Question 1.

Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues of the endomorphism $T: V \rightarrow V$.

Let $v_j$ be an eigenvector of $T$ for $\lambda_j$ ($j = 1, \ldots, m$).

We use mathematical induction to show that $v_1, \ldots, v_m$ are linearly independent.

$m = 1$. Since, by definition, $v_1 \neq 0_V$, it is linearly independent.

$m \geq 1$. Suppose that if $v_1, \ldots, v_m$ are eigenvectors of $T: V \rightarrow V$ for the distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, then $v_1, \ldots, v_m$ are linearly independent.

Let $v_1, \ldots, v_{m+1}$ be eigenvectors of $T$ for the distinct eigenvalues $\lambda_1, \ldots, \lambda_{m+1}$.

Suppose that for some $\alpha_1, \ldots, \alpha_{m+1} \in \mathbb{F}$,

$$\sum_{j=1}^{m+1} \alpha_j v_j = 0_V,$$  \hspace{1cm} (i)

Since $T$ is a linear transformation,

$$\sum_{j=1}^{m+1} \alpha_j T(v_j) = 0_V$$

Since $v_j$ is an eigenvector of $T$ for the eigenvalue $\lambda_j$,

$$\sum_{j=1}^{m+1} \alpha_j \lambda_j v_j = 0_V,$$ \hspace{1cm} (ii)

Subtracting $\lambda_{m+1}$ times (i) from (ii), we obtain

$$\sum_{j=1}^{m} \alpha_j (\lambda_j - \lambda_{m+1}) v_j = 0_V$$

As the vectors $v_1, \ldots, v_m$ are linearly independent, we have

$$\alpha_j (\lambda_j - \lambda_{m+1}) = 0$$

for $j = 1, \ldots, m$. Moreover, $\lambda_j - \lambda_{m+1} \neq 0$ for $j \neq m + 1$, whence

$$\alpha_j = 0$$

for $1 \leq j \leq m$. Substitution in (i), together with the fact that $v_{m+1} \neq 0_V$, then shows that

$$\alpha_{m+1} = 0$$
Question 2.

a. The matrix of $T : \mathbb{R}^3 \to \mathbb{R}^3$, $(x, y, z) \mapsto (-2x + 4y + 4z, -2x + 4y + 2z, -x + y + 3z)$ of $T$ with respect to the standard basis for $\mathbb{R}^3$ is

$$A = \begin{pmatrix} -2 & 4 & 4 \\ -2 & 4 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

We therefore apply elementary row operations to $A - \lambda I$.

$$\begin{pmatrix} -2 - \lambda & 4 & 4 \\ -2 & 4 - \lambda & 2 \\ -1 & 1 & 3 - \lambda \end{pmatrix} \quad \mapsto \quad R_1 - (\lambda + 2)R_3 \quad \begin{pmatrix} 0 & 2 - \lambda & \lambda^2 - \lambda - 2 \\ 0 & 2 - \lambda & 2\lambda - 4 \\ -1 & 1 & 3 - \lambda \end{pmatrix}$$

$$R_1 - R_2 \quad \mapsto \quad \begin{pmatrix} 0 & 0 & (\lambda - 1)(\lambda - 2) \\ 0 & 2 - \lambda & 2(\lambda - 2) \\ -1 & 2 & 3 - \lambda \end{pmatrix}$$

It follows that the eigenvalues of $A$ are $\lambda = 1, 2$ and that defining equations for the corresponding eigenvectors are

$$(\lambda - 1)(\lambda - 2)z = 0$$

$$(2 - \lambda)y + 2(\lambda - 2)z = 0$$

$$_{-x + y + (3 - \lambda)z} = 0$$

$\lambda = 1$: From the defining equations for eigenvectors of $A$,

$$y = 2z$$

$$x = y + 2z = 4z$$

whence the corresponding eigenvectors of $A$ are

$$r \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \quad (r \in \mathbb{R} \setminus \{0\})$$

$\lambda = 2$: From the defining equations for eigenvectors of $A$,

$$-x + y + z = 0$$

whence the corresponding eigenvectors of $A$ are

$$s \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (s, t \in \mathbb{R} \setminus \{0\})$$

Writing $V_\lambda$ for $\{(x, y, z) \in \mathbb{R}^3 \mid T(x, y, z) = \lambda(x, y, z)\}$, we see that

$V_1 = \{(4r, 2r, r) \mid r \in \mathbb{R}\}$
\[ V_2 = \{(s + t, t, s) \mid s, t \in \mathbb{R}\} \]

