iii) $T(f) := \frac{d^2 f}{dx^2} - 4 \frac{df}{dx}$
Tutorial 10

Question 1.

Show that in each of the cases below, $\beta : V \times V \rightarrow \mathbb{R}$ defines an inner product on the real vector space $V$.

(a) 

$V := \{ f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous} \}$

$\beta(f, g) := \int_0^1 f(t)g(t) \, dt$

(b) 

$V := \mathbb{R}_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

$\beta(\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}) := \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

(c) 

$V := \mathcal{P}_2,$

$\beta(p, q) := p(-1)q(-1) + p(0)q(0) + p(1)q(1)$

(d) 

$V := \mathbf{M}(m \times n; \mathbb{R})$, 

$\beta(\mathbf{A}, \mathbf{B}) := \text{tr}(\mathbf{A}' \mathbf{B})$

Question 2.

Let $\langle \langle , \rangle \rangle$ be an inner product on the real vector space $V$.

Prove that if $u, v \neq 0_V$ and $\langle \langle u, v \rangle \rangle = 0$ then $u$ and $v$ are linearly independent.

Question 3.

Let $\beta$ be a bilinear form on the finitely generated real vector space $V$.

Let $T : V \rightarrow V$ be a linear transformation.

Show that 

$\gamma : V \times V \rightarrow \mathbb{R}$, \quad \langle (x, y) \rangle \mapsto \beta(T(x), T(y))$

is also a bilinear form on $V$.

Choose a fixed basis for $V$.

Show that if $\mathbf{A}$ is the matrix of $\beta$, and $\mathbf{B}$ the matrix of $T$, then the matrix of $\gamma$ is $\mathbf{B}' \mathbf{A} \mathbf{B}$. 
Question 4.
Classify each of the following real quadratic forms according to its definiteness properties:

(a) \( q(x, y) := x^2 + 4xy + 5y^2 \)
(b) \( q(x, y, z) := 4x^2 - 4xy + 5y^2 - 2yz + 3z^2 - 4zx \)
(c) \( q(x, y, z) = 2x^2 + 3y^2 + 2z^2 + 6xy + 6yz + 4zx \)

Question 5.
Let \( \langle \cdot, \cdot \rangle \) be an inner product on the real vector space \( V \).
Let \( u \in V \) be a fixed non-zero vector in \( V \).
Let \( \ell \) be the line determined by \( u \), so that \( \ell = \{ \lambda u \mid \lambda \in \mathbb{R} \} \).
Show that if \( T: V \to V \) is reflection in \( \ell \), then
\[
T(x) = \frac{2\langle u, x \rangle}{\langle u, u \rangle} u - x.
\]
Tutorial 11

Question 1.
Find the matrix of the inner product 
\[ \langle \langle p, q \rangle \rangle : P_2 \times P_2 \to \mathbb{R}, \quad (p, q) \mapsto p(-1)q(-1) + p(0)q(0) + p(1)q(1) \]
with respect to the basis \( \{1, t, t^2\} \) of \( P_2 \).
Do the same for 
\[ \langle | \rangle : P_2 \times P_2 \to \mathbb{R}, \quad (p, q) \mapsto \int_{-1}^{1} p(x)q(x)dx. \]

Question 2.
The matrix \( B = [b_{ij}]_{n \times n} \in M(n; \mathbb{R}) \) is orthogonal if and only if \( B^t B = I_n \) and it is upper triangular if and only if \( b_{ij} = 0 \) whenever \( i > j \).
Prove that if \( A \) is an invertible real \( n \times n \) matrix, then there are an orthogonal matrix \( Q \) and an upper triangular matrix \( R \) such that 
\[ A = QR. \]
Find an orthogonal matrix, \( Q \), and an upper triangular matrix, \( R \), such that 
\[ QR = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}. \]

Question 3.
Take the real vector space \( V := \{ \varphi : [0, 2\pi] \to \mathbb{R} \mid \varphi \text{ is continuous} \} \).
Show that 
\[ \langle \langle \varphi, \psi \rangle \rangle := \frac{1}{\pi} \int_0^{2\pi} \varphi(x)\psi(x)dx \]
is an inner product on \( V \).
For \( n \in \mathbb{N} \setminus \{0\} \), define 
\[ \varphi_n(x) := \cos(nx), \quad \psi_n(x) := \sin(nx) \]
Show that \( \{ \varphi_n, \psi_n \mid n = 1, 2, \ldots \} \) is a family of orthonormal elements of \( (V, \langle \langle \cdot, \cdot \rangle \rangle) \).