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Lecture 1 Introduction

In one-variable calculus you have studied functions of one real variable, in particular the concepts of continuity, differentiation and integration. Functions of one variable can capture the dependence of some quantity by only one other quantity. In practice however, one often needs to investigate the dependence on one or more quantities on many variables, such as time, location, temperature, air pressure, or costs of different products etc. Therefore it in natural to consider functions that depend on many variables. We will write $f(x, y)$ for a function of two variables, $f(x, y, z)$ for a function of three variables or, more generally, $f(x_1, x_2, \ldots, x_n)$ for a function of n variables. Instead of f other letters can be used (such as g, h, F, G, H or f_1, f_2 etc.). Here it is assumed that a domain is specified to which the argument variables belong. We will write $\mathbb{R}^2$ for the set of all *pairs* (x, y) of real numbers, $\mathbb{R}^3$ for all *triples* of real numbers and $\mathbb{R}^n$ for all *n-tuples* of real numbers. The domains of multivariable functions are subsets of those.

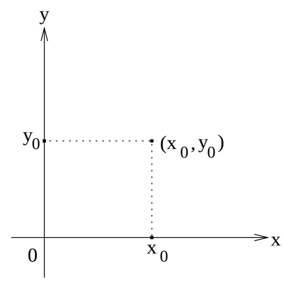
It is convenient to plot functions of one variable as a graph in the two-dimensional plane (and, vice versa, one can study curves in the plane using functions). Similar to this, a function of two variables can be plotted as a surface in three-dimensional space, and two-dimensional curved surfaces can be studied by functions of two variables. Visualising functions of more than two variables is more difficult.

In this unit we will cover the concepts of continuity, differentiation and integration of functions of many variables, as well as applying these concepts to study the geometry of curves and surfaces.

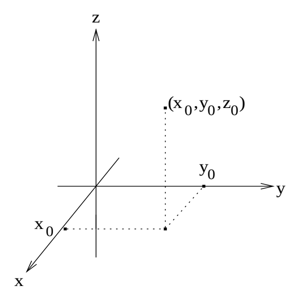
1.1 Rectangular Coordinate Systems

First we will recall some concepts from linear algebra that were introduced in Math101. Recall that a point P_0 in 2-space can be described by a pair of numbers, namely its Cartesian coordinates (x_0, y_0)[1]. We use a coordinate system of two perpendicular axes, the x-axis and y-axis. Now, x_0 is the number on the x-axis that corresponds to the perpendicular projection of P_0 to the x-axis (parallel to the y-axis) and y_0 is the number on the y-axis that corresponds to the perpendicular projection of P_0 to the y-axis (parallel to the x-axis).

[1] Here we use the lower indices 0 to indicate that these are the coordinates of the point P_0.



In 3-space, the situation is similar. Here we need an extra axis, the z-axis. Usually we make the z-axis vertical and pointing upward, while we make the x-axis and y-axis to form a horizontal xy-plane as follows



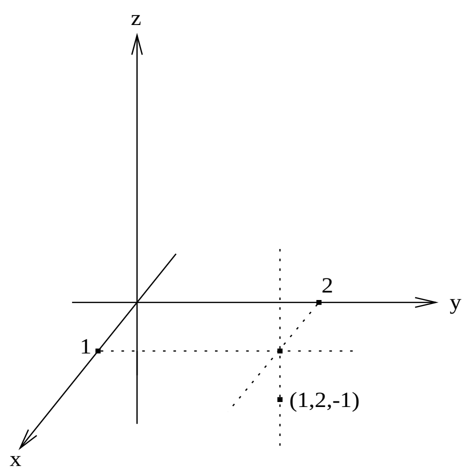
To find the coordinates a point P_0 we need to project it perpendicularly to the corresponding axis (parallel to the plane spanned by the remaining axes). This can be done in two steps, we first find the point P_0' that is the perpendicular projection of P to the xy-plane. Then x_0, y_0 are the coordinates of P_0' in the xy-plane and z_0 is the distance between P_0 and P_0' with a positive sign if P is located above the xy-plane and with a negative sign in the opposite case.

To find a point with given coordinates (x_0, y_0, z_0) first find P_0' with coordinates (x_0, y_0) in the xy-plane and the go up by the distance of $|z_0| = z_0$ if $z_0 \geq 0$ or down by $|z_0| = -z_0$ if $z_0 < 0$ to find P_0. We will use the notation $P_0(x_0, y_0, z_0)$ to indicate that the point P_0 has coordinates (x_0, y_0, z_0).

Note: The coordinate system as drawn above is called a right-handed system (when the fingers of the right hand are cupped so that they curve from the positive x-axis toward the positive y-axis, the thumb points (roughly) in the direction of the positive z-axis). If we interchange the positions of the x-axis and y-axis, then we obtain a left-handed system. It can be shown that rectangular coordinate systems in 3-space fall into just these two categories: right-handed and left-handed. In this unit, we will always use right-handed systems, and we will always draw the coordinates with the z-axis vertical and pointing upward. We will call this the xyz-coordinate system.

Example 1 Find the point with coordinates $(1, 2, -1)$ in the xyz-coordinate system.

Solution: We first draw the coordinate axes to obtain the xyz-coordinate system. Then find $(1, 2)$ on the xy-plane, draw a vertical line through the point, and move down one unit (since $z_0 = -1$), and we arrive at $(1, 2, -1)$. □



1.2 Vectors

Shifts in the plane or space can be described by saying to what point P_2 a given point P_1 has been shifted. The line ℓ connecting P_1 and P_2 gives the direction of the shift. Any other point Q_1 would be shifted along a line parallel to ℓ by a distance equal to the distance between P_1 and P_2 to a point Q_2, so that P_1, P_2, Q_1, Q_2 form a parallelogram. For this description it did not matter whether we started at P_1 or Q_1, so instead of the pairs $P_1(x_1, y_1, z_1)P_2(x_2, y_2, z_2)$ or $Q_1(X_1, Y_1, Z_1)Q_2(X_2, Y_2, Z_2)$ we may consider the object

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \langle X_2 - X_1, Y_2 - Y_1, Z_2 - Z_1 \rangle$$

called *vector* (in 3-space)[2]. To indicate that a vector $\vec{v}$ is represented by an *initial point* P_1 and *terminal point* P_2 we write $\vec{v} = \overrightarrow{P_1P_2}$.

When the initial and terminal point are the same, no shift occurs and the corresponding vector is the zero vector

$$\vec{0} = \langle 0, 0, 0 \rangle.$$

[2]The same concept works in the 2-dimensional plane and any n-dimensional space. To keep things simple, but not too simple, we restrict here to 3-dimensional space.

Points and vectors are described by in a similar way by coordinates. In fact, a point $P(x, y, z)$ can be identified with a vector $\vec{v} = \langle x, y, z \rangle$ that shifts the origin $O(0, 0, 0)$ to $P(x, y, z)$. Nevertheless, points and vectors are different objects. We cannot add points or multiply points with numbers, but we can add vectors:

$$\vec{v}_1 + \vec{v}_2 = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle.$$

The geometric meaning of adding two vectors is to perform two shifts, first by $\vec{v}_1$ and then by $\vec{v}_2$. The result is a single shift by $\vec{v}_1 + \vec{v}_2$. Notice that the sum of two vectors does not depend on the order of the two shifts being performed.

We can also multiply a vector $\vec{v} = \langle x, y, z \rangle$ by a number α (in this context called a *scalar* as opposed to a vector):

$$\alpha\vec{v} = \alpha\langle x, y, z \rangle = \langle \alpha x, \alpha y, \alpha z \rangle.$$

Geometrically, the shift $\alpha\vec{v}$ is in the same direction as $\vec{v}$, if $\alpha > 0$ and in the opposite direction, if $\alpha < 0$, by $|\alpha|$ times the original distance.

It follows easily from the arithmetic of numbers that the following properties are satisfied:

(a) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(b) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(c) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

(d) $\vec{u} + (-\vec{u}) = \vec{0}$

(e) $k(\ell\vec{u}) = (k\ell)\vec{u}$

(f) $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{u}$

(g) $(k + \ell)\vec{u} = k\vec{u} + \ell\vec{u}$

(h) $1\vec{u} = \vec{u}$.

In our computations with vectors we will rely on these properties. One can define a more abstract notion of *vector spaces* by stipulating theses properties as axioms. This will be discussed in more detail in Linear Algebra Pmth213. You may verify that the set of polynomials, or the set of polynomials of degree less or equal to 4 also constitute an abstract linear space.

Lecture 2 Length, Dot Product, Cross Product

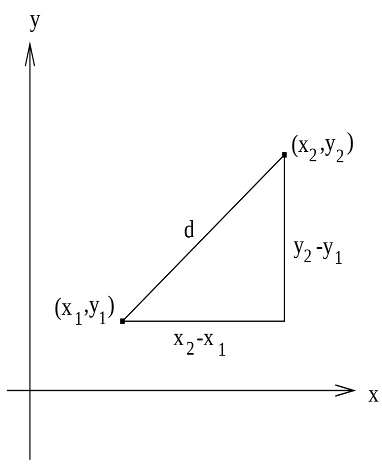
2.1 Length

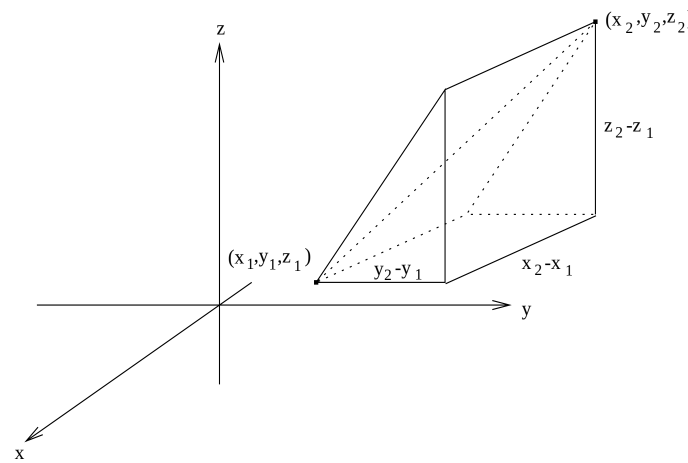
We know that in 2-space, the distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

This is at the same time the **length** of the vector $\vec{v} = \overrightarrow{P_1P_2}$, denoted by $||\vec{v}||$.

In 3-space, there is a similar formula: the distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$
which again is the length $||\vec{v}||$ of the vector $\vec{v} = \overrightarrow{P_1P_2}$.

Can you prove these formulas using elementary geometry and the following diagrams? (Hint: Pythagoras theorem).





Thus, the length of a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ (or $\vec{v} = \langle v_1, v_2, v_3, \ldots v_n \rangle$), also called the **norm** of $\vec{v}$, is given by

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2} \quad \left(\text{or } ||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2} \right).$$

The **zero vector** $\vec{0}$ has length 0.

A vector of length 1 is called a **unit vector**. For example, if $\vec{v} \neq \vec{0}$, then $\frac{1}{||\vec{v}||}\vec{v}$ is a unit vector (why? Please check). The following unit vectors are of special importance:

$$\begin{aligned}\vec{i} &= \langle 1,0\rangle, \quad \vec{j} = \langle 0,1\rangle \text{ in 2-space,}\\ \vec{i} &= \langle 1,0,0\rangle, \quad \vec{j} = \langle 0,1,0\rangle, \quad \vec{k} = \langle 0,0,1\rangle \text{ in 3-space.}\end{aligned}$$

This is because for any vector $\vec{v} = \langle v_1, v_2\rangle$, we have

$$\begin{aligned}\vec{v} = \langle v_1, v_2\rangle &= \langle v_1, 0\rangle + \langle 0, v_2\rangle\\ &= v_1\langle 1,0\rangle + v_2\langle 0,1\rangle\\ &= v_1\vec{i} + v_2\vec{j};\end{aligned}$$

and for any vector $\vec{v} = \langle v_1, v_2, v_3\rangle$, we have

$$\vec{v} = \langle v_1, v_2, v_3\rangle = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}.$$

These vectors are sometimes called the unit coordinate vectors.

2.2 Dot Product

The **dot product** assigns a scalar (number) to two vectors. It can be defined in any dimension.

Let $\vec{u} = \langle u_1, u_2\rangle, \vec{v} = \langle u_1, v_2\rangle$. Then the **dot product** of $\vec{u}$ and $\vec{v}$ is given by

$$\boxed{\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2.}$$

Similarly, for $\vec{u} = \langle u_1, u_2, u_3\rangle, \vec{v} = \langle v_1, v_2, v_3\rangle$,

$$\boxed{\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3,}$$

or, generally, for $\vec{u} = \langle u_1, u_2, \ldots, u_n\rangle, \vec{v} = \langle v_1, v_2, \ldots, v_n\rangle$,

$$\boxed{\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.}$$

Notice that the length of a vector $\vec{v}$ can be expressed as

$$||\vec{v}|| = \sqrt{\vec{v}\cdot\vec{v}}.$$

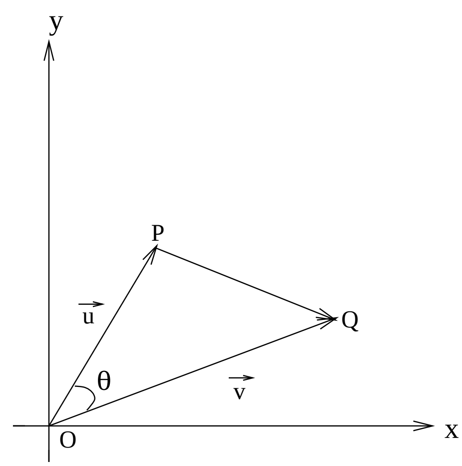
The relation between the dot product and the Euclidean geometric notion of angle is the subject of the Theorem below.

Theorem 1 Let $\vec{u}, \vec{v}$ be nonzero vectors in 2-space or 3-space[3], and let θ be the angle between $\vec{u}$ and $\vec{v}$. Then

$$\boxed{\vec{u}\cdot\vec{v} = ||\vec{u}||\,||\vec{v}||\cos\theta, \quad \text{i.e.} \qquad \cos\theta = \tfrac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}.}$$

Proof: We prove the case that $\vec{u}\cdot\vec{v}$ are 2-space vectors. The 3-space case can be proved similarly.

As in the diagram, we can use $\vec{u}$ and $\vec{v}$ to form the triangle OPQ, where $\vec{u} = \langle u_1, u_2\rangle, \vec{v} = \langle v_1, v_2\rangle$, and $\overrightarrow{OP} = \vec{u}, \overrightarrow{OQ} = \vec{v}$. Note that the length of OP is $||\vec{u}||$, that of OQ is $||\vec{v}||$ and the length of PQ is $||\overrightarrow{PQ}|| = ||\vec{v}-\vec{u}||$ since $\overrightarrow{PQ} = \langle v_1 - u_1, v_2 - u_2\rangle = \vec{v}-\vec{u}$. The law of cosines applied to the triangle OPQ gives
$||\vec{v}-\vec{u}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||\,||\vec{v}||\cos\theta$
which is equivalent to



$$\begin{aligned} ||\vec{u}||\,||\vec{v}||\cos\theta &= \frac{1}{2}\left(||\vec{u}||^2 + ||\vec{v}||^2 - ||\vec{v}-\vec{u}||^2\right) \\ &= \frac{1}{2}\left\{u_1^2 + u_2^2 + v_1^2 + v_2^2 - \left[(v_1-u_1)^2 + (v_2-u_2)^2\right]\right\} \\ &= u_1v_1 + u_2v_2 \\ &= \vec{u}\cdot\vec{v} \end{aligned}$$

i.e.

$$\vec{u}\cdot\vec{v} = ||\vec{u}||\,||\vec{v}||\cos\theta.$$

□

Theorem 1 shows that dot product can be used to calculate the angle between two vectors. In particular, two non-zero vectors $\vec{u}, \vec{v}$ are **perpendicular** if and only if $\vec{u}\cdot\vec{v} = 0$, i.e. $\cos\theta = 0$.

Another consequence of Theorem 1 is the important **Cauchy-Schwarz inequality**

$$|\vec{u}\cdot\vec{v}| \le ||\vec{u}||\,||\vec{v}||$$

[3]In higher dimensional spaces the formula $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\,||\vec{v}||}$ can be used to define the notion of angle.

with equality occuring only if the two vectors are parallel or one of them is the zero vector.

Example 1. Let $P_1 = (1,1,1), P_2 = (3,0,2)$ and $P_3 = (2,2,3)$. Find the angle between $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$.

Solution: Let θ denote the angle. Then, by Theorem 1,

$$\cos\theta = \frac{\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1P_3}}{||\overrightarrow{P_1P_2}||||\overrightarrow{P_1P_3}||}$$

$$\begin{array}{rcl} \text{We have } \overrightarrow{P_1P_2} & = & \langle 3-1, 0-1, 2-1\rangle = \langle 2, -1, 1\rangle \\ \overrightarrow{P_1P_3} & = & \langle 2-1, 2-1, 3-1\rangle = \langle 1, 1, 2\rangle \\ \overrightarrow{P_1P_2} \cdot \overrightarrow{P_1P_3} & = & 2 \times 1 + (-1) \times 1 + 1 \times 2 = 3 \\ ||\overrightarrow{P_1P_2}|| & = & \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6} \\ ||\overrightarrow{P_1P_3}|| & = & \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6} \end{array}$$

Hence

$$\cos\theta = \frac{3}{\sqrt{6}\sqrt{6}} = \frac{1}{2}, \ \theta = 60^o \ (\text{or } \frac{\pi}{3})$$

□

Dot product has the usual arithmetic properties which we list below, the proof follows from the definition of dot product directly, and therefore will not be provided. You are encouraged to prove some of them, at least (d).

If $\vec{u}, \vec{v}$, and $\vec{w}$ are vectors and k is a scalar, then

(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(b) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

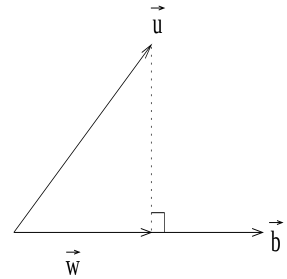
(c) $k(\vec{u} \cdot \vec{v}) = (k\vec{u}) \cdot \vec{v} = \vec{u} \cdot (k\vec{v})$

(d) $\vec{v} \cdot \vec{v} = ||\vec{v}||^2 \geq 0$ (and equality holds only for $\vec{v} = \vec{0}$).

Property (a) is called **symmetry**, (b) and (c) together **bilinearity** and (d) **positivity**. In a more abstract setting these properties are used to define dot products in arbitrary vector spaces. This concept will be developed in Pmth213.

2.3 Orthogonal Projections

Let $\vec{u}$ and $\vec{b}$ be two vectors in 2-space or 3-space. Then the vector $\vec{w}$ formed as in the diagram below is called the **orthogonal projection of $\vec{u}$ on $\vec{b}$**, and is denoted by $\text{proj}_{\vec{b}}\,\vec{u}$



If the angle θ between $\vec{u}$ and $\vec{b}$ is less than $90°$, i.e. $\frac{\pi}{2}$, then $\text{proj}_{\vec{b}}\,\vec{u}$ is in the direction of $\vec{b}$. Therefore $\text{proj}_{\vec{b}}\,\vec{u}$ has the form

$$\text{proj}_{\vec{b}}\,\vec{u} = k\vec{b} \text{ for some scalar } k > 0.$$

Using the definition, we find that the length of $\text{proj}_{\vec{b}}\,\vec{u}$ is

$$\begin{aligned} ||\vec{u}||\cos\theta &= ||\vec{u}||\frac{\vec{u}\cdot\vec{b}}{||\vec{u}||\cdot||\vec{b}||} \quad \text{(by Theorem 1)} \\ &= \frac{\vec{u}\cdot\vec{b}}{||\vec{b}||}. \end{aligned}$$

On the other hand, $||\,\text{proj}_{\vec{b}}\,\vec{u}|| = ||k\vec{b}|| = k||\vec{b}||$. Therefore

$$k||\vec{b}|| = \frac{\vec{u}\cdot\vec{b}}{||\vec{b}||}, \quad \text{that is } k = \frac{\vec{u}\cdot\vec{b}}{||\vec{b}||^2}.$$

Substituting back, we obtain

Theorem 2 $\boxed{\text{proj}_{\vec{b}}\,\vec{u} = \frac{\vec{u}\cdot\vec{b}}{||\vec{b}||^2}\vec{b}}$

The above formula is also true if θ is greater than $90°$. Please check this by modifying the above proof.

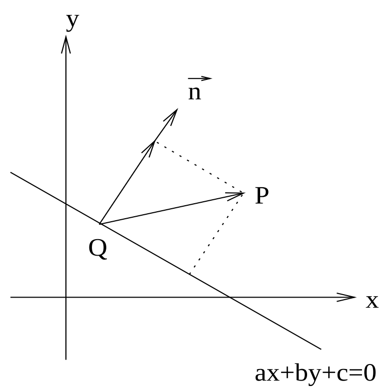
Let us now look at some of the uses of this formula.

By definition, the distance between two sets $\mathfrak{P}$ and $\mathfrak{Q}$ in 2- or 3-space is the infimum of all distances of a point $P \in \mathfrak{P}$ and $Q \in \mathfrak{Q}$, i.e., roughly speaking, the distance between the two points in P and Q that are closest to each other in the respective sets:

$$\operatorname{dist}(\mathfrak{P}, \mathfrak{Q}) = \inf_{P \in \mathfrak{P}, Q \in \mathfrak{Q}} ||PQ||.$$

Example 2. Find a formula for the distance D between the point $P(x_0, y_0)$ and the line $Ax + By + C = 0$.

Solution: Let us use the graph at the right to help with the argument. The distance from P to an arbitrary point Q on the line is the hypotenuse of a right triangle with one leg being the distance from P to the orthogonal projection (say Q') and the other being the distance between Q and Q'. From $||PQ|| \geq ||PQ'||$ we see that the infimum of distances $||PQ||$ is attained for $Q = Q'$. Therefore we need to find the orthogonal projection of P on the line.



Let $Q = (x_1, y_1)$ be any point on the line (i.e. $Ax_1 + By_1 + C = 0$) and position $\vec{n}$ so that Q is its initial point. Then

$$\begin{aligned} D &= ||\operatorname{proj}_{\vec{n}} \overrightarrow{QP}|| = ||\frac{\overrightarrow{QP} \cdot \vec{n}}{||\vec{n}||^2}\vec{n}|| \text{ (using Theorem 2)} \\ &= \left| \frac{\overrightarrow{QP} \cdot \vec{n}}{||\vec{n}||^2} \right| \cdot ||\vec{n}|| \quad \text{(using } ||k\vec{v}|| = |k|||\vec{v}||\text{)} \\ &= \frac{|\overrightarrow{QP} \cdot \vec{n}|}{||\vec{n}||} \end{aligned}$$

But $\overrightarrow{QP} = \langle x_0 - x_1, y_0 - y_1 \rangle, \vec{n} = \langle A, B \rangle$. Hence

$$\begin{aligned} \overrightarrow{QP} \cdot \vec{n} &= A(x_0 - x_1) + B(y_0 - y_1) \\ &= Ax_0 + By_0 - Ax_1 - By_1 \\ &= Ax_0 + By_0 + C \quad \text{(using } Ax_1 + By_1 + C = 0\text{)}, \\ ||\vec{n}|| &= \sqrt{A^2 + B^2} \quad \text{and} \\ D &= \frac{|\overrightarrow{QP} \cdot \overrightarrow{n}|}{||\overrightarrow{n}||} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}} \end{aligned}$$

□

Note: The point Q is introduced just for an intermediate step, it does not appear in the final formula.

Notice that the distance of a point $P(x, y)$ to the y axis is just $|x|$ and the distance to the x-axis is just $|y|$. This can be verified by the formula with $A = 0, B = 1, C = 0$ and $A = 1, B = 0, C = 0$, respectively.

2.4 Cross Product

For vectors in 3-space only, another kind of product, called the cross product, is defined. If $\vec{u}\langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$, then the **cross product** $\vec{u} \times \vec{v}$ is a vector given by

$$\boxed{\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ u_1 & v_2 \end{vmatrix} \vec{k} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}}$$

Note that $\vec{u} \cdot \vec{v}$ is a number, but $\vec{u} \times \vec{v}$ is a vector. The following theorem shows some of the properties of cross product are similar to dot product, but many are different.

Theorem 3. Cross product has the following properties:

(1) $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$, i.e.

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = 0, \;\; \vec{v} \cdot (\vec{u} \times \vec{v}) = 0.$$

(2) $||\vec{u} \times \vec{v}|| = ||\vec{u}|| \cdot ||\vec{v}|| \sin\theta$ (compare $\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta$)
Notice that this is the area of the parallelogram spanned by the vectors $\vec{u}$ and $\vec{v}$.

It follows $||\vec{u} \times \vec{v}|| \leq ||\vec{u}|| \cdot ||\vec{v}||$ where the equality holds if and only if $\vec{u} \perp \vec{v}$.

