Polynomials and Polynomial Functions

In order to keep the discussion close to the familiar, we restrict attention to \( \mathbb{R} \), the field of real numbers. The general results, but not the specific examples, hold for all fields.

Polynomials

**Definition 1.** A polynomial in the indeterminate \( t \) with real coefficients is an expression of the form

\[
c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n
\]

where \( n \) is a natural number, each \( c_j \) is a real number, and \( c_n \neq 0 \), unless \( n = 0 \).

The \( c_j \)s are the coefficients of the polynomial.

When \( c_n \neq 0 \), the polynomial is said to have degree \( n \).

The polynomials \( c_0 + c_1 t + \cdots + c_n t^n \) and \( b_0 + b_1 t + \cdots + b_m t^m \) are equal if and only if \( m = n \) and \( b_j = c_j \) for every \( j \).

The set of all polynomials in the indeterminate \( t \) with real coefficients is

\[
\mathbb{R}[t] = \{c_0 + c_1 t + \cdots + c_n t^n \mid n \in \mathbb{N}, c_j \in \mathbb{R} \text{ for } 0 \leq j \leq n \text{ and } c_n \neq 0 \text{ if } n \neq 0\}
\]

**Example 2.**

\[
t^2 + 2t + 3
\]

is the polynomial \( c_0 + c_1 t + \cdots + c_n t^n \) with \( n = 2 \), \( c_0 = 3 \), \( c_1 = 2 \) and \( c_2 = 1 \).

**Convention 3.** It is common to write \( c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n \) as

\[
\sum_{j=0}^{n} c_j t^j
\]

treating \( c_0 \) as \( c_0 t^0 \). This useful convention needs to be treated with care until the reader is comfortable with it.

**Remark 4.** It is essential to note that in the polynomial

\[
c_0 + c + 1t + \cdots + c_n t^n
\]

\( t \) is not a real number and that the + does not represent addition of real numbers.

When \( n = 0 \), a polynomial comprises the single coefficient \( c_0 \), which is just a real number. Such a polynomial is sometimes referred to as a constant polynomial.

Since there is precisely one constant polynomial for each real number, we can think of the real numbers as just special polynomials — the constant polynomials.
This makes the set of all polynomials with real coefficients an extension of the set of real numbers: think of \( \mathbb{R}[t] \) as an extension of \( \mathbb{R} \) obtained by adjoining \( t \). This point of view is central to number theory and algebra.

We can define several operations on polynomials: “addition” and “multiplication of polynomials” as well as the “multiplication of a polynomial by a real number”. The reader is certainly familiar with these. While these are important and extremely useful, we do not discuss them now as they distract from our main topic.

**Polynomial Functions**

**Definition 5.** Given \( X \subseteq \mathbb{R} \), the function

\[
f : X \rightarrow \mathbb{R}, \quad x \mapsto f(x)
\]

is a polynomial function if and only if there is a polynomial \( c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n \in \mathbb{R}[t] \) such that for all \( x \in X \)

\[
f(x) = c_0 + c_1 x + c_2 t^2 + \cdots + c_n x^n
\]

**Remark 6.** In the expression \( c_0 + c_1 x + c_2 t^2 + \cdots + c_n x^n \), the \( x \) is a real number and the + does denote addition of real numbers.

**Example 7.** The function

\[
f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto x^2 + 2x + 3
\]

is, plainly, a polynomial function.

**Example 8.**

\[
f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \cos^2 x + \sin^2 x
\]

is a polynomial function, even though

\[
\cos^2 t + \sin^2 t
\]

is not a polynomial in the indeterminate \( t \) with real coefficients.

This a consequence of Pythagoras’ Theorem, since for every real number \( x \)

\[
\cos^2 x + \sin^2 x = 1
\]

Hence the functions

\[
f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \cos^2 x + \sin^2 x
\]

\[
g : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto 1
\]

are one and the same function, even thought they are defined using radically different expressions. Of course, the second expression plainly use the polynomial of degree 0, \( c_0 \) where \( c_0 = 1 \).
Example 8 illustrates the difference between a polynomial and a polynomial function: a function can be a polynomial function even if it is not defined using a polynomial.

Example 8 also illustrates that it can require a major mathematical achievement to recognise that a given function is, in fact, a polynomial function.

It can happen in applications that a function is defined through a more or less complex procedure, which masks its “true” nature.

**Example 9.** Consider the function

$$g : \mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \int_{4}^{x} \frac{du}{u}$$

and define

$$f : \mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \frac{1}{g'(x)}$$

where $g'(x)$ is the derivative of $g$ at $x$.

While it is not immediate that this $f$ is also a polynomial function, follows from the Fundamental Theorem of Calculus that $g$ is a differentiable function, with

$$g'(x) = \frac{1}{x}$$

for every $x \in \mathbb{R}^+$. Since $\mathbb{R}$ is a field, $f(x) = x$ for every $x \in \mathbb{R}^+$, showing that our function $f$ is a polynomial function.