Question 1.

\[ a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \]
\[ \vdots \quad \vdots \quad \vdots \quad \vdots \]
\[ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \]

if and only if

\[
\begin{bmatrix}
  a_{11} \\
  \vdots \\
  a_{m1}
\end{bmatrix}
+ \begin{bmatrix}
  a_{12} \\
  \vdots \\
  a_{m2}
\end{bmatrix}
+ \cdots +
\begin{bmatrix}
  a_{1n} \\
  \vdots \\
  a_{mn}
\end{bmatrix}
= \begin{bmatrix}
  b_1 \\
  \vdots \\
  b_m
\end{bmatrix}
\]

if and only if

\[ x_1(a_{11}, \ldots, a_{m1}) + x_2(a_{12}, \ldots, a_{m2}) + \cdots + x_n(a_{1n}, \ldots, a_{mn}) = (b_1, \ldots, b_m) \]

if and only if \((b_1, \ldots, b_m)\) is a linear combination of \((a_{11}, \ldots, a_{m1}), \ldots, (a_{1n}, \ldots, a_{mn})\)

if and only if \((b_1, \ldots, b_m)\) is in the sub-space of \(F^m\) generated by

\[ \{(a_{11}, \ldots, a_{m1}), \ldots, (a_{1n}, \ldots, a_{mn})\} \]

\((x_1, \ldots, x_n), (y_1, \ldots, y_n) \in F^n\) both solve (*) if and only if for all \(1 \leq i \leq m\)

\[
\sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n} a_{ij}y_j = b_i
\]

or, equivalently,

\[
\sum_{j=1}^{n} a_{ij}z_j = 0 \quad (**)
\]

where \(z_j := x_j - y_j\).

Thus, (*) has at most one solution if and only if (**) has at most one solution, which must then be the trivial solution, \(x_j = 0\) for every \(j\).

Thus, the solution of (**) is unique if and only if

\[ (a_{11}, \ldots, a_{m1}), (a_{12}, \ldots, a_{m2}), \ldots, (a_{1n}, \ldots, a_{mn}) \]

are linearly independent, that is, if and only if they form a basis for the subspace of \(F^m\) they generate.
Question 2.

Let the matrix of the linear transformation \( T: V \rightarrow W \) with respect to the basis \( \{e_1, e_2, e_3\} \) for \( V \) and \( \{f_1, f_2\} \) for \( W \) be
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Then \( T(xe_1 + ye_2 + ze_3) = xf_1. \)

Thus,

(i) \( T \) is not injective, because \( e_2, e_3 \in \ker(T) \)

(ii) \( T \) is not surjective, because \( f_2 \not\in \text{im}(T). \)

Question 3.

Showing that \( \{(1, 2), (3, 4)\} \) is a basis for \( \mathbb{R}^2 \), is equivalent to showing that for each \((a, b) \in \mathbb{R}^2\) there are unique \(x, y \in \mathbb{R}\) with
\[
x(1, 2) + y(3, 4) = (a, b)
\]
which is equivalent to showing that given \(a, b \in \mathbb{R}\), the system of equations
\[
\begin{align*}
x + 3y &= a \\
2x + 4y &= b
\end{align*}
\]
has a unique solution.

We apply “elementary row operations” to this pair of equations to obtain equivalent pairs of equations.

Subtracting twice (i) from (ii), we obtain
\[
\begin{align*}
x + 3y &= a \\
-2y &= b - 2a
\end{align*}
\]
Adding three halves of (iv) to (iii), and dividing (iv) by -2, we obtain
\[
\begin{align*}
x &= -2a + \frac{3}{2}b \\
y &= a - \frac{1}{2}b
\end{align*}
\]
Since \(x\) and \(y\) are uniquely determined, \( \{(1, 2), (3, 4)\} \) is a basis for \( \mathbb{R}^2 \).

Similarly, \( \{(2, 1), (4, 3)\} \) is a basis for \( \mathbb{R}^2 \) if and only if for each \((a, b) \in \mathbb{R}^2\) there are unique \(x, y \in \mathbb{R}\) with
\[
x(2, 1) + y(4, 3) = (a, b)
\]
that is, given \(a, b \in \mathbb{R}\), there are unique \(x, y \in \mathbb{R}\) with
\[
\begin{align*}
2x + 4y &= a \\
x + 3y &= b
\end{align*}
\]
But this is just Equations (i) and (ii) with \(a\) and \(b\) interchanged, and so has the unique solution
\[
\begin{align*}
x &= \frac{3}{2}a - 2b \\
y &= -\frac{1}{2}a + b
\end{align*}
\]
Thus, \{(2,1), (4,3)\} is also a basis for \(\mathbb{R}^2\).

By the calculations above, the co-ordinates with respect to \(B\) of \(u(2,1)+v(4,3) = (2u+4v, u+3v)\) are \(x, y\), where

\[
x = -2(2u + 4v) + \frac{3}{2}(u + 3v) = \frac{-5}{2}u - \frac{7}{2}v
\]

\[
y = (2u + 4v) - \frac{1}{2}(u + 3v) = \frac{3}{2}u + \frac{5}{2}v
\]

so that

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix}
  -5 & -7 \\
  3 & 5
\end{bmatrix}
\begin{bmatrix}
  u \\
  v
\end{bmatrix}.
\]

Since the inverse of \(\frac{1}{2}
\begin{bmatrix}
  -5 & -7 \\
  3 & 5
\end{bmatrix}\) is \(\frac{1}{2}
\begin{bmatrix}
  -5 & -7 \\
  3 & 5
\end{bmatrix}\), the matrix of \(T\) with respect to \(B'\) is

\[
\frac{1}{4}
\begin{bmatrix}
  -5 & -7 \\
  3 & 5
\end{bmatrix}
\begin{bmatrix}
  1 & 2 \\
  3 & 4
\end{bmatrix}
\begin{bmatrix}
  -5 & -7 \\
  3 & 5
\end{bmatrix} =
\begin{bmatrix}
  4 & -2 \\
  -3 & 1
\end{bmatrix}
\]