(3) (a) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ (compare $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$)

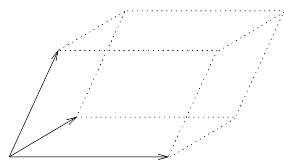
$$\left.\begin{array}{lll} (b) & \vec{u} \times (\vec{v} + \vec{w}) & = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w}) \\ (c) & (\vec{u} + \vec{v}) \times \vec{w} & = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w}) \\ (d) & k(\vec{u} \times \vec{v}) & = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{v}) \\ (e) & \vec{u} \times \vec{0} & = \vec{0} \times \vec{u} = \vec{0} \end{array}\right\} \text{similar to dot product}$$

(f) $\vec{u} \times \vec{u} = \vec{0}$ (compare $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$)

(4) If $\vec{a} = \langle a_1, a_2, a_3\rangle, \vec{b} = \langle b_1, b_2, b_3\rangle$ and $\vec{c} = \langle c_1, c_2, c_3\rangle$, then

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(5) $|\vec{a}\cdot(\vec{b}\times\vec{c})|$ is the volume of the parallelepiped spanned on the vectors $\vec{a}, \vec{b}$ and $\vec{c}$.



Proof: We only prove (4) and (1).

We prove (4) first and then use (4) to deduce (1).

Proof of (4):

$$\begin{aligned} \vec{b}\times\vec{c} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}\vec{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}\vec{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\vec{k} \end{aligned}$$

Hence

$$\begin{aligned} \vec{a}\cdot(\vec{b}\times\vec{c}) &= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

Proof of (1) (using (4)):

$$\vec{u}\cdot(\vec{u}\times\vec{v}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0,$$

because the first and second rows are the same. $\square$

Lecture 3 Linear Functions, Lines and Planes

3.1 Linear Functions

A linear function of one variable is given by an equation of the form

$$y = f(x) = mx + b,$$

where m, b are real parameters. Its graph is a line in the xy-plane with slope m and y-intercept b.

Linear functions are easy to handle, yet they can serve as good models for many processes in science, nature and economy. Even non-linear processes can often be modelled approximately using linear functions. The means to do this is differential calculus. A function $f(x)$ that is differentiable at some point x_0 can be expressed as

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + e(x, x_0),$$

where $e(x, x_0$ is a small error term. The function $f(x_0) + f'(x_0)(x - x_0) = mx + b$ is a linear function with $m = f'(x_0)$ and $b = f(x_0) - f'(x_0)x_0$. The error term is small in the sense that it tends 'faster' to zero than the linear function $x - x_0$ when x approaches x_0. More precisely, even the ratio of the two small quantities tends to zero:

$$\frac{e(x, x_0)}{x - x_0} \to 0 \text{ as } x \to x_0.$$

Indeed,

$$\lim_{x\to x_0} \frac{e(x, x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) = 0$$

holds if and only if

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0),$$

which is the definition of the derivative $f'(x_0)$.

Linear functions of several variables have the form

$$f(x, y) = ax + by + d$$
$$f(x, y, z) = ax + by + cz + d$$
$$f(x_1, x_2, \dots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n + d$$

in the case 2, 3 or n variables, respectively. Here $a, b, c, d, a_1, \dots, a_s$ are parameters. We will investigate their geometric meaning later.

The role of linear functions of several variables is similar to linear functions of one variable.

3.2 Linear mappings

A linear mapping from n-dimensional space $\mathbb{C}^n$ to m-dimensional space $\mathbb{R}^m$ is given by m linear functions

$$
\begin{aligned}
y_1 &= f_1(x_1, \dots, x_n) = a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n + b_1 \\
y_2 &= f_2(x_1, \dots, x_n) = a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n + b_2 \\
&\vdots \\
y_m &= f_m(x_1, \dots, x_n) = a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n + b_n
\end{aligned}
$$

In matrix notation this can be written as

$$\vec{\mathbf{y}} = \mathbf{A}\vec{\mathbf{x}} + \vec{\mathbf{b}},$$

where $\vec{\mathbf{x}}$ is a column vector in $\mathbb{R}^n$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{y}}$ are column vectors in $\mathbb{R}^m$ and $\mathbf{A}$ is an $m \times n$ matrix.

The geometric meaning of $\vec{\mathbf{b}}$ is a parallel displacement in direction of the vector $\vec{\mathbf{b}}$. The columns of $\mathbf{A}$ are the images of the standard vectors

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Important examples of linear mappings are rotations. A rotation in the plane about the origin at an angle ϕ is described by

$$
\begin{aligned}
y_1 &= \cos\phi x_1 - \sin\phi x_2 \\
y_2 &= \sin\phi x_1 + \cos\phi x_2.
\end{aligned}
$$

Here $\vec{\mathbf{b}} = \vec{0}$ and

$$\mathbf{A} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

Notice that $\det \mathbf{A} = 1$.

Rotations in 3-dimensional space are a bit more complicated. Any rotation in 3-dimensional space would map the standard vectors $\vec{\imath}$, $\vec{\jmath}$, $\vec{k}$ to a triple of mutually perpendicular unit vectors $\vec{\imath}'$, $\vec{\jmath}'$, $\vec{k}'$. Such rotation is completely determined by those vectors.

Simple examples are rotations about one of the coordinate axes. E.g. a rotation about the $\vec{k}$-axis corresponds to

$$\mathbf{A} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Geometrically we can decompose any rotation into a sequences of 3 simple ones about the three axes. The determinant of any rotation matrix in 3-dimensional space is 1.

We can now prove the fact stated in the previous lecture that $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\| \sin\theta$. Assume $\vec{u} \times \vec{v} \neq \vec{0}$. Let $\vec{w}$ be the unit vector

$$\vec{w} = \frac{1}{\|\vec{u} \times \vec{v}\|} \vec{u} \times \vec{v}.$$

Then

$$\|\vec{u} \times \vec{v}\| = \vec{w} \cdot \vec{u} \times \vec{v} = \det \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Now we apply a rotation in space that maps $\vec{w}$ to $\vec{k}$ and $\vec{u}$ to a multiple of $\vec{i}$. Since $\vec{v}$ is perpendicular to $\vec{w}$ its image is in the $\vec{i}, \vec{j}$ plane. The determinant of the matrix above does not change since the determinant of the rotation matrix is 1. But in the new coordinates we may assume that

$$\begin{aligned} \vec{u} &= \langle u_1, 0, 0 \rangle \\ \vec{v} &= \langle v_1, v_2, 0 \rangle \\ \vec{w} &= \langle 0, 0, 1 \rangle \end{aligned}$$

and

$$\|\vec{u} \times \vec{v}\| = \det \begin{vmatrix} 0 & 0 & 1 \\ u_1 & 0 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = u_1 v_2.$$

This is clearly the area of the parallelogram spanned by $\vec{u}$, $\vec{v}$ since u_1 is the length of the base side and v_2 is the hight.

The following fact will be used later: The cube of volume 1 spanned by the standard vectors

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{j} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is mapped under a linear mapping with matrix $\mathbf{A}$ to a parallelepiped spanned by the vectors

$$\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, \quad \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix}, \quad \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix}$$

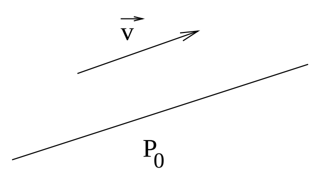
of volume $\det A$. Hence a linear mapping stretches the volume of a solid by a factor of $\det A$.

3.3 Lines

In 2-space or 3-space, a line is determined by a point P_0 on it, and a direction parallel to it. Here the point P_0 will be given by its coordinates and the direction by a non-zero vector $\vec{v}$.

First, consider the case of a line in 2-space, which passes through $P_0 = (x_0, y_0)$ and parallel to $\vec{v} = \langle a, b\rangle$. Let $P = (x, y)$ be a general point on the line. Then $\overrightarrow{P_0P} = \langle x - x_0,\, y - y_0\rangle$ is parallel to $\vec{v} = \langle a, b\rangle$. Now we use the fact that two vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} = t\vec{v}$ for some scalar t.

Hence, $\overrightarrow{P_0P} = t\vec{v}$ for some scalar t,
i.e. $\langle x - x_0,\, y - y_0\rangle = t\langle a, b\rangle$
or $x - x_0 = ta, \quad y - y_0 = tb.$



Now, when we let t run through $(-\infty, \infty)$, the point (x, y, z) determined by the above formulas runs through the entire line. We say

$$\boxed{x = x_0 + ta, \quad y = y_0 + tb, \quad t \text{ a parameter}}$$

is the **parametric equation** for the line passing through $P_0 = (x_0,\, y_0)$, and parallel to $\vec{v} = \langle a, b\rangle$.

In the case of 2 dimension, he two parametric equations can be reduced to one single equation by eliminating the parameter t. Multiplying the first equation by b and subtracting the second equation multiplied by a we get

$$b(x - x_0) - a(y - y_0) = 0.$$

Notice that the vector $\vec{n} = \langle b, -a\rangle$ is perpendicular to $\vec{v}$, since the dot product $\vec{n} \cdot \vec{v} = 0$. Such vector is called **normal** vector for the line and therefore this equation is called the **point-normal** equation. It can be rewritten in the form

$$Ax + By + C = 0 \tag{1}$$

with $A = b$, $B = -a$, $C = ay_0 - bx_0$. If $B \neq 0$ we obtain the usual 'slope-y-intercept equation' by solving for y:

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

If $B = 0$ the line is vertical and has no 'slope-y-intercept equation'.

Remark. If we divide equation (1) by the length of the normal vector $\sqrt{A^2 + B^2}$ we obtain the so-called Hesse normal form

$$\frac{A}{\sqrt{A^2 + B^2}}x + \frac{B}{\sqrt{A^2 + B^2}}y + \frac{C}{\sqrt{A^2 + B^2}} = 0.$$

There are angles $\phi_1, \phi_2 = \frac{\pi}{2} - \phi_1$ such that $\frac{A}{\sqrt{A^2+B^2}} = \cos\phi_1$ and $\frac{B}{\sqrt{A^2+B^2}} = \sin\phi_1 = \cos\phi_2$. The angles ϕ_1, ϕ_2 are the ones formed by the normal and the x and y-axes, respectively. The absolute value $\frac{|C|}{\sqrt{A^2+B^2}}$ gives the distance of the line to the origin. (Try to prove this.)

We rewrite now the parametric equation as **vector equation**. Let O be the origin and denote $\vec{r} = \overrightarrow{OP}$, $\vec{r_0} = \overrightarrow{OP_0}$. Then $\overrightarrow{P_0P} = \vec{r} - \vec{r_0}$ and the line can be represented by

$$\boxed{\vec{r} - \vec{r_0} = t\vec{v}, \quad \text{or} \quad \vec{r} = \vec{r_0} + t\vec{v}, \quad t \ \text{ a parameter.}}$$

This equation can be used in 3 or, more generally, in any n-dimensional space.

In 3-space, with $P_0 = (x_0, y_0, z_0)$, $\vec{v} = \langle a, b, c\rangle$, $\vec{r} = \langle x, y, z\rangle$ and $\vec{r_0} = \langle x_0, y_0, z_0\rangle$, the parametric equations become:

$$\begin{aligned} x &= x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc \quad \text{— parametric equation.} \\ \vec{r} &= \vec{r_0} + t\vec{v} \quad \text{— vector equation.} \end{aligned}$$

Notice that eliminating the parameter t from these equations leaves us still with 2 equations. If $a \neq 0$ we find

$$\begin{aligned} y &= \frac{b}{a}x + y_0 - \frac{b}{a}x_0 \\ z &= \frac{c}{a}x + z_0 - \frac{c}{a}x_0 \end{aligned}$$

A 1-dimensional line in 3-space cannot be described by a single linear equation because one equation lowers the dimension only by 1, so 2 equations are needed[4].

[4]This wouldn't be true if we permitted non-linear equations. E.g. $y^2 + z^2 = 0$ describes the x-axis $\{y = 0, z = 0\}$.

Example 1. Find an equation of the line L which passes through $P_1(x_1, y_1, z_1)$ and $P_2(x_2,\, y_2,\, z_2)$ where $P_1 \neq P_2$.

Solution: If we can find a vector to which the line is parallel, then we can use the formulas discussed above to find the equation. Clearly $\overrightarrow{P_1P_2} = \langle x_2 - x_1,\, y_2 - y_1,\, z_2 - z_1\rangle$ is such a vector. Hence the parametric equation is

$$x = x_1 + t(x_2 - x_1),\; y = y_1 + t(y_2 - y_1),\; z = z_1 + t(z_2 - z_1).$$

In vector form

$$\vec{r} = \langle x_1,\, y_1,\, z_1\rangle + t\overrightarrow{P_1P_2}.$$

□

Example 2. Show that the vector $\vec{n} = \langle A, B\rangle$ is perpendicular to the line $Ax + By + C = 0$.

Proof: Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points on the line, i.e.

$$Ax_1 + By_1 + C = 0, \quad Ax_2 + By_2 + C = 0.$$

Then it suffices to show $\vec{n} = \langle A, B\rangle$ is perpendicular to $\overrightarrow{P_1P_2} = \langle x_2 - x_1,\; y_2 - y_1\rangle$, i.e. to show $\vec{n}\cdot\overrightarrow{P_1P_2} = 0$.

We calculate

$$\begin{aligned}\vec{n}\cdot\overrightarrow{P_1P_2} &= A(x_2 - x_1) + B(y_2 - y_1)\\ &= Ax_2 + By_2 - (Ax_1 + By_1)\\ &= -C - (-C)\\ &= 0\end{aligned}$$

and hence prove what we wanted. □

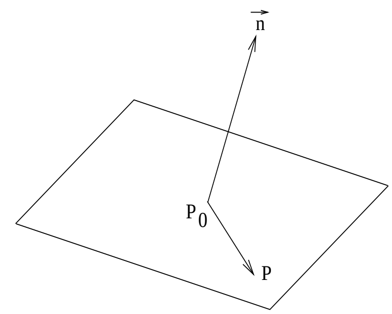
3.4 Planes in 3-space

A plane in 3-space is uniquely determined by a point $P_0 = (x_0, y_0, z_0)$ on it and a vector $\vec{n} = \langle A, B, C\rangle$ perpendicular to it. Such a vector is called a **normal vector** of the plane.

A general point $P = (x, y, z)$ is on the plane if and only if $\overrightarrow{P_0P}$ is perpendicular to $\vec{n}$, i.e. $\overrightarrow{P_0P} \cdot \vec{n} = 0$. Since $\overrightarrow{P_0P} = \langle x - x_0,\ y - y_0,\ z - z_0 \rangle$ and $\vec{n} = \langle A, B, C \rangle$, we have

$$\overrightarrow{P_0P} \cdot \vec{n} = A(x - x_0) + B(y - y_0) + C(z - z_0),$$

and the equation of the plane is



$$\boxed{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$$

This is called the **point-normal form** of the equation of the plane.

Note that on simplifying the above equation for the plane, we see that the equation has the form

$$Ax + By + Cz + D = 0 \quad \text{where} \quad D = -Ax_0 - By_0 - Cz_0. \tag{2}$$

Now, if $C \neq 0$ this can be rewritten as a 'graph' equation

$$z = -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C}. \tag{3}$$

Notice that one linear equation in 3-space lowers the dimension down by one to a 2-dimensional plane.

Remark. There is also a Hesse normal form for planes in 3-space. Divide equation (2) by $\sqrt{A^2 + B^2 + C^2}$. Then there are angles ϕ_1, ϕ_2, ϕ_3 such that $\frac{A}{\sqrt{A^2+B^2+C^2}} = \cos\phi_1$, $\frac{B}{\sqrt{A^2+B^2+C^2}} = \cos\phi_2$, $\frac{C}{\sqrt{A^2+B^2+C^2}} = \cos\phi_3$. The angles ϕ_1, ϕ_2, ϕ_3 are the angles between the normal and the respective axes. The number $\frac{|D|}{\sqrt{A^2+B^2+C^2}}$ is the distance of the plane to the origin $O(0, 0, 0)$.

Example 3. If $\langle A, B, C \rangle \neq \vec{0}$, then the equation

$$Ax + By + Cz + D = 0$$

describes a plane having normal vector $\vec{n} = \langle A, B, C \rangle$. (Compare with example 2 above: $Ax + By + C = 0$ is a line having normal vector $\vec{n} = \langle A, B \rangle$.)

Proof: Since $\langle A, B, C \rangle \neq 0$, at least one of the three components A, B, C is not zero. Suppose $C \neq 0$ (the other cases can be proved similarly). Then for any given x_0, y_0, we can find a unique z_0 such that the graph equation (3) and hence the plane equation is satisfied. Substituting this into

$$Ax + By + Cz + D = 0$$

we see the equation is equivalent to

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

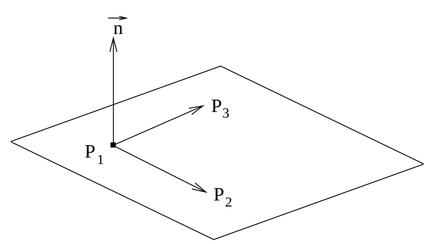
This is a point-normal form of the equation of a plane with normal vector $\vec{n} = \langle A, B, C\rangle$ passing through the point (x_0, y_0, z_0). $\square$.

Example 4. Find an equation of the plane through three different points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$ and $P_3(x_3, y_3, z_3)$.

Solution We need to find a normal vector for the plane.

With the three given points, we can form two vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$, both lying on the plane. According to the properties of cross product, $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is perpendicular to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$, hence to the plane.

Thus we can use $\vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ as a normal vector.



Since $\overrightarrow{P_1P_2} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1\rangle$ and $\overrightarrow{P_1P_3} = \langle x_3 - x_1,\ y_3 - y_1,\ z_3 - z_1\rangle$ and the equation can be written as

$$\langle x - x_1,\ y - y_1,\ z - z_1\rangle \cdot \vec{n} = 0$$

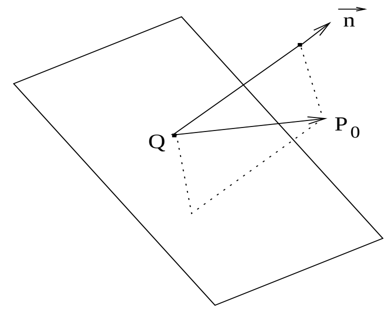
we can use property (4) for cross product to write the equation above in the following neat form:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

$\square$

Example 5. Find the distance d between $P_0(x_0, y_0, z_0)$ and the plane $Ax + By + Cz + D = 0$.

Solution We generalize the method used in Example 2, Lecture 2.



Let $Q = (x_1, y_1, z_1)$ be any point on the plane, and hence it satisfies

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

Then

$$d = \|\text{proj}_{\vec{n}} \overrightarrow{QP_0}\|,$$

where $\vec{n} = \langle A, B, C \rangle$ is a normal vector. We have

$$\begin{aligned}
\text{proj}_{\vec{n}} \overrightarrow{QP_0} &= \frac{\vec{n} \cdot \overrightarrow{QP_0}}{\|\vec{n}\|^2} \vec{n} \\
&= \frac{a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)}{a^2 + b^2 + c^2} \vec{n} \\
\|\text{proj}_{\vec{n}} \overrightarrow{QP_0}\| &= \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{A^2 + B^2 + C^2} \|\vec{n}\| \\
&= \frac{|Ax_0 + By_0 + Cz_0 - (Ax_1 + By_1 + Cz_1)|}{A^2 + B^2 + C^2} \sqrt{A^2 + B^2 + C^2} \\
&= \frac{|Ax_0 + Cy_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}} \qquad \text{(using } Ax_1 + By_1 + Cz_1 = -D)
\end{aligned}$$

Thus

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{|\sqrt{A^2 + B^2 + C^2}}.$$

$\square$

Lecture 4 Quadratic Surfaces

4.1 Quadratic Functions

Quadratic functions of one variable

$$q(x) = ax^2 + bx + c$$

with parameters $a \neq 0, b, c$ are the next simplest after linear functions. They are also often used to model processes in nature, science and economy. The Taylor formula tells us that a function $f(x)$ that has continuous derivatives up to third order can be approximated by a quadratic function in the following way:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + e(x, x_0) = ax^2 + bx + c + e(x, x_0),$$

where $a = \frac{f''(x_0)}{2}$, $b = f'(x_0) - f''(x_0)x_0$, $c = f(x_0) - f'(x_0)x_0 + \frac{f''(x_0)}{2}x_0^2$ and the error term

$$e(x, x_0) = \frac{1}{6}f'''(\xi)(x - x_0)^3$$

tends to 0 even faster than the quadratic function $(x - x_0)^2$ as x tends to x_0, i.e.

$$\lim_{x \to x_0} \frac{e(x, x_0)}{(x - x_0)^2} = 0.$$

This has been used to investigate critical points for local maxima and minima. Since the graph of a quadratic function is a parabola with an absolute minimum at its vertex if $a > 0$ or a maximum if $a < 0$ one concludes that a function that is two times differentiable has a local minimum or maximum at a critical point if $a = \frac{f''(x_0)}{2} > 0$ or $a = \frac{f''(x_0)}{2} < 0$. Indeed, if $f'(x_0) = 0$ (for x_0 being critical) we have

$$f(x) \approx \frac{1}{2}f''(x_0)(x - x_0)^2 + f(x_0).$$

The situation of many variables is similar, but slightly more complicated because of the large amount of possible quadratic terms. Let us look first into the case of two variables.

$$f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f.$$

We introduce a more systematic notation that uses indices and turns out to be even more useful in higher dimensions. Denote the first variable x by x_1 and the second variable y by x_2. Then denote $a = a_{11}$, $b = 2a_{12} = 2a_{21}$, $c = a_{22}$, $d = a_1$, $e = a_2$, $f = a$. The indices of the new a's tell us immediately how many and which factors x_1 or x_2 follow. Also we don't need to invent new letters for the huge amount of coefficients that occur in higher dimensions, and we may use sigma notation. The equation becomes

$$f(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + a_1x_1 + a_2x_2 + a = \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}x_ix_j + \sum_{i=1}^{2} a_ix_i + a.$$

In three dimensions we get

$$\begin{aligned} f(x_1, x_2, x_3) =& a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + \\ & + a_1x_1 + a_2x_2 + a_3x_3 + a \\ =& \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_ix_j + \sum_{i=1}^{3} a_ix_i + a. \end{aligned}$$

In higher dimension we only need to replace 3 as upper bound of the summation by the dimension n

$$f(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} a_ix_i + a.$$

4.2 Quadratic Curves

In Math102 we have studied implicit equations of curves $F(x, y) = 0$

A second-degree equation in x, y has the general form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{4}$$

It gives a quadratic curve[5]. By switching to an alternative coordinate system we can significantly simplify the equation (4). In a first step we will apply a rotation of the plane about the origin by a suitable angle ϕ in order to get rid of the mixed term Bxy.

Recall that such rotation is performed by a mapping

$$\begin{aligned} x &= \cos\phi\, u + \sin\phi\, v \\ y &= -\sin\phi\, u + \cos\phi\, v \end{aligned} \tag{5}$$

where x, y are the old coordinates and u, v are the new coordinates. You may check that the unit vectors $\vec{i} = \langle 1, 0\rangle$ and $\vec{j} = \langle 0, 1\rangle$ are mapped to a pair of mutually perpendicular unit vectors.

[5]Quadratic curves are often called 'conic sections' because they arise when a cone and a plane intersect in 3-space.

Theorem. *By a suitable rotation of the form (5) the equation (4) turns into*

$$A'u^2 + C'v^2 + D'u + E'v + F' = 0,$$

where A', C', D', E', F' are some new coefficients.

Proof. By plugging the expressions (5) for x, y into the equation (4) we find the coefficient in front of uv to be

$$B' = 2A\cos\phi\sin\phi + B(\cos^2\phi - \sin^2\phi) - 2C\sin\phi\cos\phi = (A - C)\sin 2\phi + B\cos 2\phi.$$

Here we used standard trigonometric formulae. To make this expression zero we need

$$\cot\phi = \frac{C - A}{B}.$$

If $B \neq 0$ this determines a unique angle ϕ between 0 and π. If $B = 0$ the mixed term didn't occur in the first place. □

Without developing the relevant theory (this will be left to Linear Algebra Pmth213) we notice that the new coefficients written as a matrix can be computed in the following way:

$$\begin{pmatrix} A' & \frac{B'}{2} \\ \frac{B'}{2} & C' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A & \frac{B}{2} \\ \frac{B}{2} & C \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$$

(You may check this by direct computation.) By our choice of ϕ we have $B' = 0$. In the classification of quadratic curves below you will see that the type of curve depends on whether A' and C' are positive, negative or zero. Notice that A' and C' are the eigenvalues[6] Of the matrix at the left hand side. It turns out that the old and new coefficient matrices have the same eigenvalues, determinant (which is the product of the eigenvalues) and trace (which is the sum of the entries at the main diagonal, hence the sum of the eigenvalues). It follows that the eigenvalues have the same sign if the determinant $A'C' = AC - \frac{B^2}{4}$ is positive and opposite signs if the determinant is negative. If both eigenvalues have the same sign then it is the sign of A.

Assume now that the quadratic equation has no mixed term. The second step depends on whether A, C are zero or not. If $A \neq 0$ and $C \neq 0$ we use a shift of the coordinate system to get rid of D and E. By completing the squares we find

$$Ax^2 + Cy^2 + Dx + Ey + F = A(x + \frac{D}{2A})^2 + C(y + \frac{E}{2C})^2 + F - \frac{D^2}{4A} - \frac{E^2}{4C} = 0.$$

[6]You may recall the concept of eigenvalues from Math101. A more thorough theory of eigenvalues will be developed in Pmth213.

Hence the equation takes the form

$$A(x - x_0)^2 + C(y - y_0)^2 = F',$$

where $x_0 = -\frac{D}{2A}$, $y_0 = -\frac{E}{2C}$, $F' = \frac{D^2}{4A} + \frac{E^2}{4C} - F$.