Then \{\(4, 2, 1\), \((1, 0, 1)\), \((1, 1, 0)\)\}, a set of eigenvectors of \(T\), is a basis for \(\mathbb{R}^3\) and the matrix of \(T\) with respect to this basis is
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

b. The matrix of
\[ T: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad (x, y, z) \mapsto (-2x + 8y, -2x + 6y, -x + 2y + 2z) \]
with respect to the standard basis for \(\mathbb{R}^3\) is
\[
\mathbf{A} =
\begin{bmatrix}
-2 & 8 & 0 \\
-2 & 6 & 0 \\
-1 & 2 & 2
\end{bmatrix}
\]

We apply elementary row operations to \(\mathbf{A} - \lambda \mathbf{1}_3\).
\[
\begin{bmatrix}
-2 - \lambda & 8 & 0 \\
-2 & 6 - \lambda & 0 \\
-1 & 2 & 2 - \lambda
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2R_2 - (\lambda + 2)R_2 & 0 & (\lambda - 2)^2 & 0 \\
-2 & 6 - \lambda & 0 \\
2R_3 - R_2 & 0 & \lambda - 2 & 2 - \lambda
\end{bmatrix}
\]

Plainly, the only eigenvalue of \(\mathbf{A}\) is \(\lambda = 2\).

The defining equation for the eigenvectors of \(\mathbf{A}\) for its only eigenvalue is
\[ x = 2y \]
so that the eigenvectors are all of the form
\[
\begin{bmatrix}
2r \\
r \\
s
\end{bmatrix} =
\begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix} +
\begin{bmatrix}
0 \\
r \\
s
\end{bmatrix} \quad (r, s \in \mathbb{R}, \ r^2 + s^2 \neq 0)
\]

Using the notation from above, the only eigenspace for \(T\) is
\[ V_2 = \{(2r, r, s) \mid r, s \in \mathbb{R}\} \]

Since no three eigenvectors of \(T\) can be linearly dependent, there is no basis for \(\mathbb{R}^3\) consisting of eigenvectors of \(T\).
Hence, there is no basis with respect to which the matrix of \(T\) is in diagonal form.

c. The matrix of
\[ T: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad (x, y, z) \mapsto (-3x + 9y + 2z, -3x + 7y + 2z, -x + 2y + 2z) \]
with respect to the standard basis for \(\mathbb{R}^3\) is
\[
\mathbf{A} =
\begin{bmatrix}
-3 & 9 & 2 \\
-3 & 7 & 2 \\
-1 & 2 & 2
\end{bmatrix}
\]

We apply elementary row operations to \(\mathbf{A} - \lambda \mathbf{1}_3\).
Because of the first two columns, this matrix has rank at least two, whence it is singular if and only if the second row is the zero row, so that the only eigenvalue of $A$ is $\lambda = 2$ and the defining equations for the corresponding eigenvectors are

\[ y = 2z \]
\[ x = 2y \]

Hence the eigenvectors of $A$ are

\[ r \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \quad (r \in \mathbb{R} \setminus \{0\}). \]

Using the notation from above, the only eigenspace for $T$ is

\[ V_2 = \{(4r, 2r, r) \mid r \in \mathbb{R}\} \]

Since any two eigenvectors are linearly dependent, there is no basis for $\mathbb{R}^3$ consisting of eigenvectors of $T$, and hence no basis with respect to which the matrix of $T$ is in diagonal form.

**Question 3.**

(i) Consider

\[ D: C^\infty(\mathbb{R}) \to C^\infty, \quad f \mapsto \frac{df}{dx} \]

Then $\lambda \in \mathbb{R}$ is an eigenvalue of $D$ if and only if the ordinary differential equation

\[ \frac{df}{dx} = \lambda f \quad (a) \]

has a non-trivial solution.

Given $\lambda \in \mathbb{R}$,

\[ f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto e^{\lambda x} \]

is a non-trivial solution of (a).

Let $h: \mathbb{R} \to \mathbb{R}$ be an arbitrary solution of (a), and define

\[ \varphi: \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{h(x)}{e^{\lambda x}} \]
Since $e^{\lambda x} \neq 0$ for all $\lambda, x \in \mathbb{R}$, $\varphi$ is everywhere differentiable, and
\[
\varphi'(x) = \frac{h'(x)e^{\lambda x} - h(x)\lambda e^{\lambda x}}{e^{2\lambda x}} = 0 \quad \text{as } h'(x) = \lambda h(x)
\]
Thus, by the Mean Value Theorem of Calculus, there is an $A \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $\varphi(x) = A$.

Hence $h(x) = Ae^{\lambda x}$ for all $Rx \in \mathbb{R}$.

Thus each $\lambda \in \mathbb{R}$ is an eigenvalue of $T$, and $h$ is an eigenvector for $\lambda$ if and only if $h(x) = Ae^{\lambda x}$ for some $A \in \mathbb{R}$.

