Assignment Problems

The assignment problems are the most challenging part of this course. Some revise earlier material, some require you to work through details of material in the lecture notes, some ask you to apply the material and some require you to see familiar material from a new perspective.

Treat them as (miniature) research projects, where you have the advantage of knowing that everything you need, other than thinking, is either in the lecture notes you have or in the pre-requisites for enrolling in this course.

You must post your assignments by the dates listed.

<table>
<thead>
<tr>
<th>Assignment</th>
<th>Post by</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>14(^{th}) July</td>
</tr>
<tr>
<td>2</td>
<td>28(^{th}) July</td>
</tr>
<tr>
<td>3</td>
<td>11(^{th}) August</td>
</tr>
<tr>
<td>4</td>
<td>1(^{st}) September</td>
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<tr>
<td>5</td>
<td>15(^{th}) September</td>
</tr>
<tr>
<td>6</td>
<td>29(^{th}) September</td>
</tr>
</tbody>
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ASSIGNMENT 1

Question 1.
Find all real $2 \times 2$ matrices, $A$, such that $A^2 = \mathbf{0}_2 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Question 2.
Let $\mathbf{1}_r$ be the $r \times r$ identity matrix.
Let $N_r := [x_{ij}]_{r \times r}$ be the real $r \times r$ matrix given by
\[ x_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise}. \end{cases} \]
Put $A = a \mathbf{1}_r + N_r$.
Find $A^m$ for $m \in \mathbb{N}$.

Question 3.
Find $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}^n$ for $n \in \mathbb{N} \setminus \{0\}$.

Question 4.
Let $X$ be the set $\{r, s, t, u\}$, with all elements distinct.
Define binary operations $+$ and $\cdot$, on $X$ by

\[
\begin{array}{c|cccc}
+ & r & s & t & u \\
\hline
r & r & s & t & u \\
s & s & r & u & t \\
t & t & u & r & s \\
u & u & t & s & r \\
\end{array} \quad \text{and} \quad \begin{array}{c|cccc}
. & r & s & t & u \\
\hline
r & r & r & r & r \\
s & r & s & l & u \\
t & t & u & l & s \\
u & u & t & s & r \\
\end{array}
\]

Show that $X$ is a field with respect to these operations.
[This field is usually denoted by $F_4$, or $F_{2^2}$.]
ASSIGNMENT 2

Question 1.
Let $V$ and $W$ be vector spaces over $F$ and $T: V \to W$ a linear transformation. Prove that $\ker(T) := \{v \in V \mid T(v) = 0_W\}$ and $\text{im}(T) := \{T(v) \mid v \in V\}$ are both vector subspaces of $V$.

Question 2.
Consider $\mathbb{C}^2$ and $\mathbb{C}^3$ as real vector spaces with respect to component-wise operations. Prove that the function $\varphi: \mathbb{C}^3 \to \mathbb{C}^2$ is a linear transformation if and only if there are complex numbers $a, b, c, d, e, f$ such that for all $(u, v, w) \in \mathbb{C}^3$

$$\varphi(u, v, w) = (au + bv + cw, du + ev + fw).$$

Question 3.
Let $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ be rotation through an angle of $\theta$ radians about the origin. Prove that $\varphi$ is an isomorphism.

Question 4.
Let $\mathbb{R}[t]$ denote the set of all polynomials in the indeterminate $t$ with real coefficients, so that

$$\mathbb{R}[t] = \{a_0 + a_1 t + \cdots + a_n t^n \mid n \in \mathbb{N}, a_0, \ldots, a_n \in \mathbb{R} \text{ and } a_n \neq 0 \text{ if } n > 0\}$$

Show that $\mathbb{R}[t]$ is a real vector space with respect to the usual operations on polynomials. The polynomial $p(t) \in \mathbb{R}[t]$ determines the function

$$p: \mathbb{R} \to \mathbb{R}, \quad x \mapsto p(x).$$

Prove that

$$I: \mathbb{R}[t] \to \mathbb{R}[t], \quad p(t) \mapsto q(t)$$

where $q(t)$ defines the function

$$q: \mathbb{R} \to \mathbb{R}, \quad x \mapsto \int_0^x p(u) \, du$$

defines an injective linear transformation. Find a left inverse for $I$. 
Question 1.

Let \{e_1, e_2, e_3\} be a basis for the vector space \(V\) over the field \(F\).

Put \(f_1 := -e_1, f_2 := e_1 - e_2\) and \(f_3 := e_1 - e_3\).

Prove that \(\{f_1, f_2, f_3\}\) is also a basis for \(V\).

Question 2.

Let \(P_2\) be the set of all real polynomials of degree no greater than 2, so that
\[ P_2 = \{at^2 + bt + c \mid a, b, c \in \mathbb{R}\} \]

It is a real vector space with respect to the usual addition of polynomials and multiplication of a polynomial by a constant.

Show that both \(B := \{1, t, t^2\}\) and \(B' := \{t, t^2 + t, t^2 + t + 1\}\) are bases for \(P_2\).

