THE RELATION BETWEEN MATHEMATICS AND PHYSICS

PAUL ADRIEN MAURICE DIRAC

The physicist, in his study of natural phenomena, has two methods of making progress:

(1) the method of experiment and observation, and
(2) the method of mathematical reasoning.

The former is just the collection of selected data; the latter enables one to infer results about experiments that have not been performed. There is no logical reason why the second method should be possible at all, but one has found in practice that it does work and meets with reasonable success. This must be ascribed to some mathematical quality in Nature, a quality which the casual observer of Nature would not suspect, but which nevertheless plays an important role in Nature's scheme.

One might describe the mathematical quality in Nature by saying that the universe is so constituted that mathematics is a useful took in its description. However, recent advances in physical science show that this statement of the case is too trivial. The connection between mathematics and the description of the universe goes far deeper than this, and one can get an appreciation of it only from a thorough examination of the various facts that make it up. The main aim of my talk to you will be to give you such an appreciation. I propose to deal with how the physicist's views on this subject have been gradually modified by the succession of recent developments in physics, and then I would like to make a little speculation about the future.

Let us take as our starting-point that scheme of physical science which was generally accepted in the last century — the mechanistic scheme. This considers the whole universe to be a dynamical system (of course an extremely complicated dynamical system), subject to laws of motion which are essentially of the Newtonian type. The role of mathematics in this scheme is to represent the laws of motion by equations, and to obtain solutions of the equations referring to observed conditions.

The dominating idea in this application of mathematics to physics is that the equations representing the laws of motion should be of a simple form. The whole success of the scheme is due to the fact that equations of simple form do seem to work. The physicist is thus provided with a principle of simplicity, which he can use as an instrument of research. If he obtains, from some rough experiments, data which fit in roughly with certain simple equations, he infers that if he performed the experiments more accurately he would obtain data fitting in more accurately with the equations. The method is much restricted, however, since the principle of simplicity applies only to fundamental laws of motion, not to natural phenomena in general. For example, rough experiments about the relation between the pressure and volume of a gas at a fixed temperature give results fitting in with a law of inverse proportionality, but it would be wrong to infer that more accurate experiments would confirm this law with greater accuracy, as one is here dealing

---

Date: Lecture delivered on the presentation of the James Scott Prize, 6$^{\text{th}}$ February, 1939.

Published in *Proceedings of the Royal Society (Edinburgh)* Vol.59, 1038–39, Part II pp.112–129.

with a phenomenon which is not connected in any very direct way with the fundamental laws of motion.

The discovery of the theory of relativity made it necessary to modify the principle of simplicity. Presumably one of the fundamental laws of motion is the law of gravitation which, according to Newton, is represented by a very simple equation, but, according to Einstein, needs the development of an elaborate technique before its equation can even be written down. It is true that, from the standpoint of higher mathematics, one can give reasons in favour of the view that Einstein's law of gravitation is actually simpler than Newton's, but this involves assigning a rather subtle meaning to simplicity, which largely spoils the practical value of the principle of simplicity as an instrument of research into the foundations of physics.

What makes the theory of relativity so acceptable to physicists in spite of its going against the principle of simplicity is its great mathematical beauty. This is a quality which cannot be defined, any more than beauty in art can be defined, but which people who study mathematics usually have no difficulty in appreciating. The theory of relativity introduced mathematical beauty to an unprecedented extent into the description of Nature. The restricted theory changed our ideas of space and time in a way that may be summarised by stating that the group of transformations to which the space-time continuum is subject must be changed from the Galilean group to the Lorentz group. The latter group is a much more beautiful thing than the former — in fact, the former would be called mathematically a degenerate special case of the latter. The general theory of relativity involved another step of a rather similar character, although the increase in beauty this time is usually considered to be not quite so great as with the restricted theory, which results in the general theory being not quite so firmly believed in as the restricted theory.

We now see that we have to change the principle of simplicity into a principle of mathematical beauty. The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. He should still take simplicity into consideration in a subordinate way to beauty. (For example Einstein, in choosing a law of gravitation, took the simplest one compatible with his space-time continuum, and was successful.). It often happens that the requirements of simplicity and of beauty are the same, but where they clash the latter must take precedence.

Let us pass on to the second revolution in physical thought of the present century - the quantum theory. This is a theory of atomic phenomena based on a mechanics of an essentially different type from Newton's. The difference may be expressed concisely, but in a rather abstract way, by saying that dynamical variables in quantum mechanics are subject to an algebra in which the commutative axiom of multiplication does not hold. Apart from this, there is an extremely close formal analogy between quantum mechanics and the old mechanics. In fact, it is remarkable how adaptable the old mechanics is to the generalization of non-commutative algebra. All the elegant features of the old mechanics can be carried over to the new mechanics, where they reappear with an enhanced beauty.

Quantum mechanics requires the introduction into physical theory of a vast new domain of pure mathematics — the whole domain connected with non-commutative multiplication. This, coming on top of the introduction of new geometries by the theory of relativity, indicates a trend which we may expect to continue. We may expect that in the future further big domains of pure mathematics will have to be brought in to deal with the advances in fundamental physics.