(ii) Consider
\[
T: \mathcal{C}^\infty(\mathbb{R}) \rightarrow \mathcal{C}^\infty, \quad f \mapsto \frac{d^2 f}{dx^2}
\]
Then $\lambda \in \mathbb{R}$ is an eigenvalue if and only if the ordinary differential equation
\[
\frac{d^2 f}{dx^2} = \lambda f
\]
has a non-trivial solution.

We consider three cases separately, $\lambda = 0$, $\lambda > 0$ and $\lambda < 0$.

$\lambda = 0$: It follows from two successive applications of the Fundamental Theorem of Calculus that there are $A, B \in \mathbb{R}$ such that for all $x \in \mathbb{R}$
\[
f(x) = Ax + B
\]
$\lambda > 0$: Then $\lambda = k^2 t$ for some $k \in \mathbb{R}^+$.

Let $f$ be a solution of (b) and define
\[
g: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto f'(x) - kf(x) \tag{*}
\]
Then
\[
\frac{dg}{dx} = \frac{d^2 f}{dx^2} - k \frac{df}{dx}
\]
\[
= k^2 f - k \frac{df}{dx}
\]
\[
= -kg
\]
Hence, by (i), there is a $C \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,
\[
g(x) = Ce^{-kx}
\]
It follows from (*) that
\[
\frac{df}{dx} - kf = C e^{-kx}
\]
or, equivalently,
\[
\frac{df}{dx} e^{-kx} - k e^{-kx} f = C e^{-2kx},
\]
that is to say,
\[
\frac{d}{dx} (e^{-kx} f(x)) = C e^{-2kx}
\]
By the Fundamental Theorem of Calculus, there is a $B \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,
\[
e^{-kx} f(x) = \frac{C}{2k} e^{-2kx} + B
\]
Putting $SA := -\frac{A_1}{2k}$, we see that for all $x \in \mathbb{R}$,
\[ f(x) = Ae^{-kx} + Be^{kx} \]

$\lambda < 0$: Then $\lambda = -k^2t$ for some $k \in \mathbb{R}^+$.

Clearly $\cos_k : \mathbb{R} \rightarrow \mathbb{R}, \; x \mapsto \cos(kx)$ is one solution.

Let $f$ be a solution of (b) and put $\phi := \frac{f}{\cos_k}$.

Notice that this introduces some “singularities”: if $x = (2n + 1)\frac{\pi}{2k}$ for some integer $n$, then $\phi$ is not defined: we have only a function $\phi : \mathbb{R} \setminus \{(2n + 1)\frac{\pi}{2k} \mid n \in \mathbb{Z}\} \rightarrow \mathbb{R}$, instead of $\phi : \mathbb{R} \rightarrow \mathbb{R}$.

Our strategy is to solve the equation on $\mathbb{R} \setminus \{(2n + 1)\frac{\pi}{2k} \mid n \in \mathbb{Z}\}$. It follows from general “topological” considerations that a solution on this set has at most one extension to a solution on $\mathbb{R}$, and we shall easily see that our solutions do, indeed, have such an extension.]

More rigorously, we define
\[ \phi : \mathbb{R} \setminus \{(2n+1)\frac{\pi}{2k} \mid k \in \mathbb{Z}\} \rightarrow \mathbb{R}, \; x \mapsto \frac{f(x)}{\cos kx} \]

Then, since for all $x \in \text{dom}(\phi)$, $f(x) = \phi(x) \cos(kx)$,
\[ f'(x) = \phi'(x) \cos(kx) - k\phi(x) \sin(kx) \]
\[ f''(x) = \phi''(x) \cos(kx) - 2k\phi'(x) \sin(kx) - k^2\phi(x) \cos(kx) \]

As $f'' = -k^2f$,
\[ \phi''(x) \cos(kx) - 2k\phi'(x) \sin(kx) = 0 \]
or, equivalently
\[ \phi''(x) \cos^2(kx) - 2k\phi'(x) \sin(kx) \cos(kx) = 0 \]

This, in turn, is equivalent to
\[ \frac{d}{dx} \left( \cos^2(kx) \phi'(x) \right) = 0 \]

By the Fundamental Theorem of Calculus, for each $n \in \mathbb{Z}$, there is a $C_n \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ with
\[ \frac{(2n-1)\pi}{2k} < x < \frac{(2n+1)\pi}{2k} \]
we have
\[ \cos^2(kx) \phi'(kx) = C_n, \]
or, equivalently,
\[ \phi'(x) = C_n \sec^2(kx) = \frac{C_n}{k} \frac{d}{dx} (\tan(kx)) \]