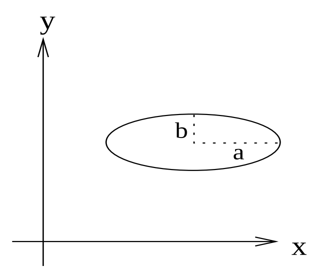
If $A = 0$ but $C \neq 0$ or $A \neq 0$ and $C = 0$ then we complete the square for the variable with the non-vanishing coefficient and leave the other unchanged to get

$$C(y - y_0)^2 + Dx = F' \text{ or } A(x - x_0)^2 + Ey = F'.$$

The following are the representative examples.

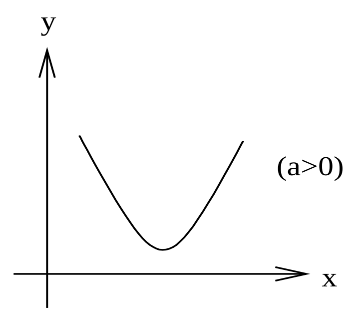
1. If $A > 0$, $B > 0$ and $F' > 0$ (or $A < 0$, $B < 0$ and $F' < 0$) we divide by F' and get the equation of an ellipse with half-axes $a = \sqrt{F'/A}$ and $b = \sqrt{F'/C}$ centred at (x_0, y_0).

   Ellipse: $\dfrac{(x - x_0)^2}{a^2} + \dfrac{(y - y_0)^2}{b^2} = 1$



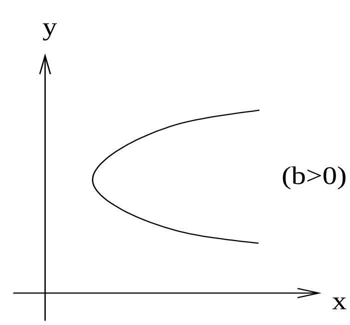
2. If $A \neq 0$ and $E \neq 0$ we may divide by E and find a vertical parabola with vertex (x_0, y_0).

   Parabola: $y - y_0 = a(x - x_0)^2$
   Here $a = -\frac{A}{E}$ and $y_0 = -\frac{F'}{E}$.



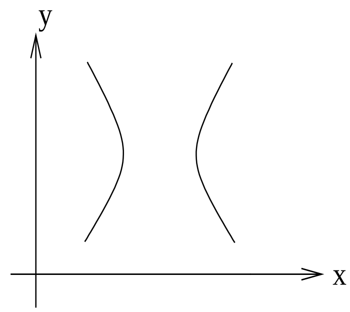
3. If $C \neq 0$ and $D \neq 0$ we may divide by D and find a horizontal parabola with vertex (x_0, y_0).

   $$x - x_0 = b(y - y_0)^2$$
   Here $b = -\frac{C}{D}$ and $x_0 = -\frac{F'}{D}$.



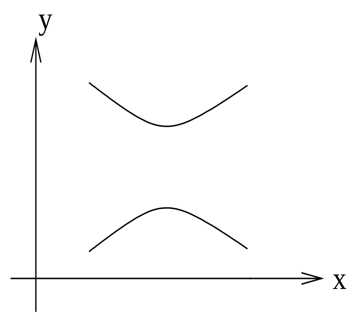
4. If $A > 0$, $C < 0$ and $F' > 0$ (or $A < 0$, $C > 0$ and $F' < 0$) we divide by F' and find a hyperbola.

   Hyperbola: $\dfrac{(x-x_0)^2}{a^2} - \dfrac{(y-y_0)^2}{b^2} = 1$
   Here $a = \sqrt{F'/A}$, $b = \sqrt{-F'/C}$.



5. If $A > 0$, $C < 0$ and $F' < 0$ (or $A < 0$, $C > 0$ and $F' > 0$) we divide by $-F'$ and find again a hyperbola.

   $$\frac{(x-x_0)^2}{a^2} - \frac{(y_0-y_0)^2}{b^2} = -1$$
   Here $a = \sqrt{-F'/A}$, $b = \sqrt{F'/C}$.



6. The remaining cases are in some sense degenerate and not really quadratic curves. We list them for the sake of completeness.

   (a) If $A > 0$, $C > 0$, $F' = 0$ (or $A < 0$, $C < 0$, $F' = 0$), the equation becomes $A(x-x_0)^2 + C(y-y_0)^2 = 0$, which is a single point. (Which?)

   (b) If $A > 0$, $C < 0$, $F' = 0$ (or $A < 0$, $C > 0$, $F' = 0$), the equation $A(x-x_0)^2 + C(y-y_0)^2 = 0$ describes a pair of crossing lines, namely, $y = y_0 \pm \sqrt{-A/C}(x-x_0)$

   (c) If $A > 0$, $C > 0$ and $F'/A < 0$ (or $A < 0$, $C < 0$ and $F'/A < 0$) we get $A(x-x_0)^2 + C(y-y_0)^2 = F'$, which is the empty set.

   (d) If $A \neq 0$, $C = 0$, $E = 0$ and $F'/A < 0$ (or $A = 0$, $C \neq 0$, $D = 0$ and $F'/C < 0$) we get $(x-x_0)^2 = F'/A$ (or $(y-y_0)^2 = F'/C$), which is the empty set.

   (e) If $A \neq 0$, $C = 0$, $E = 0$ and $F'/A > 0$ (or $A = 0$, $C \neq 0$, $D = 0$ and $F'/C > 0$) we get $x = x_0 \pm \sqrt{F'/A}$ (or $y = y_0 \pm \sqrt{F'/C}$), which is a pair of parallel lines.

   (f) If $A \neq 0$, $C = 0$, $E = 0$ and $F' = 0$ (or $A = 0$, $C \neq 0$, $D = 0$ and $F' = 0$) we get $(x-x_0)^2 = 0$ (or $(y-y_0)^2 = 0$), which is a single line.

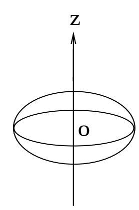
4.3 Quadratic Surfaces

A second-degree equation in x, y, z has the general form

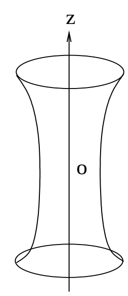
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

It gives a quadric surface (a quadric for short). A classification of these surfaces would be similar to the classification of quadratic curves carried out above, but requires more effort and time. We ignore here the degenerate cases and list the 6 important types:

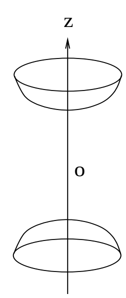
Ellipsoid: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



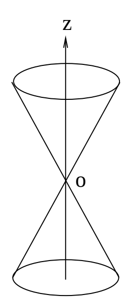
Hyperboloid of one sheet: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$



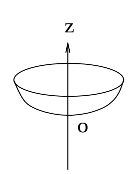
Hyperboloid of two sheets: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$



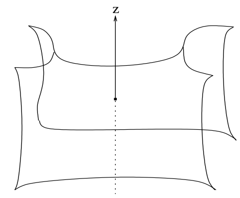
Elliptic cone: $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



Elliptic paraboloid: $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



Hyperbolic paraboloid: $z = \frac{y^2}{b_2} - \frac{x^2}{a^2}$



The most important quadratic surfaces for this unit are the paraboloids because they approximate the graphs of functions at its critical point and show whether the critical point is a maximum, a minimum or neither of them. Clearly,

$$(z - z_0) = A(x - x_0)^2 + C(y - y_0)^2$$

is a 'cup-like' elliptic paraboloid with vertex at (x_0, y_0, z_0), i.e (x_0, y_0, z_0) is a local minimum if both A, C are positive and an upside down cup-like elliptic paraboloid with vertex at (x_0, y_0, z_0), i.e (x_0, y_0, z_0) is a local maximum if both A, C are negative.

If A and C have opposite sign then the resulting surface is the 'saddle-like' hyperbolic paraboloid, i.e we have neither maximum nor minimum.

The cases when A or C (or both) are zero are indecisive.

If we started with a paraboloid of the form

$$(z - z_0) = A(x - x_0)^2 + 2B(x - x_0)(y - y_0) + C(y - y_0)^2$$

we can perform a rotation in the x, y plane about (x_0, y_0) exactly in the same way as in the classification of quadratic curves to get rid of the mixed term $2B(x-x_0)(y-y_0)$. Without doing this we can decide whether the critical point (x_0, y_0) is a maximum, minimum or saddle point by looking at the determinant

$$\det \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$$

and the coefficient A. IF $AC - B^2 > 0$ we have a maximum or minimum depending on whether $A < 0$ or $A > 0$. If $AC - B^2 < 0$ we have a saddle. The case $AC - B^2 = 0$ is indecisive.

4.4 Sketching Surfaces

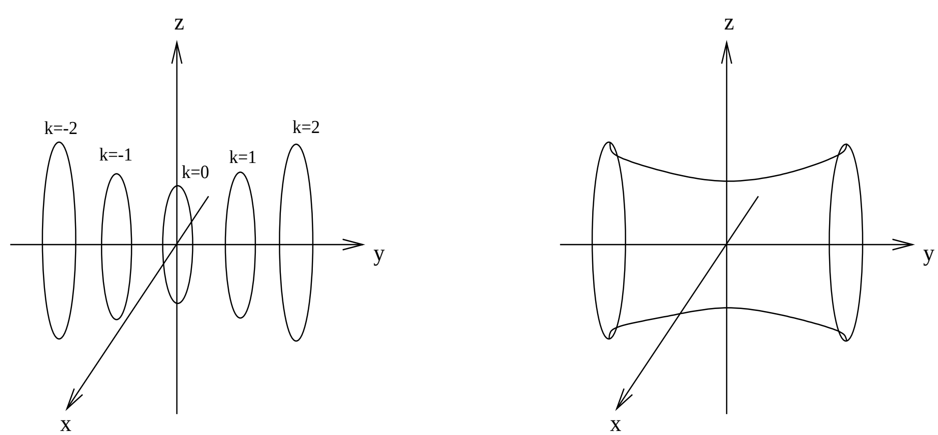
Sketching the graph of a surface is usually much more difficult than sketching a curve. One practical way to sketch a surface is by using a method called **mesh plot**: one builds up the shape of the surface using curves obtained by cutting the surface with planes parallel to the coordinate planes.

The curve of intersection of a surface with a plane is called the **trace** of the surface in the plane.

Let us now look at several examples to see how the mesh plot method can be used in sketching surfaces.

Example 1. Sketch the graph of the surface $x^2 - \dfrac{y^2}{4} + z^2 = 1$.

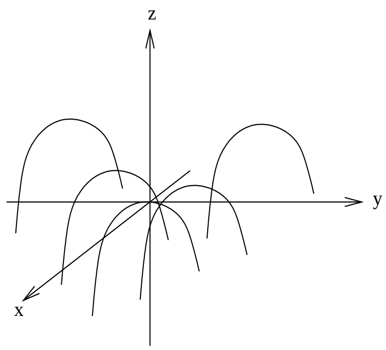
Solution: For any fixed y value, the equation can be written as $x^2 + z^2 = 1 + \dfrac{y^2}{4}$ which gives a circle (for fixed y) in the xz-plane. If we take $y = k$, and put the circle in the plane $y = k$ (by moving the one in the xz-plane and keeping the motion parallel to the y-axis), then when we choose different k, we obtain many circles which are parallel but with different radii (the circle on the plane $y = k$ has radius $\sqrt{1 + \dfrac{k^2}{4}}$).

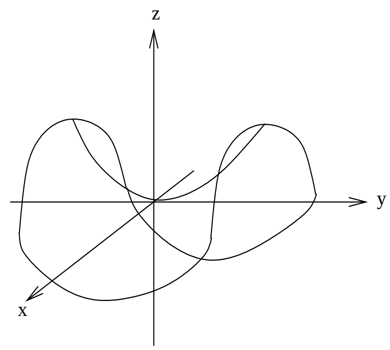


The above graphs show the circles ($k = -2, -1, 0, 1, 2$) and a sketch of the surface obtained by smoothly connecting these circles. □

Example 2. Sketch the graph of $z = \dfrac{y^2}{4} - \dfrac{x^2}{9}$.

Solution: Take $y = k$. Then the equation becomes $z - \frac{k^2}{4} = \frac{-1}{9}x^2$, which is of the form $(z - z_0) = ax^2$ and hence is a parabola. The following graphs show various parabolas obtained by varying k and a sketch of the surface by connecting these parabolas.





Example 3. Sketch the graph of $4x^2 + 4y^2 + z^2 + 8y - 4z = -4$.

Solution: We can change the equation into one of the standard forms to help us to know the shape of the surface. We use the completing squares method:

$$4x^2 + (4y^2 + 8y + 4) + (z^2 - 4z + 4) = -4 + 4 + 4,$$

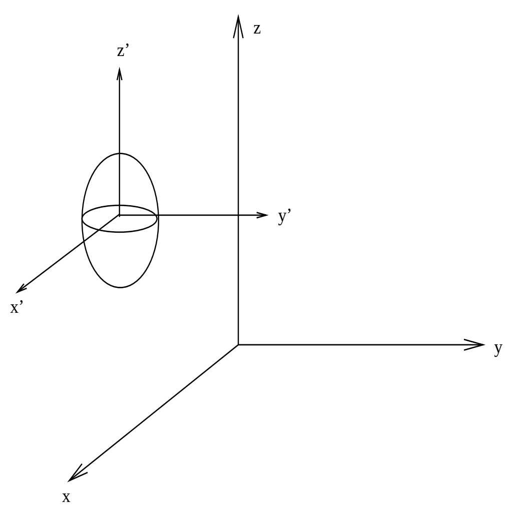
i.e.

$$4x^2 + 4(y+1)^2 + (z-2)^2 = 4,$$

or

$$x^2 + (y+1)^2 + \frac{(z-2)^2}{4} = 1.$$

Thus we know it represents an ellipsoid with center $(0, -1, 2)$.



Write $x' = x, y' = y+1$ and $z' = z-2$. Then the equation becomes $x'^2 + y'^2 + \dfrac{z'^2}{4} = 1$. We can use the mesh plot method to sketch the surface in the $x'y'z'$-coordinate system and then move it to the xyz-coordinate system according to the relationship $x' = x, y' = y + 1$ and $z' = z - 2$. $\square$

Note: $x' = x - x_0, y' = y - y_0, z' = z - z_0$ shows the x', y', z' axes are parallel to the x, y, z axes, respectively, and that the origin $(x', y', z') = (0, 0, 0)$ in the $x'y'z'$-system is at the point $(x, y, z) = (x_0, y_0, z_0)$ in the xyz-system.

Lecture 5 Vector-Valued Functions

5.1 Vector-Valued Functions and Their Graphs.

The function $y = f(x)$ assigns to each x from the domain $D \subset \mathbb{R}$ a real value y from the codomain $B \subset \mathbb{R}$, and hence it is a real-valued function. The equation of a line in 2-space has the form

$$x = x_0 + ta, \;\; y = y_0 + tb,$$

which can also be written in vector form

$$\vec{r} = \vec{r}_0 + t\vec{v},$$

where $\vec{r} = \langle x, y\rangle, \vec{r}_0 = \langle x_0, y_0\rangle, \vec{v} = \langle a, b\rangle$.

We can regard $\vec{r} = \vec{r}_0 + t\vec{v}$ as a function which assigns to each t a vector $\vec{r} = \vec{r}_0 + t\vec{v}$. Thus we have a vector-valued function here.

In general, a vector-valued function in 2-space has the form

$$\vec{r}(t) = \langle x(t), y(t)\rangle = x(t)\vec{i} + y(t)\vec{j},$$

in 3-space, a vector-valued function has the form

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}.$$

The real-valued functions $x(t), y(t)$ and $z(t)$ are called the **components** of $\vec{r}(t)$.

Clearly $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ is equivalent to

$$x = x(t), \;\; y = y(t), \;\; z = z(t).$$

We call this later system of equations the parametric form of the vector-valued function $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$.

Let $\vec{r}(t)$ be a vector-valued function in 2-space or 3-space defined on some closed interval $[a, b]$. Then the range of $\vec{r}$ describes a curve C, which we call the **graph** of $\vec{r}(t)$, or the graph of the equation $\vec{r} = \vec{r}(t)$. It might be helpful to visualise a vector function as the trajectory of a point moving in space in dependence of time t.

Clearly C has the parametric equation

$$\begin{array}{llll} x = x(t), & y = y(t) & & \text{(in 2-space)} \\ x = x(t), & y = y(t), & z = z(t) & \text{(in 3-space).} \end{array}$$

Example 1. Describe the graph of the function

$$\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}, \quad 0 \le t \le 2\pi.$$

Solution: The equation has the parametric form

$$x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi$$

which is just the parametric equation for the unit circle $x^2 + y^2 = 1$. Thus the graph is the unit circle. □

Example 2. Describe the graph of the vector-valued function

$$\vec{r} = (-2+t)\vec{i} + 3t\vec{j} + (5-4t)\vec{k}, \quad t \in \mathbb{R}.$$

Solution: The equation is equivalent to

$$x = -2 + t,\ y = 3t,\ z = 5 - 4t.$$

We know this is the parametric equation for the line passing through $(-2, 0, 5)$ and parallel to the vector $\langle 1, 3, -4\rangle$.

Or we can simply rewrite the vector equation as

$$\vec{r} = \langle -2, 0, 5\rangle + t\langle 1, 3, -4\rangle.$$

This is the equation of the line through $(-2, 0, 5)$ parallel to $\langle 1, 3, -4\rangle$. □

5.2 Limits, Continuity and Derivatives

The notions of limit, continuity and derivative for real-valued functions can all be passed to vector-valued functions through the components. Namely,

(1) Limit. For $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, define

$$\lim_{t \longrightarrow a} \vec{r}(t) = \left(\lim_{t \longrightarrow a} x(t)\right)\vec{i} + \left(\lim_{t \longrightarrow a} y(t)\right)\vec{j}.$$

For vectors in 3-space, the definition is similar. If at least one of the limits of the component functions does not exist, then we say $\lim_{t \longrightarrow a} \vec{r}(t)$ does not exist.

(2) Continuity. $\vec{r}(t)$ defined on some domain $D \subset \mathbb{R}$ is said to be **continuous** at $t_0 \in D$[7] if

[7]Sometimes the condition that $\vec{r}(t)$ is defined at t_0 is stated explicitly, but we take the point of view that continuity or non-continuity are notions that make sense only for points that are *a priori* in the domain of the function.

(a) $\lim\limits_{t \longrightarrow t_0} \vec{r}(t)$ exists and

(b) $\lim\limits_{t \longrightarrow t_0} \vec{r}(t) = \vec{r}(t_0)$.

Clearly $\vec{r}(t)$ is continuous at t_0 if and only if all its component functions are continuous at t_0.

(3) Derivative. The derivative of $\vec{r}(t_0)$ is defined by

$$\vec{r}\,'(t_0) = \frac{d}{dt}\vec{r}(t) = \lim_{h \longrightarrow 0} \frac{\vec{r}(t_0 + h) - \vec{r}(t_0)}{h}.$$

This can be reformulated as

$$\vec{r}(t) = \vec{r}(t_0) + \vec{r}\,'(t_0)(t - t_0) + \vec{e}(t, t_0),$$

where the error term $\vec{e}(t, t_0)$ has the property

$$\lim_{t \to t_0} \frac{||\vec{e}(t, t_0)||}{t - t_0} = 0, \tag{6}$$

i.e. $||\vec{e}(t, t_0)||$ tends to zero faster than $t - t_0$. In other words, is approximately a linear function

$$\vec{r}(t) \approx \vec{r}_0 + t\vec{v},$$

where $\vec{r}_0 = \vec{r}(t_0) - t_0\vec{r}\,'(t_0)$ and $\vec{v} = \vec{r}\,'(t_0)$. The error is small, even compared to the small quantity $t - t_0$.

We introduce here a convenient notation that we will use later, namely

$$\vec{e}(t, t_0) = \mathrm{o}(t - t_0),$$

to express (6).

More generally the o notation

$$f(t) = \mathrm{o}(g(t)) \text{ as } t \to t_0$$

has the meaning

$$\lim_{t \to t_0} \frac{|f(t)|}{|g(t)|} = 0.$$

Notice that this notation is always related to a variable (here) t approaching a certain limit. This limiting process has to be stated or to be known by default.

Example. $x^2 = \mathrm{o}(x)$ as $x \to 0$; $\cos x = \mathrm{o}(1)$ as $x \to \frac{\pi}{2}$; $x^m = \mathrm{o}(x^n)$ as $x \to 0$ if $m > n$, we say x^m tends to zero at a higher order that x^n.

The convenience of the o is the simplicity of its arithmetic. Without knowing the particular functions we can state:

1. $\mathrm{o}(x^n) \pm \mathrm{o}(x^n) = \mathrm{o}(x^n)$ (as $x \to 0$.)

2. $o(x^n) \times o(x^m) = o(x^{n+m})$.
3. $o(x^n) = o(x^m)$ if $n \geq m$.
4. If $f(x)$ is a bounded function defined in some neighbourhood of 0 then $f(x)\, o(x^n) = o(x^n)$.

Notice that $o(x)$ is not the notation for a function, but only expresses a property of a function.

In coordinates the derivative of a vector function is

$$\begin{aligned} \vec{r}\,'(t) &= x'(t)\vec{i} + y'(t)\vec{j} & \text{if } \vec{r}(t) &= x(t)\vec{i} + y(t)\vec{j}, \\ \vec{r}\,'(t) &= x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k} & \text{if } \vec{r}(t) &= x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}. \end{aligned}$$

The following theorem collects some of the most important properties of derivatives for vector-valued functions.

Theorem 1

(1) $\vec{r}\,'(t_0)$ is tangent to the curve $\vec{r} = \vec{r}(t)$ at $\vec{r}(t_0)$ and points in the direction of increasing parameter.

(2) (a) $(\vec{C})' = \vec{0}$, where $\vec{C}$ is a constant vector.
   (b) $(k\vec{r}(t))' = k\vec{r}\,'(t)$, where k is a scalar.
   (c) $[\vec{r}_1(t) \pm \vec{r}_2(t)]' = \vec{r}_1\,'(t) \pm \vec{r}_2\,'(t)$
   (d) $[f(t)\vec{r}(t)]' = f(t)\vec{r}\,'(t) + f'(t)\vec{r}(t)$ (product rule)
   (e) $\vec{r}(u(t))' = \vec{r}\,'(u(t))u'(t)$ (chain rule)

(3) (a) $[\vec{r}_1(t) \cdot \vec{r}_2(t)]' = \vec{r}_1\,'(t) \cdot \vec{r}_2(t) + \vec{r}_1(t) \cdot \vec{r}_2\,'(t)$ (product rule for dot product)
   (b) $[\vec{r}_1(t) \times \vec{r}_2(t)]' = \vec{r}_1\,'(t) \times \vec{r}_2(t) + \vec{r}_1(t) \times \vec{r}_2\,'(t)$ (product rule for cross product)

(4) If $||\vec{r}(t)||$ is constant for all t, then $\vec{r}(t) \cdot \vec{r}\,'(t) = 0$, i.e., $\vec{r}(t)$ and $\vec{r}\,'(t)$ are always perpendicular.

Proof: For (1) notice that the vector equation of a secant through $\vec{r}(t_0)$ and $\vec{r}(t_1)$ is

$$\vec{r}(t) = \vec{r}(t_0) + \vec{v}t$$

where

$$\vec{v} = \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}.$$

The tangent vector is the limiting position of secant vectors where t_1 approaches t_0. This limit is, as in single variable calculus, by definition the derivative $\vec{r}\,'(t_0)$.

The properties (2) (a)-(e) can be proved in the same way as in one variable calculus, or, even simpler, reduce to those properties.

Although (3) (a) also reduces to properties of one variable functions we will give a proof that shows the advantage of the o notation. Notice that our proof below works for any dimension. We have

$$\begin{aligned}\vec{r}_1(t) &= \vec{r}_1(t_0) + \vec{r}_1\,'(t_0)(t-t_0) + \mathrm{o}(t-t_0)\\ \vec{r}_2(t) &= \vec{r}_2(t_0) + \vec{r}_2\,'(t_0)(t-t_0) + \mathrm{o}(t-t_0)\end{aligned}$$

By forming the dots product of both sides and using dot product rules we get

$$\begin{aligned}\vec{r}_1(t)\cdot\vec{r}_2(t) =& \vec{r}_1(t_0)\cdot\vec{r}_2(t_0) + \vec{r}_1(t_0)\cdot\vec{r}_2\,'(t_0)(t-t_0) + \vec{r}_1\,'(t_0)(t-t_0)\cdot\vec{r}_2(t_0) +\\ &+ (\vec{r}_1(t_0) + \vec{r}_1\,'(t_0)(t-t_0))\cdot\mathrm{o}(t-t_0)\\ &+ (\vec{r}_2(t_0) + \vec{r}_2\,'(t_0)(t-t_0))\cdot\mathrm{o}(t-t_0)\\ =& \vec{r}_1(t_0)\cdot\vec{r}_2(t_0) + (\vec{r}_1(t_0)\cdot\vec{r}_2\,'(t_0) + \vec{r}_1\,'(t_0)\cdot\vec{r}_2(t_0))(t-t_0) + + \mathrm{o}(t-t_0)\end{aligned}$$

Here we have used that $\vec{r}_1(t_0)+\vec{r}_1\,'(t_0)(t-t_0)$ and $\vec{r}_2(t_0)+\vec{r}_2\,'(t_0)(t-t_0)$ are bounded in some neighbourhood of t_0, i.e.

$$\begin{aligned}||\vec{r}_1(t_0) + \vec{r}_1\,'(t_0)(t-t_0)|| \le M\\ ||\vec{r}_2(t_0) + \vec{r}_2\,'(t_0)(t-t_0)|| \le M\end{aligned}$$

and therefore, by the Cauchy-Schwarz inequalities,

$$\begin{aligned}|(\vec{r}_1(t_0) + \vec{r}_1\,'(t_0)(t-t_0))\cdot\mathrm{o}(t-t_0)| \le M||\,\mathrm{o}(t-t_0)|| = \mathrm{o}(t-t_0)\\ |(\vec{r}_2(t_0) + \vec{r}_2\,'(t_0)(t-t_0))\cdot\mathrm{o}(t-t_0)| \le M||\,\mathrm{o}(t-t_0)|| = \mathrm{o}(t-t_0).\end{aligned}$$

This line proves the claim.