Pure mathematics and physics are becoming ever more closely connected, though their methods remain different. One may describe the situation by saying that the mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen. It is difficult to predict what the result of all this will be. Possibly, the two subjects will ultimately unify, every branch of pure mathematics then having its physical application, its importance in physics being proportional to its interest in mathematics. At present we are, of course, very far from this stage, even with regard to some of the most elementary questions. For example, only four-dimensional space is of importance in physics, while spaces with other numbers of dimensions are of about equal interest in mathematics.

It may well be, however, that this discrepancy is due to the incompleteness of present-day knowledge, and that future developments will show four-dimensional space to be of far greater mathematical interest than all the others.

The trend of mathematics and physics towards unification provides the physicist with a powerful new method of research into the foundations of his subject, a method which has not yet been applied successfully, but which I feel confident will prove its value in the future. The method is to begin by choosing that branch of mathematics which one thinks will form the basis of the new theory. One should be influenced very much in this choice by considerations of mathematical beauty. It would probably be a good thing also to give a preference to those branches of mathematics that have an interesting group of transformations underlying them, since transformations play an important role in modern physical theory, both relativity and quantum theory seeming to show that transformations are of more fundamental importance than equations. Having decided on the branch of mathematics, one should proceed to develop it along suitable lines, at the same time looking for that way in which it appears to lend itself naturally to physical interpretation.

This method was used by Jordan in an attempt to get an improved quantum theory on the basis of an algebra with non-associative multiplication. The attempt was not successful, as one would rather expect, if one considers that non-associative algebra is not a specially beautiful branch of mathematics, and is not connected with an interesting transformation theory. I would suggest, as a more hopeful-looking idea for getting an improved quantum theory, that one take as basis the theory of functions of a complex variable. This branch of mathematics is of exceptional beauty, and further, the group of transformations in the complex plane, is the same as the Lorentz group governing the space-time of restricted relativity. One is thus led to suspect the existence of some deep-lying connection between the theory of functions of a complex variable and the space-time of restricted relativity, the working out of which will be a difficult task for the future.

Let us now discuss the extent of the mathematical quality in Nature. According to the mechanistic scheme of physics or to its relativistic modification, one needs for the complete description of the universe not merely a complete system of equations of motion, but also a complete set of initial conditions, and it is only to the former of these that mathematical theories apply. The latter are considered to be not amenable to theoretical treatment and to be determinable only from observation.

The enormous complexity of the universe is ascribed to an enormous complexity in the initial conditions, which removes them beyond the range of mathematical discussion.

I find this position very unsatisfactory philosophically, as it goes against all ideas of the unity of Nature. Anyhow, if it is only to a part of the description of the universe that mathematical theory applies, this part ought certainly to be sharply distinguished from the remainder. But in fact there does not seem to be any natural place in which to draw the line. Are such things as the properties of the elementary particles of physics, their masses and the numerical coefficients occurring in their laws of force, subject to mathematical theory? According to the narrow mechanistic view, they should be counted as initial conditions and outside mathematical theory. However, since the elementary particles all belong to one or other of a number of definite types, the members of one type being all exactly similar, they must be governed by mathematical law to some extent, and most physicists now consider it to be quite a large extent. For example, Eddington has been building up a theory to account for the masses. But even if one supposed all the properties of the elementary particles to be determinable by theory, one would still not know where to draw the line, as one would be faced by the next question — Are the relative abundances of the various chemical elements determinable by theory? One would pass gradually from atomic to astronomic questions.

This unsatisfactory situation gets changed for the worse by the new quantum mechanics. In spite of the great analogy between quantum mechanics and the older mechanics with regard to their mathematical formalisms, they differ drastically with regard to the nature of their physical consequences. According to the older mechanics, the result of any observation is determinate and can be calculated theoretically from given initial conditions; but with quantum mechanics there is usually an indeterminacy in the result of an observation, connected with the possibility of occurrence of a quantum jump, and the most that can be calculated theoretically is the probability of any particular result being obtained. The question, which particular result will be obtained in some particular case, lies outside the theory. This must not be attributed to an incompleteness of the theory, but is essential for the application of a formalism of the kind used by quantum mechanics.

Thus according to quantum mechanics we need, for a complete description of the universe, not only the laws of motion and the initial conditions, but also information about which quantum jump occurs in each case when a quantum jump does occur. The latter information must be included, together with the initial conditions, in that part of the description of the universe outside mathematical theory.

The increase thus arising in the non-mathematical part of the description of the universe provides a philosophical objection to quantum mechanics, and is, I believe, the underlying reason why some physicists still find it difficult to accept this mechanics. Quantum mechanics should not be abandoned, however, firstly, because of its very widespread and detailed agreement with experiment, and secondly, because the indeterminacy it introduces into the results of observations is of a kind which is philosophically satisfying, being readily ascribable to an inescapable crudeness in the means of observation available for small-scale experiments. The objection does show, all the same, that the foundations of physics are still far from their final form.