A real polynomial \(p(t)\) defines the differentiable function
\[ p : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto p(x) \]

As is shown in elementary calculus, differentiation is the linear transformation
\[ D : P_2 \rightarrow P_2, \quad p \mapsto p' = \frac{dp}{dx} \]

Find the matrix of \(D\) with respect to the bases
(i) \(B\) in both the domain and co-domain,
(ii) \(B\) in the domain and \(B'\) in the co-domain,
(iii) \(B'\) in the domain and \(B\) in the co-domain,
(iv) \(B'\) in both the domain and co-domain.

Question 3.

The set of all solutions of the differential equation
\[ \frac{d^2y}{dx^2} + y = 0 \]

is the real vector space
\[ V = \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f'' + f = 0\} \]

Show that \(\{e_1, e_2\}\) is a basis for \(V\), where
\[ e_1 : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \sin x \]
\[ e_2 : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \sin(x + \pi/4) \]

Show that
\[ D : V \rightarrow V, \quad y \mapsto \frac{dy}{dx} \]

is a linear transformation and find its matrix representation with respect to the basis above.
Question 1.
Find the determinant of the matrix
\[
\begin{pmatrix}
1 & 3 & 9 \\
2 & \\
4 & \\
\end{pmatrix}
\]

Question 2.
Show that the $n \times n$ matrix $A$ is invertible if and only if its determinant is non-zero.

Question 3.
Recall that $\mathbb{F}(p) := \{ \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} \mid x_1, \ldots, x_p \in \mathbb{F} \}$, and that the $m \times n$ matrix $A$ may be identified with the linear transformation
\[ F(n) \rightarrow F(m), \quad \mathbf{x} \mapsto A\mathbf{x}. \]
In each case below, find a basis for the image of $A$ as well as a basis for the kernel of $A$.

(a) \[ A = \begin{bmatrix}
4 & 2 & 7 \\
0 & 2 & 5 \\
1 & 1 & 1 \\
\end{bmatrix} \]
(b) \[ A = \begin{bmatrix}
1 & 6 & 3 & 5 \\
2 & 11 & 3 & 7 \\
6 & 13 & 9 & 13 \\
\end{bmatrix} \]

Question 4.
Find all $\lambda \in \mathbb{R}$ such that there is a non-zero $\mathbf{v} \in \mathbb{R}^2$ such that $A\mathbf{v} = \lambda \mathbf{v}$, where

(a) \[ A = \begin{bmatrix}
4 & -3 \\
1 & 0 \\
\end{bmatrix} , \]
(b) \[ A = \begin{bmatrix}
4 & -4 \\
1 & 0 \\
\end{bmatrix} , \]
(c) $A = \begin{bmatrix} 4 & -5 \\ 1 & 0 \end{bmatrix}$,

(d) $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. 
ASSIGNMENT 5

Question 1.

Let \( A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) be a real \( 2 \times 2 \) matrix. Show that

(a) \( A \) is diagonalisable whenever \((a - d)^2 + 4bc > 0\);

(b) \( A \) cannot be diagonalised (over \( \mathbb{R} \)) if \((a - d)^2 + 4bc < 0\).

Discuss what occurs when \((a - d)^2 + 4bc = 0\).

Question 2.

Recall that the matrices \( A \) and \( B \) are similar if and only if there is an invertible matrix \( C \) with \( B = C A C^{-1} \).

Show that

(i) the matrices \( \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \) and \( \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \) are similar, but

(ii) \( \begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix} \) and \( \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} \) are not similar.

Question 3.

Take \( A \in \mathbb{M}(n; \mathbb{F}) \).

Prove, or find a counter-example to, each of the following statements.

(a) \( \lambda \) is an eigenvalue of \( A \) if and only if it is an eigenvalue of \( A^t \).

(b) \( v \) is an eigenvector of \( A \) if and only if it is an eigenvector of \( A^t \).

(c) If \( v \) is an eigenvector of \( A \) for the eigenvalue \( \lambda \) and if \( p \) is any polynomial over \( \mathbb{F} \), then \( v \) is an eigenvalue of \( p(A) \) for the eigenvalue \( p(\lambda) \).

(d) \( A \) is invertible if and only if 0 is not an eigenvalue of \( A \).

Question 4.

Find a matrix \( B \) which diagonalises \( A := \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix} \).

Determine both \( B A B^{-1} \) and \( B^{-1} A B \).

Do the same for \( A := \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \).
Question 1.

We consider $P_2$, the vector space of all real polynomials of degree at most 2.

Show that

$$\langle f, g \rangle := f(-1)g(-1) + f(0)g(0) + f(1)g(1)$$

$$\langle f \mid g \rangle := \int_{-1}^{1} f(x)g(x)dx$$

both define inner products on $P_2$.

Use the Gram-Schmidt Procedure with respect to each of these to construct orthonormal bases for $P_2$ from the basis $\{t, 1 + t^2, 1 + t - t^2\}$.

Question 2.

Find an orthogonal matrix $A$ and an upper triangular matrix $B$ such that

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 9 & 81 \end{bmatrix}$$

Question 3.

Find orthogonal matrix $B$ such that

$$C := B^t A B$$

is a diagonal matrix, where

$$A := \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

and find $C$. 