The proof of (3) (b) is exactly the same with $\cdot$ replaced by $\times$.

To prove (4), let us first recall that $||\vec{r}(t)||^2 = \vec{r}(t)\cdot\vec{r}(t)$. Hence $\vec{r}(t)\cdot\vec{r}(t) \equiv C$ for all t.

It follows that

$$\begin{aligned}0 = (C)' &= (\vec{r}(t)\cdot\vec{r}(t))'\\ &= \vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) \text{ (using property (3)(a))}\\ &= 2\vec{r}(t)\cdot\vec{r}\,'(t)\end{aligned}$$

This implies $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$ for all t. The proof for (4) is complete. □

Example 3

(a) Find $\vec{r}\,'(t)$ for $\vec{r}(t) = (2+t)\vec{j} + (3t)\vec{j} + (4t-1)\vec{k}$

(b) Find $\vec{r}\,'(t)$ and $\vec{r}\,'(0)$ for $\vec{r}(t) = -\sin t\vec{i} + \cos t\vec{j}$.

Solution:

(a)
$$\begin{aligned}\vec{r}\,'(t) &= (2+t)'\vec{i} + (3t)'\vec{j} + (4t-1)'\vec{k}\\ &= \vec{i} + 3\vec{j} + 4\vec{k}\end{aligned}$$

(b)
$$\begin{aligned}\vec{r}\,'(t) &= (-\sin t)'\vec{i} + (\cos t)'\vec{j}\\ &= -\cos t\vec{i} - \sin t\vec{j}\\ \vec{r}\,'(0) &= -\cos 0\vec{i} - \sin 0\vec{j}\\ &= -\vec{i}\end{aligned}$$

□

Note: In part (a), the graph of $\vec{r}(t)$ is a straight line and $\vec{r}\,'(t)$ is a constant vector which is parallel to the line. Recall that property (1) in Theorem 1 says $\vec{r}\,'(t)$ is tangent to the line and our example confirms this. In part (b), $||\vec{r}(t)|| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$ and property (4) in Theorem 1 says $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$. Here we can also check directly:

$$\begin{aligned}\vec{r}(t)\cdot\vec{r}\,'(t) &= (-\sin t\vec{i} + \cos t\vec{j})\cdot(-\cos t\vec{i} - \sin t\vec{j})\\ &= (-\sin t)(-\cos t) + (\cos t)(-\sin t)\\ &= \sin t\cos t - \cos t\sin t = 0.\end{aligned}$$

Lecture 6 Integration of Vector-Valued Functions, Arc Length

6.1 Integration

Recall that if $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, then the derivative of $\vec{r}(t)$ is the vector function whose components are the derivatives of the components of $\vec{r}(t)$:

$$\vec{r}\,'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}.$$

The integral for $\vec{r}(t)$ is defined in the same fashion:

$$\int_a^b \vec{r}(t)dt = \int_a^b x(t)dt\ \vec{i} + \int_a^b y(t)dt\ \vec{j} + \int_a^b z(t)dt\ \vec{k}.$$

The fundamental theorem of calculus is also true for vector functions, i.e. if $\vec{R}(t)\colon [a,b] \to \mathbb{R}^3$ is a vector function such that

$$\vec{R}\,'(t) = \vec{r}(t)$$

then

$$\int_a^b \vec{r}(t)dt = \vec{R}(b) - \vec{R}(a).$$

Any such vector function is called antiderivative and the set of all antiderivatives is denoted by

$$\int \vec{r}(t)dt = \int x(t)dt\ \vec{i} + \int y(t)dt\ \vec{j} + \int z(t)dt\ \vec{k}.$$

Notice that any two antiderivatives differ by a constant vector $\vec{C} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$, i.e. an integration constant in each component.

Using the definition and properties for integrals of real-valued functions, one can prove easily the following properties:

(1) $\displaystyle\int C\vec{r}(t)dt = C\int \vec{r}(t)dt$

(2) $\displaystyle\int \left[\vec{r}_1(t) \pm \vec{r}_2(t)\right] dt = \int \vec{r}_1(t)dt \pm \int \vec{r}_2(t)dt$

(3) $\displaystyle\frac{d}{dt}\int \vec{r}(t)dt = \vec{r}(t)$

Example 1. Find $\int \vec{r}(t)dt$ and $\int_0^1 \vec{r}(t)dt$, where

$$\vec{r}(t) = t^2\vec{i} + (2t+1)\vec{j}$$

Solution:

$$\begin{aligned}
\int \vec{r}(t)dt &= \int t^2 dt\vec{i} + \int (2t+1)dt\vec{j} \\
&= \left(\frac{t^3}{3} + C_1\right)\vec{i} + \left(t^2 + t + C_2\right)\vec{j} \\
&= \frac{t^3}{3}\vec{i} + \left(t^2 + t\right)\vec{j} + \vec{C} \\
\int_0^1 \vec{r}(t)dt &= \left(\frac{t^3}{3}\vec{i} + \left(t^2 + t\right)\vec{j}\right)\Bigg|_0^1 = \frac{1}{3}\vec{i} + 2\vec{j}. \\
\text{or} \int_0^1 \vec{r}(t)dt &= \int_0^1 t^2 dt\vec{i} + \int_0^1 (2t+1)dt\vec{j} \\
&= \frac{t^3}{3}\bigg|_0^1\vec{i} + (t^2 + t)\bigg|_0^1\vec{j} \\
&= \frac{1}{3}\vec{i} + 2\vec{j}
\end{aligned}$$

□

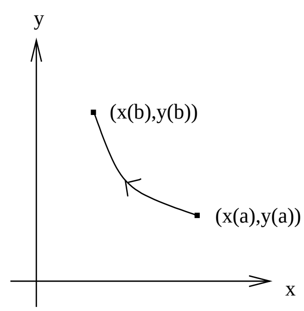
6.2 Arc Length

We know from first year calculus that if $x'(t)$ and $y'(t)$ are continuous, then the curve given by

$x = x(t),\ y = y(t),\ a \le t \le b$

has arc length

$$\boxed{L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\ dt.}$$

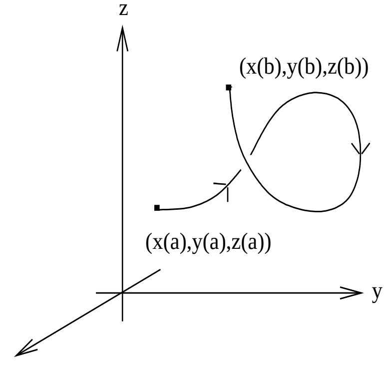


This formula generalizes to 3-space curves:
The arc length of the curve

$x = x(t), y = y(t), z = z(t), a \le t \le b$

is given by

$$\boxed{L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\ dt.}$$



Example 2. Find the arc length of the curve $\vec{r} = a\cos t\vec{i} + a\sin t\vec{j}, 0 \le t \le 2\pi$.

Solution:

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{a^2(\sin^2 t + \cos^2(t))}\, dt = \int_0^{2\pi} \sqrt{a^2}\, dt = 2\pi a. \quad \square \end{aligned}$$

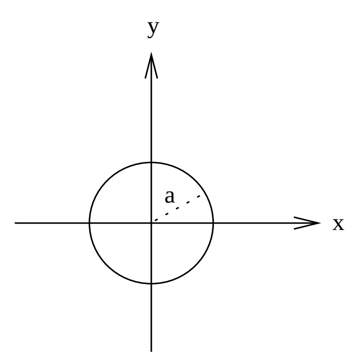
6.3 Arc Length as a Parameter

If we visualise a curve as the trajectory of a moving object it is clear that the same trajectory can be travelled at a different speed. This means that the same curve is represented in the parametric form with different parameters, and thus it has different parametric equations.

For example,

$$\begin{array}{lll} x(t) = a\cos t, & y(t) = a\sin t, & 0 \le t \le 2\pi \\ x(s) = a\cos(s^2), & y(s) = a\sin(s^2), & 0 \le s \le \sqrt{2\pi} \\ x(u) = a\cos(2\pi e^{-u}), & y(u) = a\sin(2\pi e^{-u}), & 0 \le u < \infty \end{array}$$

all represent the same curve: a circle with center $(0,0)$ and radius a.

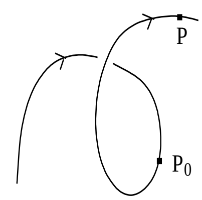


To avoid such ambiguity, it is desirable to have a universal parameter for the parametric equations. This can be done by stipulating that we travel the curve with speed 1, i.e. 1 length unit i 1 unit of time. In other words, the arc length is used as parameter. Let us now see how this can be done.

Let C be a given smooth curve. We first introduce the arc length parameter using the following three steps:

(1) choose a point P_0 on the curve, called a **reference point**;

(2) Starting from P_0, choose one direction along the curve to be the positive direction and the other to be the negative direction;



(3) If P is a point on C, let s be the "signed" arc length along C from P_0 to P, where s is positive if P is in the positive direction from P_0 and s is negative if P is in the negative direction from P_0.

Let us suppose that C is initially given by the parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t),$$

and $P_0 = (x(t_0), y(t_0), z(t_0)), P = (x(t), y(t), z(t))$, and the positive direction of C is the direction of increasing t.

Then we know from the last section that

$$s = \int_{t_0}^{t} \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2}\, du$$

This gives s as a function of t. Differentiating we obtain

$$\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}.$$

Example 3. Find parametric equations for $x = a\cos t, y = a\sin t, 0 \le t \le 2\pi$, using arc length s as a parameter, with reference point for s being $(0, a)$ in the xy-plane.

Solution: The point $(0, a)$ corresponds to $t = \frac{\pi}{2}$ on the curve.

Therefore,

$$\begin{aligned} s &= \int_{\frac{\pi}{2}}^{t} \sqrt{x'(u)^2 + y'(u)^2}\; du \\ &= \int_{\frac{\pi}{2}}^{t} \sqrt{(-a\sin u)^2 + (a\cos u)^2}\; \cdot du \\ &= \int_{\frac{\pi}{2}}^{t} \sqrt{a^2}\, dt = a\left(t - \frac{\pi}{2}\right). \end{aligned}$$

Solving for t from $s = a\left(t - \frac{\pi}{2}\right)$ we obtain

$$t = \frac{s}{a} + \frac{\pi}{2}.$$

As t varies from 0 to 2π, s varies from $-\frac{\pi}{2}a$ to $\frac{3}{2}\pi a$. Hence, the parametric equations in s are

$$x = a\cos\left(\frac{s}{a} + \frac{\pi}{2}\right), y = a\sin\left(\frac{s}{a} + \frac{\pi}{2}\right), -\frac{\pi}{2}a \le s \le \frac{3}{2}\pi a.$$

Or, in vector form

$$\vec{r} = a\cos\left(\frac{s}{a} + \frac{\pi}{2}\right)\vec{i} + a\sin\left(\frac{s}{a} + \frac{\pi}{2}\right)\vec{j}, -\frac{\pi}{2}a \le s \le \frac{3}{2}\pi a.$$

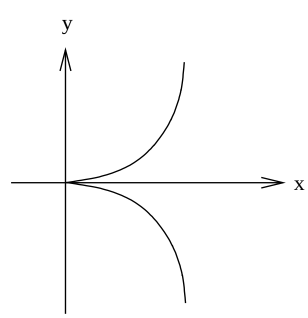
Lecture 7 Unit Tangent and Normal Vectors

7.1 Smooth curves

We call a curve **smooth** if it has a tangent at each point and the slope of the tangent changes in a continuous way from point to point. In particular this means that the curve has no edges or cusps.

If a curve in the xy-plane is described as the graph of a function $y = f(x)\colon (a, b) \to \mathbb{R}$ that has continuous derivative then the curve is a smooth function. However, the following example shows that if $\vec{r}(t)$ has continuous derivative, the curve $\vec{r} = \vec{r}(t)$ may not be smooth.

Example 1. Let $\vec{r}(t) = t^2\vec{i} + t^3\vec{j}$, then $\vec{r}\,'(t) = 2t\vec{i} + 3t^2\vec{j}$ is continuous. The parametric form for the graph of $\vec{r}(t)$ is $x = t^2$, $y = t^3$, which is equivalent to $x = |y|^{\frac{2}{3}}$ and it represents a curve which is not smooth at $(0, 0)$.



Indeed, the direction of the tangent at $\vec{r}(t_0)$ is given by $\langle x'(t_0), y'(t_0)\rangle = \langle 2t, 3t^2\rangle$. For $t_0 = 0$, the cusp point, this is the zero vector which does not give any direction. Let us try to look a this curve as a graph. Since the vertical line test gives two intersection points, the curve is not the graph of a function $y = f(x)$ but it is the graph of a function $x = g(y) = \sqrt[3]{y^2}$. Notice that the derivative of $g(y)$ is, according to the chain rule

$$g'(y) = \frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{2t}{3t^2}$$

and does not exist at $t = 0$, i.e. $(x, y) = (0, 0)$. The vanishing denominator $3t^2$ is clearly the problem here.

If we have a parametric curve $\langle x(t), y(t)\rangle$ such that $\frac{dy}{dt}(t_0) \neq 0$ then the function $y(t)$ has an inverse $t = h(y)$ on some (possibly very small) interval containing $y_0 = y(t_0)$. In this case the derivative $h'(y_0) = \frac{1}{y'(t_0)}$. This is the statement of the

inverse function theorem. It follows that $x = g(y) = x(h(y))$ is the desired graph equation.

This motivates the following definition.

A parametric curve $\vec{r} = \vec{r}(t),\ a \le t \le b$, is called **smooth** if

$$\boxed{\textbf{(a)}\ \vec{r}\,'(t) \text{ exists, } \ \textbf{(b)}\ \vec{r}\,'(t) \text{ is continuous in } (a,b) \text{ and } \ \textbf{(c)}\ \vec{r}\,'(t) \neq \vec{0} \text{ for all } t \text{ in } (a,b)}$$

If $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$, then (a)-(c) above are equivalent to

(a') $x'(t),\ y'(t),\ z'(t)$ all exist,

(b') $x'(t),\ y'(t),\ z'(t)$ are all continuous in (a,b),

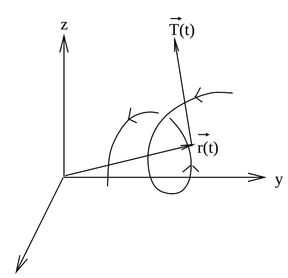
(c') at least one of $x'(t)$, $y'(t)$, $z'(t)$ is different from 0 for any t in (a,b).

7.2 Unit Tangent Vector

If $C : \vec{r} = \vec{r}(t)$ is a smooth curve (in 2-space or 3-space), then

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$$

is well-defined (why?), and it is a vector tangent to the curve with unit length (recall property (1) in Theorem 1 of Lecture 5). Such a vector is called a **unit tangent vector** to C at t.



Example 2. Find a unit tangent vector to the curve $\vec{r} = t^2\vec{i} + t^3\vec{j}$ at $t = t_0$, where $t_0 \neq 0$.

Solution: $\vec{T}(t_0) = \frac{\vec{r}\,'(t_0)}{\|\vec{r}\,'(t_0)\|}$ is such a vector.

We calculate

$$
\begin{aligned}
\vec{r}\,'(t_0) &= 2t_0\vec{i} + 3t_0^2\vec{j}, \\
\|\vec{r}\,'(t_0)\| &= \sqrt{(2t_0)^2 + (3t_0^2)^2} = \sqrt{4t_0^2 + 9t_0^4} \\
&= |t_0|\sqrt{4 + 9t_0^2}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\vec{T}(t_0) &= \frac{2}{\sqrt{4+9t_0^2}}\vec{i} + \frac{3t_0}{\sqrt{4+9t_0^2}}\vec{j} \qquad \text{if } \ t_0 > 0 \\
\vec{T}(t_0) &= -\left(\frac{2}{\sqrt{4+9t_0^2}}\vec{i} + \frac{3t_0}{\sqrt{4+9t_0^2}}\vec{j}\right) \qquad \text{if } \ t_0 < 0.
\end{aligned}
$$

□

Recall that if $\vec{r} = \vec{r}(t)$ is a smooth curve, then we can introduce the arc length parameter s at a reference point, say, at $t = t_0$.

$$
s = s(t) = \int_{t_0}^{t} ||\vec{r}\,'(u)||du
$$

From the properties of integrals,

$$
s'(t) = \frac{d}{dt}\int_0^t ||\vec{r}\,'(u)||du = ||\vec{r}\,'(t)||
$$

If the initial parameter t is already the arc length, i.e. $t = s$, then $s'(t) = t' = 1$ and thus, from the above identity,

$$
||\vec{r}\,'(t)|| = s'(t) = 1
$$

i.e. $||\vec{r}\,'(s)|| = 1$. Thus, in the arc length parameter, we always have

$$
\vec{T}(s) = \frac{\vec{r}\,'(s)}{||\vec{r}\,'(s)||} = \vec{r}\,'(s), \quad \text{or} \quad \vec{T} = \frac{d\vec{r}}{ds}.
$$

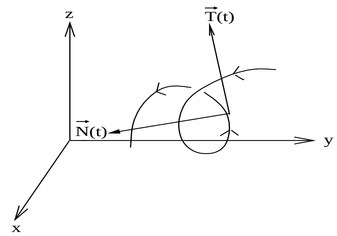
7.3 Principal Normal Vector

Recall that if $\|\vec{r}(t)\| \equiv C$, then $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$, i.e. $\vec{r}(t)$ and $\vec{r}\,'(r)$ are perpendicular: $\vec{r}(t)\perp\vec{r}\,'(t)$. Applying this result to $\vec{T}(t)$ (note $\|\vec{T}(t)\| \equiv 1$), we see $\vec{T}(t)\perp\vec{T}'(t)$. This

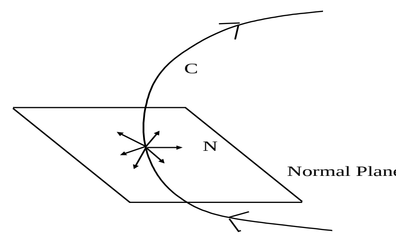
implies that $\vec{T}'(t)$ is a normal vector to the curve $\vec{r} = \vec{r}(t)$.
When $\vec{T}'(t) \neq \vec{0}$, we can normalise it to obtain

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|},$$

called the **principal unit normal vector** to C.



To a smooth curve, at any given point on it, there is a normal plane and any vector on the normal plane is a normal vector to the curve. $\vec{N}(t)$ is a special normal vector and has several useful properties in practical problems. However, we are not going to persue this matter in this unit.



Example 3. Fine $\vec{T}(t)$ and $\vec{N}(T)$ for the circular helix

$$\vec{r}(t) = a\cos t\vec{i} + a\sin t\vec{j} + ct\vec{k} \qquad \text{where} \quad a > 0,\ c > 0.$$

Solution:

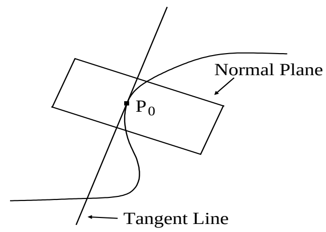
$$\begin{aligned}
\vec{r}\,'(t) &= -a\sin t\vec{i} + a\cos t\vec{j} + c\vec{k} \\
\|\vec{r}\,'(t)\| &= \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} \\
&= \sqrt{a^2(\sin^2 t + \cos^2 t) + c^2} \\
&= \sqrt{a^2 + c^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{T}(t) &= \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{-a\sin t}{\sqrt{a^2+c^2}}\vec{i} + \frac{a\cos t}{\sqrt{a^2+c^2}}\vec{j} + \frac{c}{\sqrt{a^2+c^2}}\vec{k} \\
\vec{T}'(t) &= \frac{-a\cos t}{\sqrt{a^2+c^2}}\vec{i} - \frac{a\sin t}{\sqrt{a^2+c^2}}\vec{j} + 0\vec{k} \\
\|\vec{T}'(t)\| &= \sqrt{\left(\frac{-a\cos t}{\sqrt{a^2+c^2}}\right)^2 + \left(\frac{-a\sin t}{\sqrt{a^2+c^2}}\right)^2} \\
&= \sqrt{\frac{a^2}{a^2+c^2}} = \frac{a}{\sqrt{a^2+c^2}} \\
\vec{N}(t) &= \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = -\cos t\vec{i} - \sin t\vec{j}.
\end{aligned}$$

□

Let $\vec{r} = \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ be a smooth curve in 3-space. Then at any given point $P_0 = (x_0,\ y_0,\ z_0)$ on the curve, say, $(x_0,\ y_0,\ z_0) = (x(t_0),\ y(t_0),\ z(t_0))$, we have a tangent line and a normal plane.

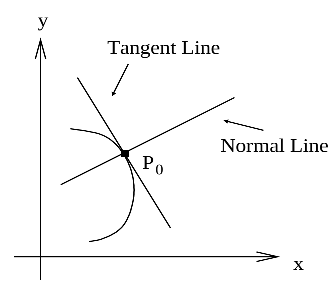


As $\vec{r}\,'(t_0) = x'(t_0)\vec{i} + y'(t_0)\vec{j} + z'(t_0)\vec{k}$ is parallel to the tangent line, and is normal to the normal plane, we have the following equations:

$$\boxed{\begin{array}{l}\text{Tangent line: } x = x_0 + x'(t_0)t,\ y = y_0 + y'(t_0)t,\ z = z_0 + z'(t_0)t \\ \text{Normal plane: } x'(t_0)(x - x_0) + y'(t_0)(y - y_0) + z'(t_0)(z - z_0) = 0.\end{array}}$$

If $\vec{r} = \vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ is a smooth curve in 2-space, then we have similar results but this time we have normal line instead of normal plane to the curve. The equations are

$$\boxed{\begin{array}{l}\text{Tangent line: } x = x_0 + x'(t_0)t,\ y = y_0 + y'(t_0)t \\ \text{Normal line: } x'(t_0)(x - x_0) + y'(t_0)(y - y_0) = 0\end{array}}$$



Example 4. Find equations for the tangent line and normal plane to the curve $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + t\vec{k}$ at $t = \frac{\pi}{2}$.

Solution:

$$\vec{r}\,'(t) = -\sin t\vec{i} + \cos t\vec{j} + \vec{k}.$$

At $t = t_0 = \frac{\pi}{2}$,

$$\begin{aligned}\vec{r}(t_0) &= \cos(\frac{\pi}{2})\vec{i} + \sin(\frac{\pi}{2})\vec{j} + \frac{\pi}{2}\vec{k} \\ &= \vec{j} + \frac{\pi}{2}\vec{k}\end{aligned}$$

i.e. $x_0 = 0,\ y_0 = 1,\ z_0 = \frac{\pi}{2}$.

$$\begin{aligned}\vec{r}\,'(t_0) &= -\sin\frac{\pi}{2}\vec{i} + \cos\frac{\pi}{2}\vec{j} + \vec{k} \\ &= -\vec{i} + \vec{k}\end{aligned}$$

i.e. $x'(t_0) = -1,\ y'(t_0) = 0,\ z'(t_0) = 1.$

Thus the tangent line is

$$x = -t,\ y = 1,\ z = \frac{\pi}{2} + t;$$

the normal plane is

$$-1(x-0) + 0(y-1) + 1(z - \frac{\pi}{2}) = 0,$$

i.e.

$$-x + z - \frac{\pi}{2} = 0.$$

□.

Example 5. If the 3-space curve $\vec{r} = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$ lies in a plane $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$, show that $\vec{T}(t)$ and $\vec{N}(t)$ also lie in this plane.

Proof: To show $\vec{T}(t)$ and $\vec{N}(t)$ lie in the plane, it suffices to show they are perpendicular to the normal $\vec{n} = \langle a, b, c\rangle$ of the plane, i.e. to show

$$\vec{n}\cdot\vec{T}(t) = 0, \quad \vec{n}\cdot\vec{N}(t) = 0.$$

As the curve lies on the plane, the general point $(x(t),\ y(t),\ z(t))$ on the curve satisfies the equation of the plane:

$$a(x(t) - x_0) + b(y(t) - y_0) + c(z(t) - z_0) = 0.$$

That is

$$\vec{n}\cdot(\vec{r}(t) - \vec{r}_0) = 0, \quad \text{where } \vec{r}_0 = \langle x_0,\ y_0,\ z_0\rangle.$$

Differentiate the above identity with respect to t, we have

$$[\vec{n}\cdot(\vec{r}(t) - \vec{r}_0)]' = 0' = 0$$

i.e.

$$\vec{n}\cdot\vec{r}\,'(t) = 0.$$

Hence

$$\vec{n}\cdot\vec{T}(t) = \vec{n}\cdot\frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{1}{\|\vec{r}\,'(t)\|}\,(\vec{n}\cdot\vec{r}\,'(t)) = 0.$$

This shows $\vec{T}$ lies in the plane.