We come now to the third great development of physical science of the present century — the new cosmology. This will probably turn out to be philosophically even more revolutionary than relativity or the quantum theory, although at present one can hardly realize its full implications. The starting-point is the observed red-shift in the spectra of distance heavenly bodies, indicating that they are receding from us with velocities proportional to their distances.[1] The velocities of the more distant ones are so enormous that it is evident we have here a fact of the utmost importance, not a temporary or local condition, but something fundamental for our picture of the universe.

If we go backwards into the past we come to a time, about 2×109 years ago, when all the matter in the universe was concentrated in a very small volume. It seems as though something like an explosion then took place, the fragments of which we now observe still scattering outwards. This picture has been elaborated by Lematre, who considers the universe to have started as a single very heavy atom, which underwent violent radioactive disintegrations and so broke up into the present collection of astronomical bodies, at the same time giving off the cosmic rays.

With this kind of cosmological picture one is led to suppose that there was a beginning of time, and that it is meaningless to inquire into what happened before then. One can get a rough idea of the geometrical relationships this involves by imagining the present to be the surface of a sphere, going into the past to be going in towards the centre of the sphere, and going into the future to be going outwards. There is then no limit to how far one may go into the future, but there is a limit to how far one can go into the past, corresponding to when one has reached the centre of the sphere. The beginning of time provides a natural origin from which to measure the time of any event. The result is usually called the epoch of that event. Thus the present epoch is 2×109 years.

Let us now return to dynamical questions. With the new cosmology the universe must have been started off in some very simple way. What, then, becomes of the initial conditions required by dynamical theory? Plainly there cannot be any, or they must be trivial. We are left in a situation which would be untenable with the old mechanics. If the universe were simply the motion which follows from a given scheme of equations of motion with trivial initial conditions, it could not contain the complexity we observe. Quantum mechanics provides an escape from the difficulty. It enables us to ascribe the complexity to the quantum jumps, lying outside the scheme of equations of motion. The quantum jumps now form the uncalculable part of natural phenomena, to replace the initial conditions of the old mechanistic view.

One further point in connection with the new cosmology is worthy of note. At the beginning of time the laws of Nature were probably very different from what they are now. Thus we should consider the laws of Nature as continually changing with the epoch, instead of as holding uniformly throughout space-time. This idea was first put forward by Milne, who worked it out on the assumptions that the universe at a given epoch is roughly everywhere uniform and spherically symmetrical. I find these assumptions not very satisfying, because the local departures from uniformity are so great and are of such essential importance for our world of life that it seems unlikely there should be a principle of uniformity overlying them. Further, as we already have the laws of Nature depending on the epoch, we should expect them also to depend on position in space, in order to preserve the beautiful idea of the theory of relativity there is fundamental similarity between space and time. This goes more drastically against Milne's assumptions than a mere lack of uniformity in the distribution of matter.

[1]The recession velocities are not strictly proved, since one may postulate some other cause for the spectral red-shift. However, the new cause would presumably be equally drastic in its effect on cosmological theory and would still need the introduction of a parameter of the order 2×109 years for its mathematical discussion, so it would probably not disturb the essential ideas of the argument in the text.

We have followed through the main course of the development of the relation between mathematics and physics up to the present time, and have reached a stage where it becomes interesting to indulge in speculations about the future. There has always been an unsatisfactory feature in the relation, namely, the limitation in the extent to which mathematical theory applies to a description of the physical universe. The part to which it does not apply has suffered an increase with the arrival of quantum mechanics and a decrease with the arrival of the new cosmology, but has always remained.

This feature is so unsatisfactory that I think it safe to predict it will disappear in the future, in spite of the startling changes in our ordinary ideas to which we should then be led. It would mean the existence of a scheme in which the whole of the description of the universe has its mathematical counterpart, and we must suppose that a person with a complete knowledge of mathematics could deduce, not only astronomical data, but also all the historical events that take place in the world, even the most trivial ones. Of course, it must be beyond human power actually to make these deductions, since life as we know it would be impossible if one could calculate future events, but the methods of making them would have to be well defined. The scheme could not be subject to the principle of simplicity since it would have to be extremely complicated, but it may well be subject to the principle of mathematical beauty.

I would like to put forward a suggestion as to how such a scheme might be realized. If we express the present epoch, 2×109 years, in terms of a unit of time defined by the atomic constants, we get a number of the order 1039, which characterizes the present in an absolute sense. Might it not be that all present events correspond to properties of this large number, and, more generally, that the whole history of the universe corresponds to properties of the whole sequence of natural numbers? At first sight it would seem that the universe is far too complex for such a correspondence to be possible. But I think this objection cannot be maintained, since a number of the order 1039 is excessively complicated, just because it is so enormous. We have a brief way of writing it down, but this should not blind us to the fact that it must have excessively complicated properties.

There is thus a possibility that the ancient dream of philosophers to connect all Nature with the properties of whole numbers will some day be realized. To do so physics will have to develop a long way to establish the details of how the correspondence is to be made. One hint for this development seems pretty obvious, namely, the study of whole numbers in modern mathematics is inextricably bound up with the theory of functions of a complex variable, which theory we have already seen has a good chance of forming the basis of the physics of the future. The working out of this idea would lead to a connection between atomic theory and cosmology.