By the Fundamental Theorem of Calculus, there is is, for each $n \in \mathbb{Z}$ a $B_n \in \mathbb{R}$ such that for $x$ with $\frac{(2n-1)\pi}{2k} < x < \frac{(2n+1)\pi}{2k}$,
\[ \phi(x) = A_n \tan(kx) + B_n \]
where $A_n := \frac{C_n}{k}$. In other words, given $x$ with $\frac{(2n-1)\pi}{2k} < x < \frac{(2n+1)\pi}{2k}$,
\[ f(x) = \cos(kx) \phi(x) = A_n \sin(kx) + B_n \cos(kx) \]

In order to extend this to a function defined on all of $\mathbb{R}$, we must choose the constants $A_n$ and $B_n$ in such a manner that both $f$ and $f'$ are continuous at $\frac{(2n+1)\pi}{2k}$ for every $n \in \mathbb{Z}$.
Direct substitution shows that the is the case if and only if for each \( n \in \mathbb{Z} \),
\[
A_n = A_{n+1} \quad \text{and} \quad B_n = B_{n+1}
\]
Thus there are \( A, B \in \mathbb{R} \) such that for all \( x \in \mathbb{R} \)
\[
f(x) = \cos(kx)\varphi(x) = A_n \sin(kx) + B_n \cos(kx)
\]
which is clearly well defined for all \( x \in \mathbb{R} \) and satisfies (b) on all of \( \mathbb{R} \).

We summarise the above.

**Theorem.** Every real number is an eigenvalue of the endomorphism

\[
T: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad f \mapsto f''
\]

and a basis for the eigenspace of \( \lambda \) is
\[
\begin{align*}
\{ x, 1 \} & \quad \text{if } \lambda = 0 \\
\{ e^{kx}, e^{-kx} \} & \quad \text{if } \lambda = k^2 \text{ for some } k > 0 \\
\{ \cos(kx), \sin(kx) \} & \quad \text{if } \lambda = -k^2 \text{ for some } k > 0
\end{align*}
\]

Here we have been loose, identifying the function \( f: \mathbb{R} \to \mathbb{R}, \ x \mapsto f(x) \) with the expression \( f(x) \).

(iii) Consider the endomorphism

\[
T: C^\infty(\mathbb{R}) \to C^\infty, \quad f \mapsto f'' - 4f'
\]

\( \lambda \in \mathbb{R} \) is an eigenvalue if and only if there is a non-trivial solution \( y \) to the ordinary differential equation

\[
f'' - 4f' = \lambda f
\]
or, equivalently,

\[
\frac{d^2f}{dx^2} - 4 \frac{df}{dx} + 4f = (\lambda + 4)f \tag{c}
\]

Plainly, (c) is equivalent to

\[
e^{-2x} f''(x) - 4e^{-2x} f'(x) + 4e^{-2x} f(x) = (\lambda + 4)e^{-2x} f(x),
\]

which, in turn, is equivalent to

\[
\frac{d^2}{dx^2} \left( e^{-2x} f(x) \right) = \mu e^{-2x} f(x), \tag{d}
\]

where \( \mu := \lambda + 4 \).

Put \( g(x) := e^{-2x} f(x) \). Then (d) becomes

\[
g'' = \mu g,
\]

so that by (iii)

\[
\begin{align*}
g(x) &= A \cos(kx) + B \sin(kx) \quad & \text{if } \mu = -k^2 \text{ for some } k > 0 \\
g(x) &= Ax + B \quad & \text{if } \mu = 0 \\
g(x) &= Ae^{kx} + Be^{-kx} \quad & \text{if } \mu = k^2 \text{ for some } k > 0
\end{align*}
\]

But \( \mu = \lambda + 4 \) and \( f(x) = e^{2x} g(x) \), so

\[
\begin{align*}
f(x) &= e^{2x} (A \cos(kx) + B \sin(kx)) \quad & \text{if } \lambda = 4 - k^2 \text{ for some } k > 0 \\
f(x) &= e^{-2x} (Ax + B) \quad & \text{if } \lambda = 4 \\
f(x) &= Ae^{(2+k)x} + Be^{(2-k)x} \quad & \text{if } \lambda = 4 + k^2 \text{ for some } k > 0
\end{align*}
\]
Comments. The last question provides an application of linear algebra to analysis.

It illustrates that eigenvalues and eigenvectors are important even when the vector spaces in question are not finitely generated.

It illustrates that it is not always possible to work with matrices and determinants.

It illustrates that while the general principles of linear algebra are quite straightforward and broadly applicable, working in specific vector spaces may require delicate arguments and techniques special to the specific vector space(s) in question.