Now differentiate $\vec{n}\cdot\vec{T}(t) = 0$ we deduce similarly $\vec{n}\cdot\vec{T}'(t) = 0$, which implies $\vec{n}\cdot\vec{N}(t) = 0.$ □.

Lecture 8 Curvature

Intuitively, curvature, say of a road bend, is reflected by how much you need to turn the steering wheel when driving along that bend. If you keep the steering wheel fixed the path along which you drive is a circle. A good measure for the curvature of a circle is the reciprocal of its radius.

For another approach to curvature we could look at the rate of change of the direction of the tangent. Bigger curvature would relate to a more rapid change of the direction. The derivative of the unit tangent vector $T(t)$

$$\frac{dT}{dt}$$

would depend on the parametrisation of the curve (the travel speed along the curve). Indeed, if τ was a another parameter, so that $t = t(\tau)$ depends on τ then

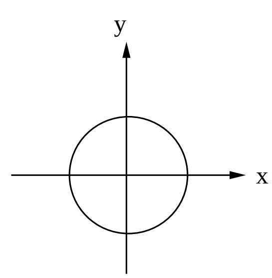
$$\frac{dT}{d\tau} = \frac{dT}{dt}\frac{dt}{d\tau}.$$

To avoid such dependence on the choice of parameter we use the arc length parameter s. Now, as s varies, only the direction of $\vec{T}(s)$ changes (its length is always 1). The derivative of $\vec{T}(s)$, namely $\frac{d\vec{T}}{ds}$, then measures the rate of change of direction of the curve $\vec{r} = \vec{r}(s)$. The **curvature** of the curve is defined by

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|.$$

Example 1 Show that the two concepts of curvature introduced above are the same for the circle of radius R: $\vec{r} = R(\cos t\,\vec{i} + \sin t\,\vec{j})$.

First change to arc length parameter. We choose the point $t = 0$ as a reference point and obtain



$$\begin{aligned} s &= \int_0^t ||\vec{r}\,'(u)||du \\ &= \int_0^t \sqrt{(-R\sin u)^2 + (R\cos u)^2}du \\ &= \int_0^t Rdu = Rt \end{aligned}$$

Therefore $\vec{r}(s) = R(\cos\frac{s}{R}\vec{i} + \sin\frac{s}{R}\vec{j})$ and

$$\begin{aligned}
\vec{T}(s) &= \vec{r}\,'(s) = -\sin\frac{s}{R}\vec{i} + \cos\frac{s}{R}\vec{j} \\
\vec{T}'(s) &= -\frac{1}{R}\cos\frac{s}{R}\vec{i} - \frac{1}{R}\sin\frac{s}{R}\vec{j} \\
\kappa(s) = ||\vec{T}'(s)|| &= \sqrt{(-\frac{1}{R}\cos\frac{s}{R})^2 + (-\frac{1}{R}\sin\frac{s}{R})^2} = \frac{1}{R}.
\end{aligned}$$

The curvature is always $\frac{1}{R}$, as expected. □

Changing to arc length parameter according to the formula

$$s = \int_0^t ||\vec{r}\,'(u)||du$$

could be difficult to use, as the integration could be hard to do.

For example, if $\vec{r}(t) = 2\cos t\vec{i} + 3\sin t\vec{j}$, then

$$||\vec{r}\,'(t)|| = \sqrt{(-2\sin t)^2 + (3\cos t)^2} = \sqrt{4\sin^2 t + 9\cos^2 t}$$

and it is difficult to find

$$\int_0^t \sqrt{4\sin^2 u + 9\cos^2 u}\,du.$$

The following theorem gives us some practical formula to calculate the curvature without changing to arc length parameter.

Theorem 1. If $\vec{r}(t)$ is a smooth vector-valued function in 2-space or 3-space, and if $\vec{T}'(t)$, $\vec{r}\,''(t)$ exist, then

$$\boxed{\text{(a) } \kappa = \kappa(t) = \frac{||\vec{T}'(t)||}{||\vec{r}\,'(t)||}, \qquad \text{(b) } \kappa = \kappa(t) = \frac{||\vec{r}\,'(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3}.}$$

Note. In (b), if $\vec{r}(t)$ is in 2-space, i.e. $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, then we write it as $\vec{r}(t) = x(t)\vec{i} + yt\vec{j} + 0\vec{k}$ in order to be able to perform the cross product (remember cross product is defined only for 3-space vectors).

Proof.

(a) Recall that from
$$s(t) = \int_{to}^{t} ||\vec{r}\,'(u)||du,$$
one always has $s'(t) = ||\vec{r}\,'(t)||$. Now by the chain rule
$$\vec{T}'(t) = \frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt} = \frac{d\vec{T}}{ds}||\vec{r}\,'(t)||.$$
Hence,
$$||\vec{T}'(t)|| = \left\|\frac{d\vec{T}}{ds}\right\| \, ||\vec{r}\,'(t)||,$$
and dividing by $||\vec{r}\,'(t)||$, we obtain
$$\kappa = \left\|\frac{d\vec{T}}{ds}\right\| = \frac{||\vec{T}'(t)||}{||\vec{r}\,'(t)||}.$$
This proves (a)

(b) Since $||\vec{T}(t)|| \equiv 1$, by property (4) in Theorem 1, Lecture 5, $\vec{T}(t) \perp \vec{T}'(t)$, i.e. the angle between these two vectors is $\theta = \frac{\pi}{2}$. It follows, as $||\vec{T}(t)|| = 1$ and $\sin\theta = \sin\frac{\pi}{2} = 1$,
$$||\vec{T}(t) \times \vec{T}'(t)|| = ||\vec{T}(t)|| \cdot ||\vec{T}'(t)|| \cdot \sin\theta = ||\vec{T}'(t)||.$$
By definition,
$$\vec{T}(t) = \left(\frac{1}{||\vec{r}\,'(t)||}\right)\vec{r}\,'(t).$$
Therefore
$$\begin{aligned}
\vec{T}'(t) &= \left(\frac{1}{||\vec{r}\,'(t)||}\right)\vec{r}\,''(t) + \left(\frac{1}{||\vec{r}\,'(t)||}\right)'\vec{r}\,'(t) \\
\vec{T}(t) \times \vec{T}'(t) &= \left(\frac{1}{||\vec{r}\,'(t)||}\right)\vec{T}(t) \times \vec{r}\,''(t) + \left(\frac{1}{||\vec{r}\,'(t)||}\right)'\vec{T}(t) \times \vec{r}\,'(t). \\
&= \left(\frac{1}{||\vec{r}\,'(t)||}\right)^2 \vec{r}\,'(t) \times \vec{r}\,''(t) + \left(\frac{1}{||\vec{r}\,'(t)||}\right)'\left(\frac{1}{||\vec{r}\,'(t)||}\right)\vec{r}\,'(t) \times \vec{r}\,'(t). \\
&= \left(\frac{1}{||\vec{r}\,'(t)||}\right)^2 \vec{r}\,'(t) \times \vec{r}\,''(t).
\end{aligned}$$

Note that we have used $\vec{r}\,'(t) \times \vec{r}\,'(t) = \vec{0}$.

Now we use the formula proved in part (a) together with what we have been proved above, namely
$$||\vec{T}'(t)|| = ||\vec{T}(t) \times \vec{T}'(t)||$$

and

$$\vec{T}(t) \times \vec{T}'(t) = \frac{\vec{r}\,'(t) \times \vec{r}\,''(t)}{||\vec{r}\,'(t)||^2}.$$

and obtain

$$\begin{aligned}\kappa &= \frac{||\vec{T}'(t)||}{||\vec{r}\,'(t)||} = \frac{||\vec{T}(t) \times \vec{T}'(t)||}{||\vec{r}\,'(t)||} \\ &= \frac{\left|\left|\frac{\vec{r}\,'(t)\times\vec{r}\,''(t)}{||\vec{r}\,'(t)||^2}\right|\right|}{||\vec{r}\,'(t)||} \\ &= \frac{||\vec{r}\,'(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3}. \quad \square\end{aligned}$$

We show now that the two approaches to curvature are equivalent for all plain curves. Without loss of generality we assume that a curve is given as a graph $y = f(x)$ and the point at which we want to compute the curvature is the origin. We choose the coordinate system in such a way that the x-axis is tangent to the curve at the origin. Thus, $f(0) = 0$ and $f'(0) = 0$ and therefore the curve has the equation

$$y = f(x) = \frac{1}{2}f''(0)x^2 + \mathrm{o}(x^2).$$

A (concave up) circle passing through the origin and tangent to the x axis has the equation

$$x^2 + (y - R)^2 = R^2$$

or

$$y = R - \sqrt{R^2 - x^2} = R - R(1 = \frac{x^2}{2R^2}) + \mathrm{o}(x^2) = \frac{x^2}{2R} + \mathrm{o}(x^2).$$

Hence the radius of the best fitting circle is $R = \frac{1}{f''(0)}$[8].

On the other hand,

$$\begin{aligned}\vec{r} &= \langle x, f(x), 0\rangle \\ \vec{r}\,' &= \langle 1, f'(x), 0\rangle \\ \vec{r}\,'' &= \langle 0, f''(x), 0\rangle\end{aligned}$$

and therefore

$$\begin{aligned}\vec{r}\,' \times \vec{r}\,'' &= \langle 0, 0, f''(x)\rangle \\ ||\vec{r}\,' \times \vec{r}\,''(0)|| &= \sqrt{(f''(0))^2} = |f''(0)| \\ ||\vec{r}\,'(0)|| &= \sqrt{1 + (f'(x))^2} = 1 \\ \kappa(0) = |f''(x)| &= \frac{1}{R}\end{aligned}$$

[8]If $f''(0) < 0$ the best fitting circle would be concave down with equation $x^2 + (y + R)^2 = R^2$. Then a similar computation shows that $R = -\frac{1}{f''(0)} = \frac{1}{|f''(0)|}$.

which shows that the curvature is the reciprocal of the radius of the best fitting circle.

Example 2. Find $\kappa(t)$ for $\vec{r} = (2\cos t)\vec{i} + (3\sin t)\vec{j}$.

Solution:

$$\begin{aligned}
\vec{r}(t) &= (2\cos t)\vec{i} + (3\sin t)\vec{j} + 0\vec{k} \\
\vec{r}\,'(t) &= (-2\sin t)\vec{i} + (3\cos t)\vec{j} + 0\vec{k} \\
\vec{r}\,''(t) &= (-2\cos t)\vec{i} + (-3\sin t)\vec{j} + 0\vec{k}
\end{aligned}$$

$$\begin{aligned}
&\vec{r}'(t) \times \vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin t & 3\cos t & 0 \\ -2\cos t & -3\sin t & 0 \end{vmatrix} = 6(\sin^2 t + \cos^2 t)\vec{k} = 6\vec{k} \\
&||\vec{r}\,'(t) \times \vec{r}\,''(t)|| = ||6\vec{k}|| = 6 \\
&||\vec{r}\,'(t)|| = \sqrt{(-2\sin t)^2 + (3\cos t)^2} = \sqrt{4\sin^2 t + 9\cos^2 t} \\
&\kappa(t) = \frac{||\vec{r}\,'(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3} = \frac{6}{(4\sin^2 t + 9\cos^2 t)^{\frac{3}{2}}}. \quad \square
\end{aligned}$$

Example 3. Find $\kappa(t)$ for $x = a\cos t, y = a\sin t, z = ct$ $(a > 0, c > 0)$.

Solution:

$$\begin{aligned}
\vec{r}(t) &= a\cos t\vec{i} + a\sin t\vec{j} + ct\vec{k} \\
\vec{r}(t) &= -a\sin t\vec{i} + a\cos t\vec{j} + c\vec{k} \\
\vec{r}\,''(t) &= -a\cos t\vec{i} + (-a\sin t)\vec{j} + 0\vec{k} \\
\vec{r}\,'(t) \times \vec{r}\,''(t) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix} \\
&= (ac\sin t)\vec{i} - (ac\cos t)\vec{j} + a^2\vec{k} \\
||\vec{r}\,'(t) \times \vec{r}\,''(t)|| &= \sqrt{(ac\sin t)^2 + (-ac\cos t)^2 + (a^2)^2} \\
&= \sqrt{a^2c^2 + a^4} = a\sqrt{a^2 + c^2} \\
||\vec{r}\,'(t)|| &= \sqrt{(-a\sin t)^2 + (a\cos t)^2 + c^2} = \sqrt{a^2 + c^2} \\
\kappa(t) &= \frac{||\vec{r}(t) \times \vec{r}\,''(t)||}{||\vec{r}\,'(t)||^3} = \frac{a\sqrt{a^2 + c^2}}{(\sqrt{a^2 + c^2})^3} = \frac{a}{(\sqrt{a^2 + c^2})^2} = \frac{a}{a^2 + c^2}
\end{aligned}$$

$\square$

Lecture 9 Multivariable Functions

9.1 Definition of multivariable functions and their natural domains

Let us first recall that a function of one variable, $y = f(x) \colon D \to \mathbb{R}$, is a rule that assigns a unique real number $f(x)$ to each point x in some set D of the x-axis. The set D is called the domain of the function. Mostly the domains of the functions we considered were open or closed intervals, the whole real line $\mathbb{R}$ or half-lines.

A function f of two (or three) real variables, x and y (and $z, \dots$), is a rule that assigns a unique number $f(x, y)$ ($f(x, y, z)$) to each point (x, y) ((x, y, z)) in some set D of the xy-plane (xyz-space). The set D is also called the **domain** of the function.

The above definition extends naturally to functions of n-real variables, usually denoted by $f(x_1,x_2, \dots, x_n)$.

If the domain of the function is not specified, then, as in the single variable case, it is understood that the domain consists of all points at which the formula in the definition of the function makes sense; this is called the **natural domain** of the function.

The procedure of finding the natural domain is similar to single variable functions. First understand the formula as a chain of operations, as you would if you computed the formula using the calculator. E.g. $f(x, y) = \dfrac{1}{\log(x^2 + y)}$ is represented by the following chain:

$$x \to x^2 \to x^2 + y \to \log(x^2 + y) \to \frac{1}{\log(x^2 + y)}.$$

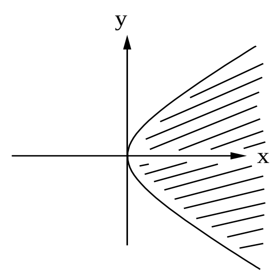
Each step must be well-defined, which gives a number of conditions: no condition for x^2+y, for log being defined $x^2+y > 0$, for $1/\log(x^2+y)$ being defined $\log x^2 + y \neq 0$, i.e. $x^2+y \neq 1$. In this case the natural domain is given by $x^2+y > 0$ and $x^2+y \neq 1$. The first condition is an inequality. To understand the geometrical shape of the corresponding domain one may look at the equality $x^2 + y = 0$ first. This is the parabola $y = -x^2$ which cuts the xy-plane into two pieces, one where $x^2 + y > 0$ and $x^2 + y < 0$. To find out which one we are interested in it suffices to check for one point. Since $(1, 0)$ belongs to the set ($1^2 + 0 > 0$), it is clearly the upper part of the two. From this we need to delete the parabola $x^2 + y \neq 1$, i.e. $y - 1 = -x^2$. The only difference to the analogous problem in single variable calculus is that we need to handle inequalities and equations that involve several variables and describe regions in 2 or higher dimensional space.

Example 1. Find the natural domain of

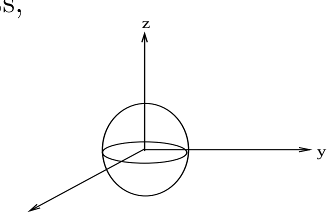
(a) $f(x, y) = \ln(x - y^2)$ (b) $f(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$.

Solution

(a) For $\ln(x-y^2)$ to be defined, we need $x-y^2 > 0$ and that is the only restriction. Therefore the natural domain consists of all the points (x, y) which satisfy $x - y^2 > 0$. We denote this set of points as $\{(x, y) : x - y^2 > 0\}$. Geometrically, this set consists of all the points lying to the right of the parabola $x = y^2$.



(b) For the formula to make sense, we need $\sqrt{1 - x^2 - y^2 - z^2} \neq 0$ and $1 - x^2 - y^2 - z^2 \geq 0$. Combining these two requirements, we need $1 - x^2 - y^2 - z^2 > 0$, i.e. $x^2 + y^2 + z^2 < 1$. Thus the natural domain is $\{(x, y, z) : x^2 + y^2 + z^2 < 1\}$. This set consists of all the points lying inside the sphere $x^2 + y^2 + z^2 = 1$.



9.2 Graphing multivariable functions

The graph of a function of two variables $z = f(x, y) \colon D \to \mathbb{R}$ is the surface

$$\{(x, y, z) \colon (x, y) \in D,\ z = f(x, y)\}$$

in the xyz-space lying in a curved way over the domain D.

Another way to visualise functions of 2 variables is the method of level curves. For a fixed value $z = k$, the equation $f(x, y) = k$ represents a curve in the xy-space, called a **level curve** of height k, or level curve with constant k. These are the curves of intersection of the graph surface with the planes $z = k$ that are parallel to the xyplane and intersect the z-axis at the level $z = k$. Such level curves are often used in practice, e.g. in meteorology the points of equal barometric pressure are connected by curves (isobars).

The level curves of the ellipsoid $x^2+2y^2+z^2=1$ are empty for $k<-1$ or $k>1$ (the planes $z=k$ do not intersect the ellipsoid). For $k=\pm 1$ the level curves are just a point $(0,0,\pm 1)$. For $-1<k<1$ the level curves are the ellipses

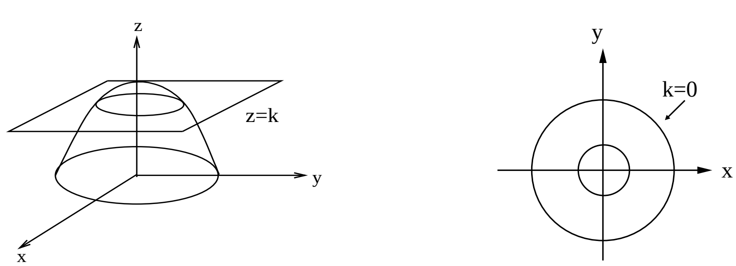
$$x^2+2y^2=1-k^2.$$

A three variable function $w=f(x,y,z)$ cannot be realised as a graph in 2 or 3-space. But, similar to level curves, for fixed values $w=k$, the equations $f(x,y,z)=k$ represent surfaces in the xyz-space, called a **level surfaces** of height k.

Example 2. Describe the graph and level curves for $z=\sqrt{1-x^2-y^2}$.

Solution: Square both sides of the equation $z=\sqrt{1-x^2-y^2}$. It results $z^2=1-x^2-y^2$, which is equivalent to $x^2+y^2+z^2=1$, and we know this equation represents a sphere of radius 1 center $(0,0,0)$. However, our initial equation $z=\sqrt{1-x^2-y^2}$ implies that z is always non-negative. Therefore the graph is the upper half of this sphere, or the upper hemisphere.

To describe the level curves, let us choose a constant k and consider $\sqrt{1-x^2-y^2}=k$. This equation can hold for some point (x,y) only if $0\le k\le 1$, as the left hand side is always between 0 and 1. On the other hand, for each constant k between 0 and 1, $\sqrt{1-x^2-y^2}=k$ is equivalent to $1-x^2-y^2=k^2$, or $x^2+y^2=1-k^2$. This last equation represents a circle with center $(0,0)$ and radius $\sqrt{1-k^2}$. □



Example 3. Describe the level surface for $w=x^2+(y-1)^2+(z-2)^2$.

Solution: Let $w=k$. Then

$$x^2+(y-1)^2+(z-2)^2=k$$

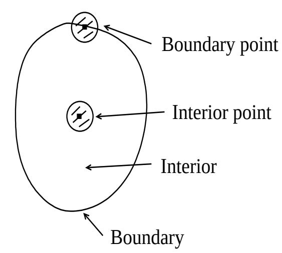
is a sphere of radius $\sqrt{k}$ and center $(0,1,2)$ if $k\ge 0$.

If $k<0$, then the equation can never be satisfied. □

9.3 Topological properties of domains

For computing limits and derivatives (which are a special type of limits) functions had to be defined in some 'neighbourhood' of the limit point. On the other hand for the theorems on extrema and intermediate values of continuous functions it was essential that the function was defined on a closed interval. To understand the domain of a multivariable function properly, we need similar notions of open and closed sets.

If D is a set of points in 2-space, then a point (x_0, y_0) is called an **interior point** of D if there is a circular disk ('neighbourhood') with center (x_0, y_0) and positive radius which lies entirely in D.
If every such disk contains both points in D and points not in D, then (x_0, y_0) is called a **boundary point** of D. The set of all interior points of D is called the **interior** of D, and the set of all boundary points of D is called the **boundary** of D.



In 3-space, the definitions are similar: we replace (x_0, y_0) by (x_0, y_0, z_0), replace circular disk by spherical ball.

If a set contains no boundary point, it is called an **open set**. If a set contains all its boundary points, it is called a **closed set**.

The following common terminology is convenient: Any open set containing a point $\vec{a}$ is called a **neighbourhood** of $\vec{a}$. A neighbourhood of $\vec{a}$ from which the point $\vec{a}$ has been deleted is called a **punctured neighbourhood** of $\vec{a}$.

A set S is said to be **bounded** if it can be put inside some circle or sphere, otherwise it is **unbounded**.

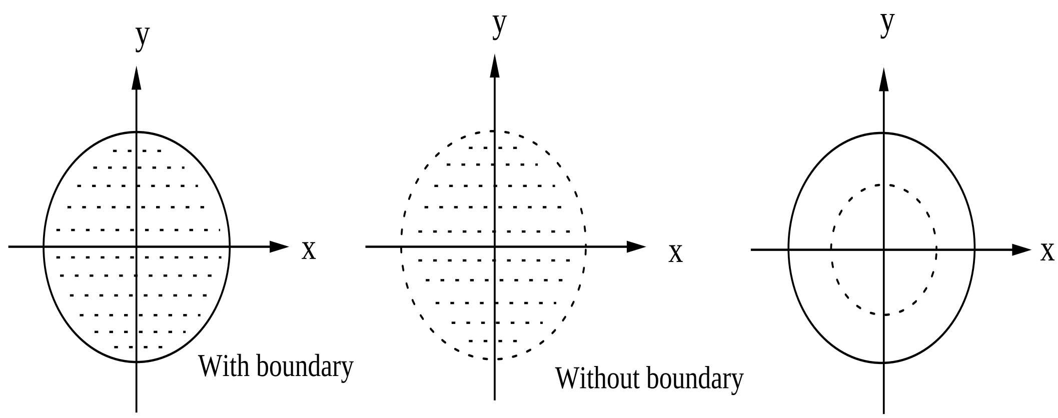
A set S is said to be **connected** if for any two points $\vec{a}$ and $\vec{b}$ of the set there are finitely many points $\vec{x}_1, \dots \vec{x}_N$ such that the segments $\overline{\vec{a}\vec{x}_1}$, $\overline{\vec{x}_1\vec{x}_2}, \dots, \overline{\vec{x}_{N-1}\vec{x}_N}$, $\overline{\vec{x}_N\vec{b}}$ are all contained in S.

Example 4.

(a) $D_1 = \{(x, y) : x^2+y^2 \leq 2\}$ is a closed set as it contains all its boundary points, its boundary is the circle $x^2 + y^2 = 2$. It is bounded.

(b) $D_2 = \{(x, y) : x^2 + y^2 < 2\}$ is an open set as it contains no boundary point. The set is bounded.

(c) $D_3 = \{(x, y) : 1 < x^2 + y^2 \leq 2\}$ is neither open nor closed, as it contains part of its boundary. It is bounded. (For sketches of the sets D_1, D_2, D_3 see the figure below.)

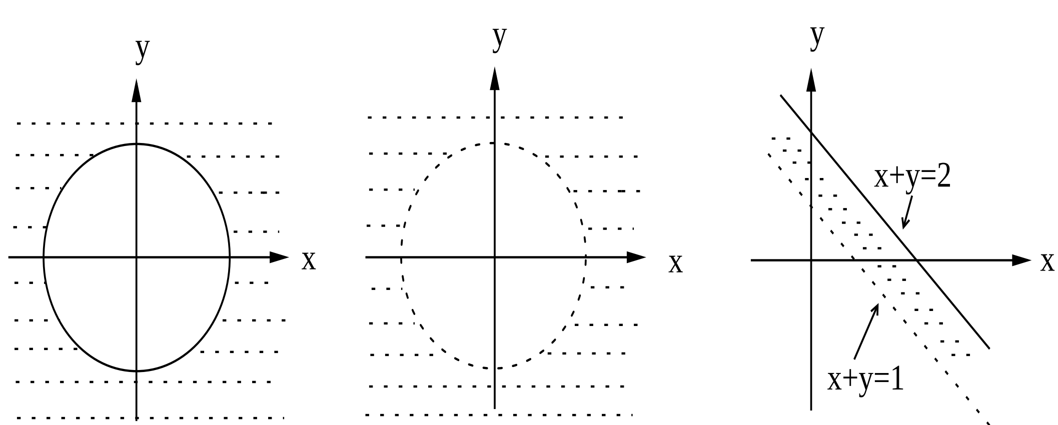


Example 5

(a) $D_1 = \{(x, y) : x^2 + y^2 \geq 2\}$ consists of all the points lying outside the sphere $x^2 + y^2 = 2$, including the sphere which is its boundary. It is closed and unbounded.

(b) $D_2 = \{(x, y) : x^2 + y^2 > 2\}$ is D_1 without boundary. It is open and unbounded.

