TUTORIAL 2, PMTH212

1. Name and sketch the surface $x^2 + y^2 - z^2 = 9$.

Solution. This is a hyperboloid of one sheet (two $+$ signs and one $-$ sign).
The traces on the horizontal planes $\{z = 0\}$ and $\{z = 4\}$ are circles of radius 3 and 5 respectively:

\[
\begin{align*}
    x^2 + y^2 - 0 &= 9, \\
    x^2 + y^2 - 16 &= 9, \\
    x^2 + y^2 &= 9 \\
    x^2 + y^2 &= 25
\end{align*}
\]

The trace on $\{x = 0\}$ is the hyperbola $y^2 - z^2 = 9$. 

![Diagram of a hyperboloid with traces labeled](image-url)
2. Find the trace of the surface $x^2 + y^2 - z^2 = 9$ on the plane \( \{x = 0\} \).

**Solution.** For $x = 3$ we get

\[
3^2 + y^2 - z^2 = 9 \\
y^2 - z^2 = 0 \\
(y - z)(y + z) = 0.
\]

The latter equation is satisfied if and only if either $y - z = 0$ or $y + z = 0$. This is a pair of straight lines that intersect at $(0, 0, 0)$. It is an interesting fact that through each point of any hyperboloid of one sheet there is a pair of straight lines that are entirely contained in the hyperboloid.
3. Sketch a.) \(4x^2 + y^2 - z^2 = 9\) b.) \(4x^2 - 8x + y^2 - z^2 = 5\).

**Solution.** a). This is a hyperboloid of one sheet. The trace on the plane \(z = 0\) is the ellipse \(4x^2 + y^2 = 9\) with vertices \(x = 0, y = \pm 3\) and \(y = 0, x = \pm \frac{3}{2}\).

b). By completing the square this equation becomes \(4(x - 1)^2 + y^2 - z^2 = 9\). This is the surface from a) shifted by one unit into x-direction. The centre is \((1, 0, 0)\).
4. Describe the graphs of the vector equations
   a) \( \vec{r} = 3 \cos t \vec{i} + 3 \sin t \vec{j} - \vec{k} \)
   b) \( \vec{r} = 2t \vec{i} - t \vec{j} + (2 + 3t) \vec{k} \)
   c) \( \vec{r} = t \vec{i} + t^2 \vec{j} + 2t \vec{k} \)

   **Solution.** a.) This is a circle of radius 3 in the plane \( z = -1 \). We have \( x^2 + y^2 = 9 \cos^2 t + 9 \sin^2 t = 9 \)
   b.) This is a line with vector equation \( \vec{r} = (0, 0, 2) + t(2, -1, 2) \).
   c.) This is a parabola in the plane \( z = 2 \). From \( x = t, y = t^2 \) we get \( y = x^2 \).

5. Find the natural domains for the functions \( r(t) \). Determine, where it describes a smooth curve, i.e. where \( r'(t) \) exists and is different from \( \vec{0} \).
   a) \( \vec{r} = e^t \vec{i} + (\ln t - 1) \vec{j} - \cos t \vec{k} \)
   b) \( \vec{r} = t^3 \vec{i} + 3t^2 \vec{j} + t^2 \vec{k} \)

   **Solution.** a.) The natural domain is \( \{ t > 0 \} \) for \( \ln t \) to be well-defined. \( r' \) exists for \( t > 0 \) and
      \[ r' = e^t \vec{i} + \frac{1}{x} \vec{j} + \sin t \vec{k}. \]
      Since \( e^t \neq 0 \) we have \( r' \neq \vec{0} \) and \( r \) is smooth.
   b.) \( r \) is well-defined for any \( t \in \mathbb{R} \) and has a derivative
      \[ r' = 3t^2 \vec{i} + 6t \vec{j} + 2t \vec{k}. \]
      For \( t = 0 \) this is \( \vec{0} \) hence the curve is not smooth for \( t = 0 \).

6. Compute
   a) \( \int (t^2 \vec{i} - 2t \vec{j} + \frac{1}{t} \vec{k}) dt. \)
   b) \( \int_0^1 (e^{-t}, te^t, 3t^2) dt \)

   **Solution.** a.)
   \[ \int (t^2 \vec{i} - 2t \vec{j} + \frac{1}{t} \vec{k}) dt = \frac{t^3}{3} \vec{i} - t^2 \vec{j} + \ln |t| \vec{k} + \vec{c} \]
   b.)
   \[ \int_0^1 (e^{-t}, te^t, 3t^2) dt = \left. (-e^{-t}, (t-1)e^t, t^3) \right|_0^1 \]
   \[ = (-e^{-1} + 1, 1, 1) \]
   \[ = (1 - \frac{1}{e}, 1, 1) \]
7. Compute the arc-length of \( r(t) = t^2 \mathbf{i} - t^2 \mathbf{j} + \frac{1}{2} \sqrt{6t^2} \mathbf{k} \) for \( 1 \leq t \leq 3 \).

Solution. We have

\[
L = \int_1^3 \sqrt{(3t^2)^2 + 1 + (\sqrt{6t^2})^2} \, dt
\]

\[
= \int_1^3 \sqrt{9t^4 + 1 + 6t^2} \, dt
\]

\[
= \int_1^3 (3t^2 + 1) \, dt
\]

\[
= t^3 + t\bigg|_1^3 = 28
\]