(c) $D_3 = \{(x, y) : 1 < x + y \leq 2\}$ consists of all the points lying between the two lines $x + y = 1$ and $x + y = 2$. The line $x + y = 2$ is in D_3 but $x + y = 1$ is not. This set is unbounded, but it is neither open, nor closed.



D_1 D_2 D_3

Lecture 10 Limits and Continuity

The idea of continuity for multivariable functions is the same as in the case of single variable functions, namely to make sure that the value of the function changes only a little if the arguments change a little. Consider a function $f(x_1, x_2)$ of two variables defined on some domain $D \subset \mathbb{R}^2$. Let $a = (a_1, a_2)$ be a point in the domain and $f(a_1, a_2) = b$. The change of $f(x_1, x_2)$ from $f(a_1, a_2) = b$ is considered to be small if

$$|f(x) - f(a)| = |f(x_1, x_2) - f(a_1, a_2)| < \varepsilon, \tag{7}$$

where ε is a small positive number.

We want now to control the magnitude of ε by restricting $x = (x_1, x_2)$ to a small neighbourhood of a, i.e. by requiring

$$||x - a|| = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2} < \delta$$

where δ is a small positive number that depends on ε chosen in such a way that (7) is satisfied.

Notice that by combining (x_1, x_2) into one symbol x and (a_1, a_2) into one symbol a the definition of continuity becomes exactly like in one variable calculus, with the only difference that

$$||x - a|| < \delta$$

is now a distance in the 2-dimensional plane. The condition means that x lies within a circle centred at a of (small) radius δ. This definition carries over to 3 and higher dimensional space by setting $x = (x_1, x_2, x_3)$ (or $x = (x_1, x_2, \ldots, x_n)$) and $a = (a_1, a_2, a_3)$ (or $a = (a_1, a_2, \ldots, a_n)$). The distance $||x - a||$ is then

$$\sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2} \quad (\text{or } \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + \cdots + (x_n - a_n)^2}).$$

The formal definition of continuity is as follows:

A function $f\colon D \to \mathbb{R}$ is continuous at a point $a \in D$ if for any positive number ε there exists a positive number δ depending on ε such that the condition $||\vec{x} - \vec{a}|| < \delta$ guarantees $|f(\vec{x}) - f(\vec{a})| < \varepsilon$.

This can be written in a concise way using the $\forall$ ('for all') and $\exists$ ('there exists') symbols:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \forall \vec{x} \in D \text{ with } ||\vec{x} - \vec{a}|| < \delta \implies |f(\vec{x}) - f(\vec{a})| < \varepsilon.$$

If a function is continuous at every point in a set R, then we say the function is **continuous on R.**

Example 1. The constant functions $f(\vec{x}) = c$ are continuous at any point $\vec{a}$ of their domain. In fact,

$$|f(\vec{x}) - f(\vec{a})| = |0| = 0 < \varepsilon$$

is satisfied for any $\vec{x}$ no matter how small ε is and no matter how far $\vec{x}$ from $\vec{a}$ is.

Example 2. Linear functions are continuous. Let us consider linear functions of 3 variables

$$f(x_1, x_2, x_3) = c_1x_1 + c_2x_2 + c_3x_3 + d,$$

where c_1, c_2, c_3, d are some constants. We want to achieve

$$\begin{aligned} |f(\vec{x}) - f(\vec{a})| &= |c_1x_1 + c_2x_2 + c_3x_3 + d - (c_1a_1 + c_2a_2 + c_3a_3 + d)| \\ &= |c_1(x_1 - a_1) + c_2(x_2 - a_2) + c_3(x_3 - a_3)| < \varepsilon \quad (8) \end{aligned}$$

for any small positive number ε. In a first step we have applied elementary algebra to simplify the expression $|f(\vec{x}) - f(\vec{a})|$. Now we need to find a condition of $||\vec{x} - \vec{a}||$ being small that guarantees (8). Rather than trying to manipulate inequality (8) we notice that the left hand side is a dot product

$$|\langle c_1, c_2, c_3\rangle \cdot \langle x_1 - a_1, x_2 - a_2, x_3 - a_3\rangle| = |\vec{c} \cdot (\vec{x} - \vec{a})|$$

which can be estimated using the Cauchy-Schwarz inequality by

$$|\vec{c} \cdot (\vec{x} - \vec{a})| \leq ||\vec{c}||\,||\vec{x} - \vec{a}||.$$

Now making

$$||\vec{x} - \vec{a}|| < \delta,$$

where δ is any positive number smaller than $\frac{\varepsilon}{||\vec{c}||}$ we achieve

$$|\vec{c} \cdot (\vec{x} - \vec{a})| \leq ||\vec{c}||\,||\vec{x} - \vec{a}|| \leq ||\vec{c}||\delta < ||\vec{c}||\frac{\varepsilon}{||\vec{c}||} = \varepsilon$$

as required.

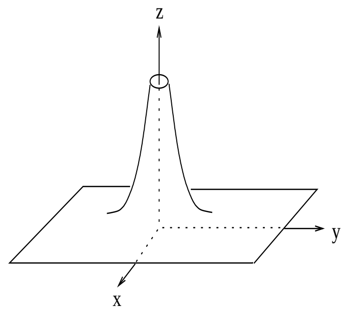
The negation of the continuity statement is

$$\exists\varepsilon > 0 \text{ such that } \forall\delta > 0\ \exists\vec{x} \text{ with } ||\vec{x} - \vec{a}|| < \delta \text{ but } |f(\vec{x} - f(\vec{a})| \geq \varepsilon.$$

Notice that for the negation the symbols $\forall$ and $\exists$ swap. To prove discontinuity we need to find one particular number ε such that, some $\vec{x}$ arbitrarily close to $\vec{a}$ such that the distance of $f(\vec{x})$ to $f(\vec{a})$ is bigger than the chosen ε.

Below we look at two examples that are not continuous.

Example 3. $f(x_1, x_2) = \dfrac{1}{x_1^2 + x_2^2}$ for $\vec{x} \neq 0$ and $f(\vec{0}) = 0$ is not continuous at $(0, 0)$.



From the picture we see that $f(\vec{x})$ becomes arbitrarily big as $\vec{x}$ approaches $\vec{0}$. We choose $\varepsilon = 1$. Now we need to find $\vec{x}$ arbitrarily close to $\vec{0}$ such that

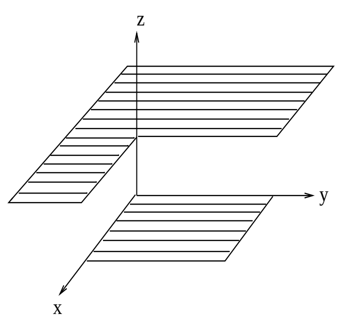
$$f(\vec{x}) = \frac{1}{x_1^2 + x_2^2} = \frac{1}{||\vec{x}||^2} \geq 1.$$

In fact, any $\vec{x}$ with $||\vec{x}|| \leq 1$ satisfies the condition. For any $\delta > 0$ choose $\vec{x} = (x_1, 0)$ with $x_1 = \min\{\delta/2, 1\}$ in order to satisfy $||\vec{x} - \vec{0}|| < \delta$.

Example 4.

$$f(x,y) = \begin{cases} 0 & \text{if } x \geq 0, y \geq 0 \\ 1 & \text{otherwise} \end{cases}$$

is not continuous along the positive x-axis and positive y-axis. Here the function



jumps by a step of 1. Therefore choose $\varepsilon = \frac{1}{2}$ (Any number smaller than 1 is good.) Now of we approach $\vec{0}$ from the region $\{x < 0\} \cup \{y < 0\}$ then $f(\vec{x}) \equiv 1$, so

$$|f(\vec{x}) - f(\vec{0})| = |1 - 0| = 1 > \varepsilon = \frac{1}{2}.$$

Proving continuity from first principles is usually difficult and it is easier to apply one of the following theorems, which are similar to one variable calculus theorems:

Theorem 1

(i) If $f(\vec{x})$ and $g(\vec{x})$ are continuous at $\vec{a}$, then so is

(a) $f(\vec{x}) \pm g(\vec{x})$,

(b) $f(\vec{x}) \cdot g(\vec{x})$

(c) $\dfrac{f(\vec{x})}{g(\vec{x})}$ (provided $g(\vec{a}) \neq 0$).

(ii) If $h(\vec{x})$ is continuous at $(\vec{a})$ and $g(u)$ is continuous at $u = h(\vec{a})$, then $f(\vec{x}) = g(h(\vec{a}))$, is continuous at $(\vec{a})$.

(iii) All the elementary functions[9] are continuous on their natural domains.

The following Theorem generalises the fact, known from Math101, that continuous functions on closed intervals are bounded and attain their minimum, maximum and intermediate values. Notice that closed intervals are the only bounded, closed, connected subsets of $\mathbb{R}$.

Theorem. If $D \subset \mathbb{R}^n$ is closed and bounded and $f\colon D \to \mathbb{R}$ is continuous on D then

1. f is bounded, i.e. $\inf_{\vec{x}\in D} f(\vec{x}) = m > -\infty$, $\sup_{\vec{x}\in D} f(\vec{x}) = M < \infty$.
2. There are points $\vec{a}$ and $\vec{b}$ in D where the infimum m and supremum M are attained, i.e. $f(\vec{a}) = m$ and $f(\vec{b}) = M$.
3. If D is connected then any intermediate value $K \in [m, M]$ is attained, i.e. there exists $\vec{c} \in D$ such that $f(\vec{c}) = K$.

As in one-variable calculus the concept of limit is related to continuity but slightly different. If a function $f(\vec{x})$ is not defined at some point $\vec{a}$ or it is defined but we ignore the value by some reason (e.g. if we doubt the reliability of some measurement) we may ask the question: What would be the 'right' value for f at $\vec{a}$ to make f continuous at $\vec{a}$? This 'right' value L is called the limit of $f(\vec{x})$ as $\vec{x}$ approaches $\vec{a}$ and denoted by

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L.$$

The difference to continuity is that we need to specify the limit L and that the point $\vec{x}$ must remain different from $\vec{a}$ itself when it approaches $\vec{a}$. The formal definition is as follows.

The limit $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ if L is a number such that

$$\forall\varepsilon > 0\ \exists\delta > 0 \text{ such that } \forall\vec{x} \in D \text{ with } 0 \neq ||\vec{x} - \vec{a}|| < \delta \implies |f(\vec{x}) - L| < \varepsilon.$$

[9] Elementary functions are: polynomials, trigonometric functions and their inverses, exponential and logarithmic functions and all their sums, products, quotients and composites.

To compute the limit of an elementary function f at a point $\vec{a}$ of its natural domain is as simple as plugging $\vec{a}$ into f. Other limits usually require some algebraic or geometric tricks or some known limits and the following rules:

Theorem 2 If $\lim_{(\vec{x})\to\vec{x}_0} f(\vec{x}) = L_1$ and $\lim_{\vec{x}\to\vec{x}_0} g(\vec{x}) = L_2$, then

(a) $\lim_{\vec{x}\to\vec{x}_0} [Cf(\vec{x})] = CL_1$, where C is a constant;

(b) $\lim_{\vec{x}\to\vec{x}_0} [f(\vec{x}) \pm g(\vec{x})] = L_1 \pm L_2$;

(c) $\lim_{\vec{x}\to\vec{x}_0} f(\vec{x}) \cdot g(\vec{x}) = L_1 L_2$;

(d) $\lim_{\vec{x}\to\vec{x}_0} \dfrac{f(\vec{x})}{g(\vec{x})} = \dfrac{L_1}{L_2}$ provided that $L_2 \neq 0$.

(e) If $f(\vec{x}) \leq g(\vec{x})$ then $L_1 \leq L_2$.

(f) (Squeezing principle) If $L_1 = L_2$ and $h(\vec{x})$ is a function such that $f(\vec{x}) \leq h(\vec{x}) \leq g(\vec{x})$. Then $\lim_{\vec{x}\to\vec{x}_0} h(\vec{x})$ exists and equals $L_1 = L_2$.

The proofs of these statements are similar to the one-variable proofs and a they are a good exercise familiarise yourself with the $\varepsilon - \delta$ technique. The proofs are not difficult, perhaps with the exception of (c) and (more so) (d).

Example 5. Find

(a) $\displaystyle\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(-1,2)} e^{\frac{xy}{x^2+y^2}}$

Solution:

(a) The function is a fraction of the elementary functions xy and x^2+y^2. Therefore it is continuous as long as the denominator is not zero. At $(-1,2), x^2+y^2 = (-1)^2+2^2 = 5 \neq 0$. Hence the function is continuous at $(-1,2)$, and the limit is simply the value of the function at $(-1,2)$, namely,

$$\lim_{(x,y)\to(-1,2)} \frac{xy}{x^2+y^2} = \frac{(-1)(2)}{(-1)^2+2^2} = \frac{-2}{5}.$$

(b) As $g(u) = e^u$ is continuous for all u and $\dfrac{xy}{x^2+y^2}$ is continuous at $(-1,2)$, $e^{\frac{xy}{x^2+y^2}} = g\left(\dfrac{xy}{x^2+y^2}\right)$ is continuous at $(-1,2)$. Thus

$$\lim_{(x,y)\to(-1,2)} e^{\frac{xy}{x^2+y^2}} = e^{-\frac{2}{5}}.$$

$\square$

If the continuity of a function $f(x, y)$ at a given point (x_0, y_0) cannot be determined through Theorem 1 above, to find whether $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists is usually not easy.

Let us note that if $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$, then $f(x, y)$ must be close to L as soon as (x, y) is close to (x_0, y_0). While in one variable calculus x could approach x_0 only in two ways, namely from the left or from the right, in the 2-dimensional plane (x, y) can approach (x_0, y_0) from infinitely many directions and even can change the direction during the approach. In particular, it can approach (x_0, y_0) along any given smooth curve $C : x = x(t), y = y(t), a \le t \le b$, which passes through (x_0, y_0), say, at $t = t_0$, i.e. $x_0 = x(t_0), y_0 = y(t_0)$. This suggests that no matter what such curve C we choose,

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ \text{(along } C)}} f(x, y) = \lim_{t\to t_0} f(x(t), y(t)) = L.$$

The limit $\lim_{t\to t_0} f(x(t), y(t))$ is usually much easier to obtain as it is the limit of a function with only one variable t.

The analysis here is particularly useful in showing that $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist. If we can find two curves C_1 and C_2 both passing through (x_0, y_0), but with

$$\lim_{\substack{(x,y)\to(x_0,y_0) \\ \text{(along } C_1)}} f(x, y) \neq \lim_{\substack{(x,y)\to(x_0,y_0) \\ \text{(along } C_2)}} f(x, y),$$

then we conclude immediately that $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist, for otherwise the limit along C_1 would have been the same as that along C_2.

On the other hand, if we choose many curves passing through (x_0, y_0) and find that the limit along all these curves are the same, say L, then we may suspect that $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$. However, checking with many curves is **not a proof** for the existence of the limit. We need to provide a rigorous proof in this case. Indeed, the limit may not exist even if the limits along many curves are the same as Example 7 below shows.

Example 6 Determining whether $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2 + y^2}$ exists.

Solution. The denominator is 0 at $(0, 0)$. Therefore we cannot find the limit through continuity. Let us check the limit along the straight line $C_k : y = kx$, or

$x = t, y = kt$. It passes through $(0,0)$ at $t = 0$. Here k is a constant.

$$\begin{aligned} \lim_{\substack{(x,y)\to(0,0)\\ \text{(along } C_k)}} \frac{xy}{x^2+y^2} &= \lim_{t\to 0} \frac{t(kt)}{t^2+(kt)^2} = \lim_{t\to 0} \frac{kt^2}{(1+k^2)t^2} \\ &= \lim_{t\to 0} \frac{k}{1+k^2} = \frac{k}{1+k^2} \end{aligned}$$

If we choose different values for k, we obtain different limit along C_k. This implies that the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist. □

Example 7 Determining whether $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exists.

Solution Let us check the limit along $C_k : x = t, y = kt$.

We have

$$\begin{aligned} \lim_{\substack{(x,y)\to(0,0)\\ \text{(along } C_k)}} \frac{xy^2}{x^2+y^4} &= \lim_{t\to 0} \frac{t(kt)^2}{t^2+(kt)^4} \\ &= \lim_{t\to 0} \frac{k^2t}{1+k^4t^2} = 0 \end{aligned}$$

For $C'_k : x = t, y = kt^2$,

$$\begin{aligned} \lim_{\substack{(x,y)\to(0,0)\\ \text{(along } C'_k)}} \frac{xy^2}{x^2+y^4} &= \lim_{t\to 0} \frac{t(kt^2)^2}{t^2+(kt^2)^4} \\ &= \lim_{t\to 0} \frac{k^2t^3}{1+k^4t^8} = 0 \end{aligned}$$

It seems to suggest that the limit exists and is 0. However, on checking with $C : x = t, y = \sqrt{t}$, we have

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{(along } C)}} \frac{xy^2}{x^2+y^4} = \lim_{t\to 0} \frac{t(\sqrt{t})^2}{t^2+(\sqrt{t})^4} = \lim_{t\to 0} \frac{t^2}{2t^2} = \frac{1}{2}.$$

As the limit along C is different from that along C_k, we conclude that the limit does not exist. □

Example 8. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^2+y^2} = 0$.

Before engaging in the proof, let us first note that we cannot just check the limit along several (or infinitely many) curves and conclude if these limits are all 0.

On the other hand we can reduce this limit to one-variable limit by switching to polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

noticing that

$$(x, y) \to (0, 0)$$

is equivalent to

$$r = ||(x, y) - (0, 0)|| = \sqrt{x^2 + y^2} \to 0.$$

Hence, the limit in Example 8. is equivalent to

$$\lim_{r\to 0} \frac{r^4\cos^3\theta\sin\theta}{r^2} = \lim_{r\to 0} r^2\cos^3\theta\sin\theta.$$

A squeezing argument shows that this limit is 0. Indeed,

$$0 \leq |r^2\cos^3\theta\sin\theta - 0| = r^2|\cos\theta|^3|\sin\theta| \leq r^2$$

since sin and cos vary between -1 and 1. As $r \to 0$ both sides of the inequality tend to 0 hence the term squeezed in the middle must converge to 0 as well.

One can guess that the limit is zero by looking at the (minimal) degrees of numerator and denominator. In Example 8. it is $4 : 2$, so the degree in the denominator is bigger, which indicates a zero limit. To make this argument rigorous we prove the following lemma.

Lemma. If $g(\vec{x})$ is bounded in some neighbourhood of $\vec{a}$ and

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = 0$$

then

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x})g(\vec{x}) = 0.$$

The proof is also based on the squeezing principle. Let M be a positive constant such that $|g(\vec{x})| < M$. Then

$$0 \leq |f(\vec{x})g(\vec{x}) - 0| = |f(\vec{x})||g(\vec{x})| \leq M|f(\vec{x})|$$

in some neighbourhood of $\vec{a}$. Now, as $\vec{x} \to \vec{a}$, both sides of the inequality tend to 0, hence the term in the middle tends to 0 as well.

Lecture 11 Differentiability of Two Variable Functions

To motivate the introduction of differentiability for two variable functions, let us recall a few facts about one variable function. By definition, $f(x)$ is differentiable at $x = x_0$ if

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \quad \text{exists.}$$

The limit is denoted by $f'(x_0)$, and is called the derivative of $f(x)$ at $x = x_0$.

Now from

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

we see that

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0) - f'(x_0)\Delta x}{\Delta x} = 0$$

when $f(x)$ is differentiable at x_0. This implies that

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x \quad \text{when} \quad \Delta x \text{ is small.}$$

More precisely, we have

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + E(\Delta x)$$

where the error term $E(\Delta x)$ tends to zero faster than Δx, i.e. using the o notation

$$E(\Delta x) = \mathrm{o}(\Delta x).$$

Geometrically this means that, up to a small error, the function f can be approximated by a linear function, as long as we stay close to x_0.

By passing to the limit for $\Delta x \to 0$ it follows that $f'(x_0)\Delta x \to 0$ and $E(\Delta x) \to 0$ hence $f(x_0 + \Delta x) \to f(x_0)$, i.e. f is continuous at x_0 when it is differentiable there.

It turns out that the natural generalization of differentiability is along the line

$$f(x_0 + \Delta x) = f(x_0) + A\Delta x + \mathrm{o}(\Delta x)$$

where the number $A = f'(x_0)$.

Definition. $f(x, y)$ is said to be **differentiable at** (x_0, y_0) if there exist numbers A, B such that

$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + A\Delta x + B\Delta y + \mathrm{o}(||\Delta r||)$$

where $||\Delta r|| = \sqrt{\Delta x^2 + \Delta y^2}$, or, equivalently,

$$\lim_{(\Delta x, \Delta y)\to(0,0)} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - A\Delta x - B\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = 0.$$

We say $f(x, y)$ is differentiable in a region R if it is differentiable at every point in R. The pair of numbers (A, B) replaces the single number $f'(x_0)$ from one variable calculus. The differential, i.e. the linear function on Δx

$$df = f'(x_0)\Delta x$$

is replaced by a linear function on two variables $\Delta x, \Delta y$

$$df = A\Delta x + B\Delta y.$$

In matrix notation this becomes

$$df = \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}.$$

Theorem 1. If $f(x, y)$ is differentiable at (x_0, y_0), then $f(x, y)$ is continuous at (x_0, y_0).

Proof. We want to show

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

i.e.,

$$\lim_{(\Delta x, \Delta y)\to(0,0)} f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0)$$

or

$$\lim_{(\Delta x, \Delta y)\to(0,0)} [f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)] = 0.$$

Denote

$$g(\Delta x, \Delta y) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

We have to show that

$$\lim_{(\Delta x, \Delta y)\to(0,0)} g(\Delta x, \Delta y) = 0.$$

From the differentiability we have

$$g(\Delta x, \Delta y) == A\Delta x + B\Delta y + E(\Delta x, \Delta y),$$

where the error function $E(\Delta x, \Delta y) = \mathrm{o}\,||\Delta r||$, i.e.

$$E(\Delta x, \Delta y) = ||\Delta r|| e(\Delta x, \Delta y),$$

where $e(\Delta x, \Delta y)$ still tends to zero as $(\Delta x, \Delta y) \to (0, 0)$.

Now all the functions $A\Delta x$, $B\Delta y$, $e(\Delta x, \Delta y)$ and $||\Delta r||$ tend to 0 and hence $g(\Delta x, \Delta y)$ tends to 0. $\square$

Lecture 12 Partial Derivatives

1. **Partial Derivatives for Two Variable Functions**

Consider a function $f(x,y)$. If we hold $y = y_0$, then $f(x,y_0)$ is a function of x only, its derivative at $x = x_0$ (when exists) is denoted by $f_x(x_0,y_0)$, and called the **partial derivative of** $f(x,y)$ **with respect to** x at the point (x_0,y_0). Similarly, holding $x = x_0$, $f(x_0,y)$ is a function of y only, and its derivative (if exists) at $y = y_0$ is denoted by $f_y(x_0,y_0)$, called the **partial derivative of** $f(x,y)$ **with respect to** y at the point (x_0,y_0). If (x_0,y_0) is a general point, we usually write $f_x(x,y)$ and $f_y(x,y)$ for the partial derivatives. They are functions of x and y.

Example 1. Find $f_x(1,2)$ and $f_y(1,2)$, where

$$f(x,y) = 2x^2y + y^3 + x + 1.$$

Solution

$$\begin{aligned}
f(x,2) &= 2x^2(2) + (2)^3 + x + 1 = 4x^2 + x + 9\\
f_x(x,2) &= (4x^2 + x + 9)' = 8x + 1\\
f_x(1,2) &= (8x+1)|_{x=1} = 8 + 1 = 9.\\
\text{or}\quad f_x(x,y) &= 4xy + 0 + 1 + 0 = 4xy + 1\\
f_x(1,2) &= 4(1)(2) + 1 = 9\\
f_y(x,y) &= 2x^2 + 3y^2 + 0 + 0 = 2x^2 + 3y^2\\
f_y(1,2) &= 2(1)^2 + 3(2)^2 = 14.
\end{aligned}$$

□

The partial derivatives have many different kinds of notations. Some of the most often used are listed below (for $z = f(x,y)$).

$$\begin{aligned}
f_x(x_0,y_0) &= \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \left.\frac{\partial z}{\partial x}\right|_{(x_0,y_0)}\\
f_x(x,y) &= \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}\\
f_y(x_0,y_0) &= \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{\partial z}{\partial y}\right|_{(x_0,y_0)}\\
f_y(x,y) &= \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}.
\end{aligned}$$

Example 2. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ if $z = x^2\sin(xy)$.

Solution:

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2x\sin(xy) + x^2 \cdot \cos(xy)\cdot y \quad \text{(product rule and chain rule)}\\ &= 2x\sin(xy) + x^2 y\cos(xy)\\ \frac{\partial z}{\partial y} &= x^2\cos(xy)x = x^3\cos(xy).\end{aligned}$$

□

It turns out that the numbers (A, B) in the definition of differentiability in the previous lecture are exactly the partial derivatives of the function f with respect to x and y at x_0, y_0. Indeed, for $\Delta y = 0$ we get

$$f(x_0 + \Delta x, y_0) = f(x_0, y_0) + A\Delta x + \mathrm{o}(|\Delta x|),$$

and hence $A = f_x(x_0, y_0)$. In a similar way we find $B = f_y(x_0, y_0)$. We can rewrite the definition of differentiability as

$$f(x_0 + \Delta x, y_0) = f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \mathrm{o}(||\Delta r||).$$

2. **Higher-Order Partial Derivatives**

For a given function $f(x, y)$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are functions of x and y, each can have partial derivatives. We define

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

These are called the **second-order partial derivatives**. Moreover, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are called the **pure second-order partial derivatives** as they are obtained by differentiation with respect to the same variable twice. $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are called the **mixed second order partial derivatives**.

In this context, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are called the **first-order partial derivatives**.

We define third-order partial derivatives as the partial derivatives of the second-order partial derivatives, and so on. For example, we have

$$\frac{\partial^3 f}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right), \quad \frac{\partial^3 f}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right),$$

$$\frac{\partial^4 f}{\partial x^4} = \frac{\partial}{\partial x}\left(\frac{\partial^3 f}{\partial x^3}\right), \quad \frac{\partial^4 f}{\partial x \partial^2 y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial^3 f}{\partial y^2 \partial x}\right).$$

Higher-order partial derivatives also have different notations. For example,

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial x \partial y} = f_{yx}, f_{xxyy} = \frac{\partial^4 f}{\partial y \partial y \partial x \partial x} = \frac{\partial^4 f}{\partial y^2 \partial x^2}.$$

Note that while $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$, we have $f_{yx} = (f_y)_x$. That is why we have $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$ (note the difference in the order for x and y in the notation).

Example 3. Let $f(x, y) = e^x \sin y + \ln x$. Find f_{xyx}.

Solution

$$\begin{aligned} f_x &= \frac{\partial}{\partial x}(e^x \sin y + \ln x) = e^x \sin y + \frac{1}{x}. \\ f_{xy} &= (f_x)_y = \frac{\partial}{\partial y}(f_x) = \frac{\partial}{\partial y}(e^x \sin y + \frac{1}{x}) = e^x \cos y \\ f_{xyx} &= (f_{xy})_x = \frac{\partial}{\partial x}(f_{xy}) = \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y. \end{aligned}$$

$\square$.

3. **Partial Derivatives of Functions of More Than Two Variables**

For functions of more than two variables, the partial derivatives and higher-order partial derivatives are defined analogously.

For example,

$\frac{\partial f}{\partial x} = f_x(x, y, z)$ is calculated by holding y and z fixed.

$\frac{\partial f}{\partial x_i} = f_{x_i}(x_1, \ldots, x_n)$ is calculated by holding all the variables except x_i fixed.

Example 4. Let $f(x, y, z) = x^2yz + yz + x$. Find f_{xyy}.

Solution:

$$f_x = 2xyz + 1, \quad f_{xy} = \frac{\partial}{\partial y}(2xyz + 1) = 2xz,$$

$$f_{xyy} = \frac{\partial}{\partial y}(2xz) = 0.$$

$\square$

Example 5. Find $\frac{\partial^3 z}{\partial \theta \partial \phi \partial \rho}$ where $z = \rho^2 \cos\phi \sin\theta$

Solution:

$$\begin{aligned} \frac{\partial z}{\partial \rho} &= 2\rho \cos\phi \sin\theta \\ \frac{\partial^2 z}{\partial \phi \partial \rho} &= \frac{\partial}{\partial \phi}(2\rho \cos\phi \sin\theta) = -2\rho \sin\phi \sin\theta \\ \frac{\partial^3 z}{\partial \theta \partial \phi \partial \rho} &= \frac{\partial}{\partial \theta}(-2\rho \sin\phi \sin\theta) = -2\rho \sin\phi \cos\theta \end{aligned}$$

□

Example 6. Find $\frac{\partial}{\partial x_1}\left(\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\right)$ and $\frac{\partial}{\partial x_i}\left(\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\right)$.

Solution

$$\begin{aligned}
\frac{\partial}{\partial x_1}\left(\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\right) &= \frac{1}{2}(x_1^2+x_2^2+\cdots+x_n^2)^{-\frac{1}{2}}\cdot 2x_1 \quad \text{(chain rule)}\\
&= \frac{x_1}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}\\
\frac{\partial}{\partial x_i}\left(\sqrt{x_1^2+x_2^2+\cdots+x_n^2}\right) &= \frac{1}{2}(x_1^2+x_2^2+\cdots+x_n^2)^{-\frac{1}{2}}2x_i \quad \text{(chain rule)}\\
&= \frac{x_i}{\sqrt{x_1^2+x_2^2+\ldots x_n^2}}
\end{aligned}$$

□

The following theorem shows that we can use information on the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ to determine the differentiability of $f(x,y)$.

Theorem 1. If $f_x(x,y), f_y(x,y)$ exist for (x,y) in some circular region centered at (x_0,y_0), and $f_x(x,y), f_y(x,y)$ are continuous at (x_0,y_0), then $f(x,y)$ is differentiable at (x_0,y_0).

Proof. We want to show

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)-f_x(x_0,y_0)\Delta x - f_y(x_0,y_0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} = 0$$

Denoting the fraction above by I, we want to find a function $g(\Delta x,\Delta y)$ such that

$$|I| \le g(\Delta x,\Delta y) \text{ and } \lim_{(\Delta x,\Delta y)\to(0,0)} g(\Delta x,\Delta y) = 0$$

By the squeezing method, the existence of such a function g implies $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} I = 0$, as required.

To use the given information about the partial derivatives, we write

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$

$$= [f(x_0+\Delta x, y_0+\Delta y) - f(x_0,+\Delta x, y_0)] + [f(x_0+\Delta x, y_0) - f(x_0,y_0)],$$

and use the mean-value theorem to obtain

$$\begin{aligned}
f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0) &= f_y(x_0+\Delta x, y_0+\theta_1\Delta y)\cdot\Delta y, 0<\theta_1<1\\
f(x_0+\Delta x, y_0) - f(x_0,y_0) &= f_x(x_0+\theta_2\Delta x, y_0)\Delta x, 0<\theta_2<1.
\end{aligned}$$

Thus

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$
$$= f_y(x_0+\Delta x, y_0+\theta_1\Delta y)\Delta y + f_x(x_0+\theta_2\Delta x, y_0)\Delta x.$$

Substituting this into the expression of I, we obtain

$$\begin{aligned} I &= \frac{[f_x(x_0+\theta_2\Delta x, y_0) - f_x(x_0, y_0)]\Delta x + [f_y(x_0+\Delta x, y_0+\theta_1\Delta y) - f_y(x_0, y_0)]\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} \\ &= [f_x(x_0+\theta_2\Delta x, y_0) - f_x(x_0, y_0)]\frac{\Delta x}{\sqrt{\Delta x^2+\Delta y^2}} \\ &\quad +[f_y(x_0+\Delta x, y_0+\theta_1\Delta y) - f_y(x_0, y_0)]\frac{\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} \end{aligned}$$

Since $\left|\frac{\Delta x}{\sqrt{\Delta x^2+\Delta y^2}}\right| = \frac{|\Delta x|}{\sqrt{\Delta x^2+\Delta y^2}} \le \frac{|\Delta x|}{\sqrt{\Delta x^2}} = \frac{|\Delta x|}{|\Delta x|} = 1$, and similarly

$$\left|\frac{\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}\right| \le 1,$$

we obtain

$$\begin{aligned} |I| &\le |f_x(x_0+\theta_2\Delta x, y_0) - f_x(x_0, y_0)|\left|\frac{\Delta x}{\sqrt{\Delta x^2+\Delta y^2}}\right| \\ &+ |f_y(x_0+\Delta x, y_0+\theta_1\Delta y) - f_y(x_0, y_0)|\left|\frac{\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}\right| \\ &\le |f_x(x_0+\theta_2\Delta x, y_0) - f_x(x_0, y_0)| + |f_y(x_0+\Delta x, y_0+\theta_1\Delta y) - f_y(x_0, y_0)|. \end{aligned}$$

Let us choose $g(\Delta x, \Delta y)$ to be the right hand side of this last inequality. Since $f_x(x,y)$ and $f_y(x,y)$ are continuous at (x_0, y_0), as $(x,y)\to(x_0,y_0)$,

$$f_x(x,y) - f_x(x_0,y_0) \to 0, \quad f_y(x,y) - f_y(x_0,y_0) \to 0.$$

But from $0<\theta_1<1,\ 0<\theta_2<1$ we know that as $(\Delta x, \Delta y)\to(0,0)$,

$$(x_0+\theta_2\Delta x, y_0) \to (x_0,y_0), (x_0+\Delta x, y_0+\theta_1\Delta y) \to (x_0,y_0).$$

Therefore, as $(\Delta x, \Delta y)\to(0,0)$,

$$\begin{aligned} |f_x(x_0+\theta_2\Delta x, y_0) - f_x(x_0, y_0)| &\to 0 \\ |f_y(x_0+\Delta x, y_0+\theta_1\Delta y) - f_y(x_0, y_0)| &\to 0 \end{aligned}$$

It follows that

$$\lim_{(\Delta x,\Delta y)\to(0,0)} g(\Delta x, \Delta y) = 0.$$

As we already have, by the definition of $g(\Delta x, \Delta y)$,

$$|I| \leq g(\Delta x, \Delta y),$$

the squeezing method implies

$$\lim_{(\Delta x,\Delta y)\to(0,0)} I = 0.$$

This finishes the proof. $\square$

In addition to guaranteeing differentiability and continuity of $f(x, y)$, continuity of the partial derivatives also ensures that the mixed second-order partial derivatives of f are equal. This is the content of the following theorem, whose proof we omit.

Theorem 2. If $f_x(x, y), f_y(x, y), f_{xy}(x, y)$ and $f_{yx}(x, y)$ are continuous on an open set, then $f_{xy}(x, y) = f_{yx}(x, y)$ at each point of the set.

For most of the functions we meet, their partial derivatives are continuous. Therefore the mixed partial derivatives are equal. Applying Theorem 3, say to f_x, we can deduce $f_{xxy} = f_{xyx}$, etc, provided the continuity conditions are met.

Example 2. For $f(x, y) = e^x(x^2 + xy)$, check that $f_{xy} = f_{yx}$ and $f_{xyy} = f_{yyx} = f_{yxy}$.

Solution:

$$\begin{aligned}
f_x &= e^x(x^2 + xy) + e^x(2x + y) = e^x(x^2 + 2x + xy + y) \\
f_{xy} &= e^x(x + 1) \\
f_{xyy} &= 0 \\
f_y &= e^x x \\
f_{yy} &= 0 \\
f_{yyx} &= 0 \\
f_{yx} &= \frac{\partial}{\partial x}(e^x x) = e^x x + e^x = e^x(x + 1) \\
f_{yxy} &= \frac{\partial}{\partial y}(e^x x + e^x) = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
f_{xyy} &= f_{yyx} = f_{yxy} = 0, \\
f_{xy} &= f_{yx} = e^x(x + 1).
\end{aligned}$$

$\square$.

Lecture 13 The Chain Rules

Recall that in the one variable function case, if $y = f(x)$, $x = x(t)$, then the derivative of the composite function $y = f(x(t))$ can be calculated by te chain rule:

$$\frac{dy}{dt} = f'(x(t))x'(t), \quad \text{or} \quad \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

For two variable functions, there are several situations where the chain rule is needed. The first situation is described in the following theorem.

Theorem 1 (Chain Rule). If $x = x(t), y = y(t)$ are differentiable at t and $z = f(x, y)$ is differentiable at $(x, y) = (x(t), y(t))$, then $z = f(x(t), y(t))$ as a function of the single variable t is differentiable at t, and

$$\boxed{\begin{array}{l} \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt}, \qquad \text{i.e} \\ \dfrac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t)y'(t) \end{array}}$$

Proof. Since $x(t)$ and $y(t)$ are differentiable at t, by definition,

$$\begin{aligned} \lim_{\Delta t \longrightarrow 0} \frac{\Delta x}{\Delta t} &= \lim_{\Delta t \longrightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = x'(t), \\ \lim_{\Delta t \longrightarrow 0} \frac{\Delta y}{\Delta t} &= \lim_{\Delta t \longrightarrow 0} \frac{y(t + \Delta t) - y(t)}{\Delta t} = y'(t). \end{aligned}$$

The existence of the above two limits implies that

$$\Delta x \longrightarrow 0 \ \text{ and } \ \Delta y \longrightarrow 0 \ \text{ as } \ \Delta t \longrightarrow 0.$$

Let us denote, for convenience of later use,

$$\begin{aligned} \varepsilon &= \frac{\Delta z - f_x \Delta x - f_y \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \qquad (9) \\ &= \frac{f(x + \Delta x, y + \Delta y) - f(x, y) - f_x(x, y)\Delta x - f_y(x, y)\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \end{aligned}$$

where we suppose $x = x(t), y = y(t), \Delta x = x(t + \Delta t) - x(t), \Delta y = y(t + \Delta t) - y(t)$. Since $f(x, y)$ is differentiable at $(x(t), y(t))$, and we know already that $\Delta x \longrightarrow$

$0, \Delta y \longrightarrow 0$ as $\Delta t \longrightarrow 0$, we obtain from the definition of differentiability that $\varepsilon \longrightarrow 0$ as $\Delta t \longrightarrow 0$.

We can rewrite equation (1) as

$$\Delta z = f_x \Delta x + f_y \Delta y + \varepsilon \cdot \sqrt{\Delta x^2 + \Delta y^2},$$

from which we get

$$\begin{aligned} \frac{\Delta z}{\Delta t} &= f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \varepsilon \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta t} \\ &= f_x(x(t), y(t)) \frac{\Delta x}{\Delta t} + f_y(x(t), y(t)) \frac{\Delta y}{\Delta t} \pm \varepsilon \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2} \end{aligned}$$

where, before the last term, we take positive sign when $\Delta t > 0$ and take negative sign when $\Delta t < 0$.

Now we take the limit $\Delta t \longrightarrow 0$ in the above identity, recalling $\dfrac{\Delta x}{\Delta t} \longrightarrow x'(t)$, $\dfrac{\Delta y}{\Delta t} \longrightarrow y'(t)$ and $\varepsilon \longrightarrow 0$, and obtain

$$\lim_{\Delta t \longrightarrow} \frac{\Delta z}{\Delta t} = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).$$

That is to say $\frac{dz}{dt}$ exists and

$$\frac{dz}{dt} = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).$$

The proof is complete. □

Example 1. Use the chain rule to find $\frac{dz}{dt}$ where $z = x^2 + y^2 + xy, x = t^2, y = t$.

Solution $\dfrac{\partial z}{\partial x} = 2x + y,\ \dfrac{\partial z}{\partial y} = 2y + x,\ \dfrac{dx}{dt} = 2t,\ \dfrac{dy}{dt} = 1.$

By the chain rule,

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = (2x + y)(2t) + (2y + x)(1) \\ &= (2t^2 + t)(2t) + (2t + t^2)(1) \\ &= 4t^3 + 2t^2 + 2t + t^2 \\ &= 4t^3 + 3t^2 + 2t. \end{aligned}$$

□

Note that we always have another way to find the derivative, that is, we can substitute $x = t^2, y = t$ into the expression $z = x^2 + y^2 + xy$ to obtain

$$z = (t^2)^2 + (t)^2 + (t^2) + (t^2)(t) = t^4 + t^2 + t^3$$

and then differentiate

$$\frac{dz}{dt} = \frac{d}{dt}(t^4 + t^3 + t^2) = 4t^3 + 3t^2 + 2t.$$

However, if the finctions are complicated, it is usually better to use the chain rule.

Another case where the chain rule arises naturally is described by the following theorem.

Theorem 2 (Chain Rule) If $x = x(u,v)$, $y = y(u,v)$ have first order partial derivatives at (u,v) and $z = f(x,y)$ is differentiable at $(x(u,v), y(u,v))$, then $z = f(x(u,v), y(u,v))$ has first order partial derivatives at (u,v), and

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$

The proof of Theorem 2 is similar to that of Theorem 1, because, for example, when you calculate $\frac{\partial z}{\partial u}$, v is held as a constant. The details are left as an exercise (please have a try!).

Example 2 Given $z = \sin(xy), x = 2u + v, y = uv$. Find

$$\frac{\partial z}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v}.$$

Solution

$$\begin{aligned}
\frac{\partial z}{\partial x} &= y\cos(xy), \ \frac{\partial z}{\partial y} = x\cos(xy), \\
\frac{\partial x}{\partial u} &= 2, \ \frac{\partial x}{\partial v} = 1, \ \frac{\partial y}{\partial u} = v, \ \frac{\partial y}{\partial v} = u.
\end{aligned}$$

By the chain rule,

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \\
&= (y\cos(xy))(2) + (x\cos(xy))(v) \\
&= 2uv\cos((2u+v)(uv)) + v(2u+v)\cos((2u+v)(uv)) \\
&= 2uv\cos(2u^2v + uv^2) + (2uv + v^2)\cos(2u^2v + uv^2) \\
&= (4uv + v^2)\cos(2u^2v + uv^2)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \\
&= (y\cos(xy))(1) + (x\cos(xy)) \cdot u \\
&= uv\cos(2u^2v + uv^2) + u(2u+v)\cos(2u^2v + uv^2) \\
&= (2u^2 + 2uv)\cos(2u^2v + uv^2).
\end{aligned}$$

$\square$

Please try to substitute $x = 2u + v, y = uv$ first and then differentiate. You should arrive at the same solutions.

Lecture 14 Tangent Planes and Total Differentials

From the last lecture, we know that if $z = f(x, y), x = x(t)$ and $y = y(t)$, then by the chain rule

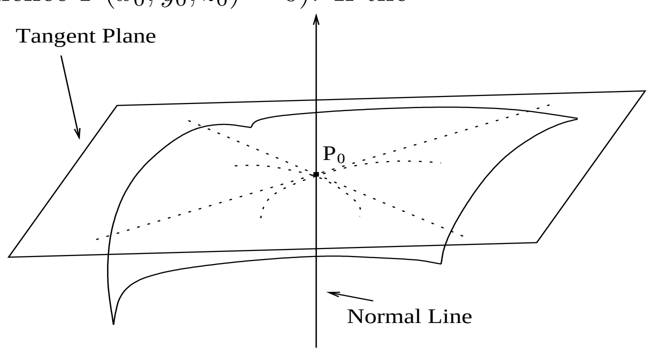
$$\frac{d}{dt}f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t).$$

For the case $w = f(x, y, z)$, $x = x(t), y = y(t), z = z(t)$, there is a natural generalization:

$$\frac{d}{dt}f(x(t), y(t), z(t)) = f_x(x, y, z)x'(t) + f_y(x, y, z)y'(t) + f_z(x, y, z)z'(t).$$

This formula (a chain rule for three variable functions) will be useful below.

Let us consider a surface determined by the equation $F(x, y, z) = 0$. Let $P_0(x_0, y_0, z_0)$ be a point on the surface (and hence $F(x_0, y_0, z_0) = 0$). If the tangent lines at P_0 to all smooth curves that pass through P_0 and lie on the surface are contained in a common plane, then this plane is called the **tangent plane** to the surface at P_0. The line through P_0 parallel to the normal vector of the tangent plane is called the **normal line** to the surface at P_0.



We now set to find an equation of the tangent plane at a given point $P_0(x_0, y_0, z_0)$ on a given surface $F(x, y, z) = 0$. As a point P_0 on the plane is already given, we need only to find a normal vector $\vec{n}$.

To this end, we let $x = x(t), y = y(t), z = z(t)$ be the parametric equation of an arbitrary curve lying on $F(x, y, z) = 0$, passing through (x_0, y_0, z_0) at $t = t_0$. This implies that

$$F(x(t), y(t), z(t)) \equiv 0 \quad \text{for all } t, \quad x(t_0) = x_0, y(t_0) = y_0, z(t_0) = z_0.$$

Recall that $\vec{r}(t_0) = \langle x'(t_0), y'(t_0), z'(t_0)\rangle$ is a tangent vector of the curve $\vec{r} = \langle x(t), y(t), z(t)\rangle$ at P_0. Thus the normal vector $\vec{n}$ should be perpendicular to $\vec{r}'(t_0)$, i.e. $\vec{n} \cdot \vec{r}'(t_0) = 0$. On the other hand, if we differentiate the identity

$$F(x(t), y(t), z(t)) = 0,$$

and use the chain rule, we obtain

$$\frac{d}{dt}F(x(t), y(t), z(t) = 0, \quad \text{i.e.}$$

$F_x(x(t), y(t), z(t))x'(t) + F_y(\ldots)y'(t) + F_z(\ldots)z'(t) = 0$, for all t. Take $t = t_0$, we get

$$F(x_0, y_0, z_0)x'(t_0) + F_y(x_0, y_0, z_0)y'(t_0) + F_z(x_0, y_0, z_0)z'(t_0) = 0.$$

That is to say, the vector $\langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0)\rangle$ and $\vec{r}\,'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0)\rangle$ are perpendicular. Therefore, we can take

$$\vec{n} = \langle F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_o), F_z(x_0, y_0, z_0)\rangle.$$

We can now write down the equation of the tangent plane:

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$

The equation of the normal line is

$$\boxed{x - x_0 = F_x(x_0, y_0, z_0)t, y - y_0 = F_y(x_0, y_0, z_0)t, z - z_0 = F_z(x_0, y_0, z_0)t}$$

It is preferable to give a surface as a graph over the xy-plane (or xz- or yz-plane) by its explicit equation $z = f(x, y)$. To find the explicit equation one has to 'solve' the implicit equation $F(x, y, z) = 0$ for z. If $F(x, y, z) = ax + by + cz + d$ is a linear equation we readily find

$$z = -\frac{a}{c}x - \frac{b}{c}y - \frac{d}{c}$$

if $c \neq 0$. If $c = 0$ we cannot solve this equation for z. For non-linear $F(x, y, z)$ it might be very difficult or even impossible to solve this equation by algebraic manipulations. The so-called 'implicit function' theorem tells us that a solution exists, even if we cannot write down a formula:

Theorem 1. If $F(x, y, z)$ is a function which has continuous derivatives in some neighbourhood of (x_0, y_0, z_0), $F(x_0, y_0, z_0) = 0$ and $F_z(x_0, y_0, z_0) \neq 0$. Then there exists a function $f(x, y)$ defined in a neighbourhood of (x_0, y_0) such that (in some neighbourhood of (x_0, y_0, z_0)) the condition $F(x, y, z) = 0$ is equivalent to $z = f(x, y)$. The function f has continuous partial derivatives

$$f_x = -\frac{F_x}{F_z}, \quad f_y = -\frac{F_y}{F_z}.$$

We do not give a proof for this but we point out that the proof is based on the fact that

$$F(x,y,z) \approx F(x_0,y_0,z_0)+F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0).$$

Taking into account $F(x,y,z) = F(x_0,y_0,z_0) = 0$ we get

$$z - z_0 \approx -\frac{F_x(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}(x-x_0) + -\frac{F_y(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)}(x-x_0)$$

from which we can also guess the formula for the partial derivatives of f. To make this argument rigorous one would need to give a precise meaning to '$\approx$'.

In practice this result is used to find approximate solution, usually by iterative methods such as Newton's method. This topic will be discussed in Amth250.

Example. Let $F(x,y,z) = x + y + z\cos z$. Then $F(0,0,0) = 0$ and $F_z = \cos z + z\sin z$, hence $F_z(0,0,0) = 1 \neq 0$. According to the theorem then the is a function $z = f(x,y)$ that solves the implicit equation $F(x,y,z) = 0$. We are not able to give an algebraic formula for f but we now that it exists and equals approximately

$$f(x,y) \approx -x - y,$$

since $f(0,0) = z_0 = 0$, $f_x(0,0) = -\dfrac{F_x(0,0,0)}{F_z(0,0,0)} = -1$, $f_y(0,0) = -\dfrac{F_y(0,0,0)}{F_z(0,0,0)} = -1$.

Remark. If $F_z(x_0,y_0,z_0) = 0$ we cannot solve the equation with respect to z (similar to $ax + by + cz = 0$ when $c = 0$), but if $F_x(x_0,y_0,z_0) \neq 0$ we solve with respect to x, i.e. find a function $g(y,z)$ such that $F(x,y,z) = 0$ is locally equivalent to $x = g(y,z)$. Or, if $F_y(x_0,y_0,z_0) \neq 0$ then there is a function $h(x,z)$ such that $F(x,y,z) = 0$ is locally equivalent to $y = h(x,z)$.

If the surface is already given as a graph $z = f(x,y)$, we can always rewrite it in the form $F(x,y,z) = 0$ by simply letting

$$F(x,y,z) = f(x,y) - z.$$

We have $F_x(x_0,y_0,z_0) = f_x(x_0,y_0), F_y(x_0,y_0,z_0) = f_y(x_0,y_0)$ and $F_z(x_0,y_0,z_0) = -1$. Hence, on substituting into the above equations for tangent plane and normal line, we obtain

(i) Tangent plane at (x_0,y_0,z_0) for $y = f(x_0,y_0)$:
$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$

(ii) Normal line:
$x - x_0 = f_x(x_0,y_0)t, y - y_0 = f_y(x_0,y_0)t, z - z_0 = -t$

The above equation for the tangent plane can be written in the form

$$\begin{aligned} z - z_0 &= f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0), \text{ or} \\ z &= z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \end{aligned}$$

Geometrically, we know the tangent plane is the closest plane to the surface near P_0. Analytically, this is to say the linear function (in x and y)

$$z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is the best linear approximation of the (in general nonlinear) function $f(x, y)$ near (x_0, y_0). If we denote $x - x_0$ by Δx, $y - y_0$ by Δy and $z - z_0$ by Δz, then the best linear approximation for Δz is

$$\Delta z \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.$$

Write $\Delta x = dx, \Delta y = dy$. Then the quantity

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy$$

is called the **total differential** of $f(x, y)$ at (x_0, y_0).

Our above discussion shows that when $dx = \Delta x$ and $dy = \Delta y$ are small,

(i) dz is the best linear approximation of Δz,

(ii) $z_0 + dz$ is the best linear approximation of $f(x_0 + dx, y_0 + dy)$

Example 1. Find an equation of the tangent plane to the surface $z = x^2 + y^2$ at $(1, 1, 2)$.

Solution $f(x, y) = x^2 + y^2$. At $(x, y) = (1, 1)$,
$f_x(1, 1) = 2x|_{(1,1)} = 2, \quad f_y(1, 1) = 2y|_{(1,1)} = 2$

Therefore, the equation is

$$2(x - 1) + 2(y - 1) - (z - 2) = 0.$$

i.e. $2x + 2y - z - 2 = 0$. □

Example 2 Use total differential to approximate $f(1.01, 1.99)$ where $f(x, y) = x^3y^4$.

Solution $f(1.01, 1.99) = f(1 + 0.01, 2 - 0.01)$

$$\begin{aligned}
&\approx f(1,2) + f_x(1,2)(0.01) + f_y(1,2)(-0.01) \\
&= (1)^3(2)^4 + 3(1)^2(2)^4(0.01) + 4(1)^3(2)^3(-0.01) \\
&= 16 + 0.48 - 0.32 \\
&= 16.16
\end{aligned}$$

Lecture 15 Directional Derivatives and Gradients

Let us recall that if $C : x = x(t), y = y(t)$ is a smooth curve passing through (x_0, y_0) at $t = t_0$, i.e., $x(t_0) = x_0$, $y = (t_0) = y_0$, then the limit of $f(x, y)$ at (x_0, y_0) along C is

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\ \text{(along } C)}} f(x, y) = \lim_{t\to t_0} f(x(t), y(t))$$

Let $\vec{u} = \langle u_1, u_2\rangle$ be a given unit vector. Then the straight line $l : x = x_0 + u_1t$, $y = y_0 + u_2t$ passes through (x_0, y_0) at $t = 0$ and has positive direction $\vec{u}$. Of course we can calculate the limit of $f(x, y)$ at (x_0, y_0) along l. Moreover, we can also find the derivative of $f(x, y)$ along l as follows.

Along l, $f(x, y) = f(x_0 + u_1t, y_0 + u_2t)$. Therefore,

$$\frac{d}{dt} f(x_0 + u_1t, y_0 + u_2t) = f_x(x_0 + u_1t, y_0 + u_2t)u_1 + f_y(x_0 + u_1t, y_0 + u_2t)u_2.$$

At $t = 0$ this derivative is $f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2$, which is called the **directional derivative** of $f(x, y)$ at (x_0, y_0) in the direction of $\vec{u}$, and is denoted by $D_{\vec{u}}f(x_0, y_0)$

Remark: We always require $\vec{u}$ to be a unit vector here.

Example 1 Find the directional derivative of $f(x, y) = e^{xy}$ at (1, 1) in the direction of $\vec{a} = 3\vec{i} + 4\vec{j}$.

Solution Since $\vec{a}$ is not a unit vector, we need firstly to normalise it:

$$\vec{u} = \frac{\vec{a}}{||\vec{a}||} = \frac{3\vec{i} + 4\vec{j}}{\sqrt{3^2 + 4^2}} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

The directional derivative is

$$\begin{aligned} D_{\vec{u}}f(1, 1) &= \left(f_x\frac{3}{5} + f_y\frac{4}{5}\right)\Big|_{(1,1)} \\ &= \left(ye^{xy}\frac{3}{5} + xe^{xy}\frac{4}{5}\right)\Big|_{(1,1)} \\ &= \frac{7}{5}e. \end{aligned}$$

If we introduce the vector $\vec{u}_0 = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}$, then

$$\begin{aligned} D_{\vec{u}} f(x_0, y_0) &= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \\ &= \vec{u}_0 \cdot \vec{u} \qquad \text{(by definition of dot product)} \\ &= ||\vec{u}_0|| ||\vec{u}|| \cos\theta \\ &= ||\vec{u}_0|| \cos\theta \qquad \text{(since } \vec{u} \text{ is a unit vector)} \end{aligned}$$

where θ is the angle between $\vec{u}$ and $\vec{u}_0$. It follows, as $-1 \leq \cos\theta \leq 1$,

$$-||\vec{u}_0|| \leq D_{\vec{u}} f(x_0, y_0) \leq ||\vec{u}_0||$$

and $D_{\vec{u}} f(x_0, y_0)$ takes the maximum $||\vec{u}_0||$ if $\theta = 0$, i.e. $\vec{u}$ is in the direction of $\vec{u}_0$; it takes the minimum $-||\vec{u}_0||$ if $\theta = \pi$, i.e. $\vec{u}$ is in the opposite direction of $\vec{u}_0$.

Recall that the derivative measures the rate of change. Therefore the above observation can be interpreted as the following:

The function $f(x, y)$ increases most rapidly near (x_0, y_0) in the direction of $\vec{u}_0 = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}$, and it decreases most rapidly near (x_0, y_0) in the opposite direction of $\vec{u}_0$.

We call the vector $f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}$ the **gradient** of $f(x, y)$ at (x_0, y_0), and denote it by $\bigtriangledown f(x_0, y_0)$. Thus

$$\boxed{\begin{array}{l} \bigtriangledown f(x_0, y_0) = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j} \\ D_{\vec{u}} f(x_0, y_0) = \bigtriangledown f(x_0, y_0) \cdot \vec{u} \end{array}}$$

Denote $k_0 = f(x_0, y_0)$. Then $f(x, y) = k_0$ gives a curve, known as the level curve passing through (x_0, y_0). Clearly the value of $f(x, y)$ does not change along this level curve (the value of $f(x, y)$ is fixed at k_0 there). We have already known that the value of $f(x, y)$ changes most rapidly near (x_0, y_0) in the direction of $\bigtriangledown f(x_0, y_0)$. Let us show in the following that $\bigtriangledown f(x_0, y_0)$ is in fact perpendicular to the level curve at (x_0, y_0), i.e. $\bigtriangledown f(x_0, y_0)$ is perpendicular to the tangent line of this level curve at (x_0, y_0).

Suppose $x = x(t),\ y = y(t)$ with $x(t_0) = x_0$, $y(t_0) = y_0$ is a parametric equation for the level curve $f(x, y) = k_0$. Then $k_0 = f(x(t), y(t))$ for all t.

We differentiate this identity and obtain

$$0 = \frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t)y'(t) \quad \text{for all } t.$$

Take $t = t_0$. It gives

$$\begin{aligned} 0 &= f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0) \\ &= \bigtriangledown f(x_0, y_0) \cdot \langle x'(t_0), y'(t_0)\rangle. \end{aligned}$$

But $\langle x'(t_0), y'(t_0)\rangle$ is a vector parallel to the tangent line of the level curve at (x_0, y_0). Hence the above identity implies that $\bigtriangledown f(x_0, y_0)$ is perpendicular to the tangent line of the level curve at (x_0, y_0).

To summarise, we have proved the following theorem.

Theorem 1

(1) $D_{\vec{u}}f(x_0, y_0)$ takes the largest value $||\bigtriangledown f(x_0, y_0)||$ among all possible directions when $\vec{u}$ is in the same direction of $\bigtriangledown f(x_0, y_0)$; it takes the smallest value $-||\bigtriangledown f(x_0, y_0)||$ when $\vec{u}$ is in the opposite direction of $\bigtriangledown f(x_0, y_0)$.

(2) $\bigtriangledown f(x_0, y_0)$ is perpendicular, at (x_0, y_0), to the level curve of $f(x, y)$ passing through (x_0, y_0).

Example 2 Find the equation of the level curve of the function $f(x, y) = x^3 + y^3$ passing through (1, 2) and then find the equation of the normal line of this level curve at (1, 2).

Solution. $f(1, 2) = (1)^3 + (2)^3 = 9$. Therefore the equation of the level curve is

$$x^3 + y^3 = 9.$$

By theorem 1, we know $\bigtriangledown f(1, 2) = (3x^2\vec{i} + 3y^2\vec{j})|_{(1,2)} = 3\vec{i} + 12\vec{j}$ is perpendicular to the level curve at (1,2) and hence is parallel to the normal line. Therefore, the normal line has equation

$$x = 1 + 3t, \quad y = 2 + 12t$$

or

$$\frac{x-1}{3} = \frac{y-2}{12}$$

or

$$4x - y - 2 = 0$$

$\square$

Lecture 16 Functions of Three and n Variables

All the definitions and results for two variable functions developed so far can be extended to functions of three or n variables.

In the following, we list some of the most important definitions and properties for two and three variable functions:

(a) $f(x, y)$ is **differentiable** at (x_0, y_0) if $f_x(x_0, y_0), f_y(x_0, y_0)$ exist and

$$\lim_{(\Delta x, \Delta y) \longrightarrow (0,0)} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) - f_x(x_0, y_0)\Delta x - f_y(x_0, y_0)\Delta y)}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$$

(a′) $f(x, y, z)$ is **differentiable** at (x_0, y_0, z_0) if $f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)$ exist and

$$\lim_{(\Delta x, \Delta y, \Delta z) \longrightarrow (0,0,0)} \frac{\Delta f - f_x(x_0, y_0, z_0)\Delta x - f_y(x_0, y_0, z_0)\Delta y - f_z(x_0, y_0, z_0)\Delta z}{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}} = 0$$

where $\Delta f = f(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z) - f(x_0, y_0, z_0)$.

(b) If $f_x(x, y), f_y(x, y)$ exist for (x, y) near (x_0, y_0) and they are continuous at (x_0, y_0) then $f(x, y)$ is differentiable at (x_0, y_0).

(b′) If $f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)$ exist for (x, y, z) near (x_0, y_0, z_0) and they are continuous at (x_0, y_0, z_0), then $f(x, y, z)$ is differentiable at (x_0, y_0, z_0).

(c) If $z = f(x, y), x = x(t), y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

(c′) If $w = f(x, y, z), x = x(t), y = y(t), z = z(t)$, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

(d) If $\vec{u} = u_1\vec{i} + u_2\vec{j}$ and $||\vec{u}|| = 1$, then

$$\begin{aligned} D_{\vec{u}} f(x_0, y_0) &= f_x(x_0, y_0)u_1 + f_y(x_0, y_0)u_2 \\ &= \nabla f(x_0, y_0) \cdot \vec{u}, \end{aligned}$$

where $\nabla f(x_0, y_0) = f_x(x_0, y_0)\vec{i} + f_y(x_0, y_0)\vec{j}$ is the **gradient** of $f(x, y)$ at (x_0, y_0).

(d′) If $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $||\vec{u}|| = 1$, then

$$\begin{aligned} D_{\vec{u}}f(x_0,y_0,z_0) &= f_x(x_0,y_0,z_0)u_1 + f_y(x_0,y_0z_0)u_2 + f_z(x_0,y_0,z_0)u_3 \\ &= \nabla f(x_0,y_0,z_0)\cdot\vec{u}, \end{aligned}$$

where $\nabla f(x_0,y_0,z_0) = f_x(x_0,y_0,z_0)\vec{i} + f_y(x_0,y_0,z_0)\vec{j} + f_z(x_0,y_0,z_0)\vec{k}$ is the **gradient** of $f(x,y,z)$ at (x_0,y_0,z_0).

(e) $\nabla f(x_0,y_0)$ is the direction when the directional derivative of $f(x,y)$ at (x_0,y_0) takes maximum value among all the directions.

(e′) $\nabla f(x_0,y_0,z_0)$ is the direction when the directional derivative of $f(x,y,z)$ at (x_0,y_0,z_0) takes maximum value among all the directions.

(f) $\nabla f(x_0,y_0)$ is perpendicular to the level curve of $f(x,y)$ through (x_0,y_0).

(f′) $\nabla f(x_0,y_0,z_0)$ is perpendicular to the level surface of $f(x,y,z)$ through (x_0,y_0,z_0).

(g) $dz = f_x(x_0,y_0)dx + f_y(x_0,y_0)dy$ is called the **total differential** of $f(x,y)$ at (x_0,y_0), and dz is the best linear approximation of Δz when dx and dy are small.

(g′) $dw = f_x(w_0,y_0,z_0)dx + f_y(x_0,y_0,z_0)dy + f_z(x_0,y_0,z_0)$ is called the **total differential** of $f(x,y,z)$ at (x_0,y_0,z_0), and dw is the best linear approximation of Δw when dx, dy and dz are small.

The generalization to n–variable functions is similar. For example, the directional derivative is given by

$$\begin{aligned} D_{\vec{u}}f(x_1^0,\ldots,x_n^0) &= f_{x_1}(x_1^0,\ldots,x_n^0)u_1 + \ldots + f_{x_n}(x_1^0,\ldots,x_n^0)u_n \\ &= \nabla f(x_1^0,\ldots,x_n^0)\cdot\vec{u} \end{aligned}$$

$\nabla f(x_1^0,\ldots,x_n^0) = f_{x_1}(x_1^0,\ldots,x_n^0)\vec{e}_1 + \ldots + f_{x_n}(x_1^0,\ldots,x_n^0)\vec{e}_n$ is the **gradient** of $f(x_1,\ldots,x_n)$ at $(x_1^0,\ldots,x_n^0)$, where

$$\vec{e}_1 = \langle 1,0,\ldots,0\rangle, \vec{e}_2 = \langle 0,1,0,\ldots,0\rangle, \ldots, \vec{e}_n = \langle 0,\ldots,0,1\rangle.$$

$dw = f_{x_1}(x_1^0,\ldots,x_n^0)dx_1 + \ldots + f_{x_n}(x_1^0,\ldots,x_n^0)dx_n$ is the **total differential** of $f(x_1,\ldots,,x_n)$ at $(x_1^0,\ldots,x_n^0)$.

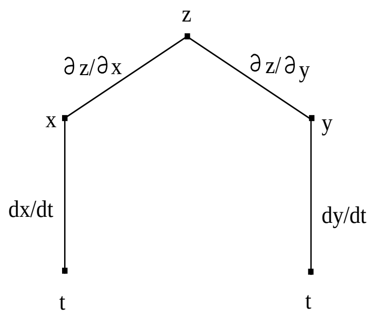
For functions of three or more variables, there are more versions of the chain rule than for two variable functions. It is impossible to give a complete list of such chain rules. However, the so called **Tree Diagrams For the Chain Rules** are very useful and convenient. Let us see how these diagrams are used through several examples.

Example 1

(a) The chain rule

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

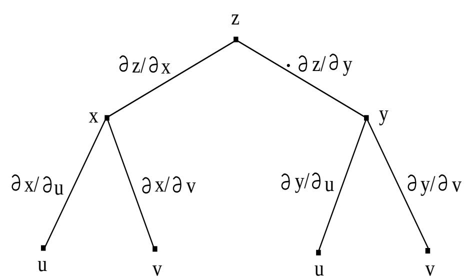
can be explained by the following tree diagram:



(b) The chain rule

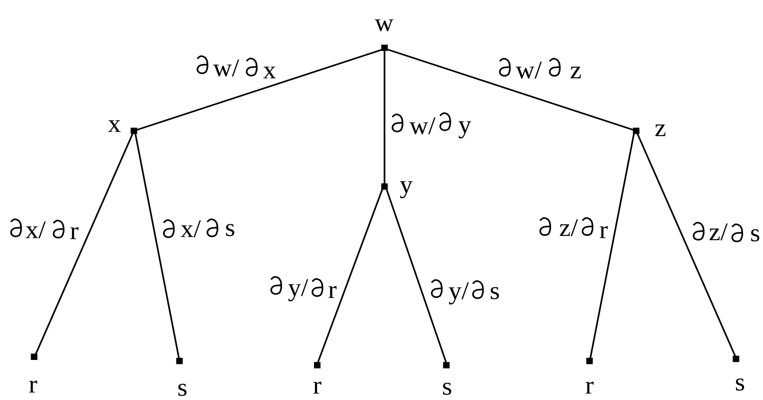
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

can be expressed by the tree diagram on the right.



Example 2 Let $w = f(x, y, z), x = x(r, s), y = y(r, s), z = z(r, s)$. Use a tree diagram to find the corresponding chain rule.

Solution. The tree diagram should look like the following:



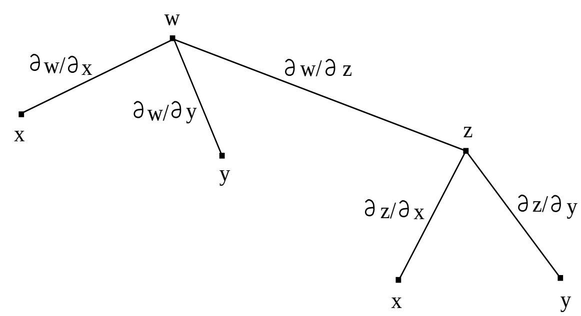
The chain rules are

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

□

Example 3. Let $w = f(x, y, z), z = z(x, y)$. Find $\frac{\partial}{\partial x} f(x, y, z(x, y))$.

Solution. The tree diagram is the following



The chain rule is

$$\frac{\partial}{\partial x} f(x, y, z(x, y)) = \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}.$$

□

The chain rules are perhaps the most difficult part in the calculation of partial derivatives. You should have plenty of practice with them in order to master this skill.

Lecture 17 Multivariable Taylor formula

Taylor's formula tells us that a function that has $n+1$ continuous derivatives at x_0 can be approximated by a polynomial of degree n so that the error term E_n tends to 0 faster than $(x-x_0)^n$. This can be generalised to multivariable functions. For simplicity we start with the second order expansion of a function of two variables $f(x,y)$ with reference point $(0,0)$. We have

$$f(x,y) = f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2}(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}y^2)+E_2(x,y).$$

The error term $E_2 = \mathrm{o}(x^2+y^2) = \mathrm{o}(||\langle x,y\rangle||)$.

This follows from the one-variable Maclaurin formula for the auxiliary function $g(t) = f(tx,ty)\colon [0,1] \to \mathbb{R}$

$$g(t) = g(0)+g'(0)t+\frac{1}{2}g''(0)t^2+\frac{1}{6}g'''(\theta)t^3 = g(0)+g'(0)t+\frac{1}{2}g''(0)t^2+E_2 \quad (10)$$

where θ is some unknown number between 0 and 1. Now

$$\begin{aligned}
g(0) =& f(0,0)\\
g'(0) =& \frac{\partial f}{\partial x}(0,0)\frac{d}{dt}(tx)|_{t=0}+\frac{\partial f}{\partial y}(0,0)\frac{d}{dt}(ty)|_{t=0} = f_y(0)y\\
g''(0) =& \frac{\partial^2 f}{\partial x^2}(0,0)(\frac{d}{dt}(tx)|_{t=0})^2+\frac{\partial^2 f}{\partial x\partial y}(0,0)\frac{d}{dt}(tx)|_{t=0}\frac{d}{dt}(ty)|_{t=0}+\frac{\partial f}{\partial x}(0,0)\frac{d^2}{dt^2}(tx)|_{t=0}\\
&+\frac{\partial^2 f}{\partial y\partial x}(0,0)\frac{d}{dt}(ty)|_{t=0}\frac{d}{dt}(tx)|_{t=0}+\frac{\partial^2 f}{\partial y^2}(0,0)(\frac{d}{dt}(ty)|_{t=0})^2+\frac{\partial f}{\partial y}(0,0)\frac{d^2}{dt^2}(ty)|_{t=0}\\
=& f_{xx}(0,0)x^2+f_{xy}(0,0)xy+f_{yx}(0,0)yx+f_{yy}(0,0)y^2\\
=& f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2.
\end{aligned}$$

A similar computation shows that $g'''(\theta)$ is polynomial of pure 3rd order in x,y whose coefficients are third order partial derivatives of f, which are bounded by a constant M. Hence

$$|E_2| \le M||\langle x,y\rangle||^3 = \mathrm{o}(||\langle x,y\rangle||^2).$$

From equation (10) with $t=1$ we get now the desired formula

$$f(x,y) = f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2}(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2)+\mathrm{o}(||\langle x,y\rangle||^2).$$

Example. Find the second order Maclaurin formula $f(x,y) = \ln(1+x^2+y^2)$. This can be done by substituting $t = x^2+y^2$ into the one-variable Maclaurin formula $\ln(1+t) = t-\frac{1}{2}t^2+\mathrm{o}(t^2)$. Since $\frac{1}{2}t^2 = \frac{1}{2}(x^2+y^2)^2 = \mathrm{o}(||\langle x,y\rangle||^2)$ the result is

$$f(x,y) = x^2+y^2+\mathrm{o}(||\langle x,y\rangle||^2).$$

This can be used to compute limits, e.g.

$$\lim_{(x,y)\to(0,0)} \frac{\ln(1+x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2+y^2+\mathrm{o}(||\langle x,y\rangle||^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} 1+\mathrm{o}(1) = 1.$$

The second order Taylor formula with reference point (x_0, y_0) is

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{1}{2}(f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2) \\ &\qquad + \mathrm{o}(||\langle x-x_0, y-y_0\rangle||^2). \end{aligned}$$

The corresponding formula for fuctions of three variables is

$$\begin{aligned} f(x,y,z) &= f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) \\ &+ \frac{1}{2}(f_{xx}(x_0,y_0,z_0)(x-x_0)^2 + f_{yy}(x_0,y_0,z_0)(y-y_0)^2 + f_{zz}(x_0,y_0,z_0)(z-z_0)^2) \\ &+ f_{xy}(x_0,y_0,z_0)(x-x_0)(y-y_0) + f_{xz}(x_0,y_0,z_0)(x-x_0)(z-z_0) + f_{yz}(x_0,y_0,z_0)(y-y_0)(z-z_0) \\ &\qquad + \mathrm{o}(||\langle x-x_0, y-y_0, z-z_0\rangle||^2). \end{aligned}$$

Lecture 18 Parametric Problems (optional)

For $z = f(x, y)$ or $w = f(x, y, z)$, we have defined partial derivatives like $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial w}{\partial y}$, $\dfrac{\partial w}{\partial z}$, etc. These are **differentiations** with all except one variable held constant. Naturally, we can also perform integrations to multivariable functions but with all except one variable held constant, for example, $\int f(x, y)\ dy$ means that x is held constant and the integration is for the variable y only; similarly, $\int_a^b f(x, y, z)dx$ means y and z are held constant, and the definite integral is performed for the variable x only.

Thus, if $f(x, y) = x^2 + y^2 + xy$, then

$$\begin{aligned}\int_a^b f(x,y)dy &= \int_a^b (x^2 + y^2 + xy)dy \\ &= \left[x^2 y + \frac{y^3}{3} + \frac{x}{2}y^2\right]\Bigg|_a^b \\ &= x^2(b-a) + \frac{b^3 - a^3}{3} + \frac{x}{2}(b^2 - a^2)\end{aligned}$$

We see that $\int_a^b f(x, y)dy$ is a function of x, which we can denote as $F(x)$, where a and b are regarded as given constants.

Let us note that $F'(x) = 2x(b-a)+\frac{1}{2}(b^2-a^2)$ by differentiation of the expression of $F(x)$, i.e. $F(x) = x^2(b-a) + \frac{b^3-a^3}{3} + \frac{x}{2}(b^2 - a^2)$.

On the other hand, we have

$$\begin{aligned}f_x(x,y) &= 2x + y \qquad \text{and} \\ \int_a^b f_x(x,y)dy &= \int_a^b (2x + y)dy = \left(2xy + \frac{y^2}{2}\right)\Bigg|_a^b \\ &= 2x(b-a) + \frac{b^2 - a^2}{2}\end{aligned}$$

Therefore, we have

$$F'(x) = \int_a^b f_x(x,y)dy, \text{ i.e.}$$
$$\frac{d}{dx}\left[\int_a^b f(x,y)dy\right] = \int_a^b \frac{\partial}{\partial x} f(x,y)dy \tag{1}$$

This is to say, whether we perform integration first and differentiation second, or differentiation first and integration second, we may arrive at the same result.

Let us note, however, that our above discussion is not a rigorous proof that identity (1) holds for any function $f(x, y)$ as we merely checked that this identity happens to be true for the particular function $f(x, y) = x^2 + y^2 + xy$.

Nevertheless, identity (1) might be true in general, and it is an interesting problem to find an answer for this. The following theorem is a result of some research along this line.

Theorem 1. Suppose that for every $x \in (c, d)$, where $-\infty \leq c < d \leq \infty$, the following hold:

(i) $\int_a^b f(x, y)dy$ and $\int_a^b \frac{\partial}{\partial x} f(x, y)dy$ exist,

(ii) $f_{xx}(x, y)$ exists and satisfies

$$|f_{xx}(x, y)| \leq g(y) \quad \text{for} \quad a \leq y \leq b,\ c < x < d,$$

where $g(y)$ is some function with the property that

$$\int_a^b g(y)dy = K < \infty$$

Then for each $x \in (c, d)$,

$$\frac{d}{dx} \int_a^b f(x, y)dy = \int_a^b \frac{\partial}{\partial x} f(x, y)dy$$

Remark. Let us note that condition (i) in Theorem 1 is very natural. In fact, it is necessary for identity (1) to make any sense. However, condition (ii) makes a restriction to the function $f(x, y)$. For example, if $f(x, y) = \frac{x^2}{2} g(y)$, then $f_{xx}(x, y) = g(y)$ and such an $f(x, y)$ satisfies (ii) only if $\int_a^b g(y)dy < \infty$. Nevertheless, many functions satisfy (ii).

Proof of Theorem 1. Let us denote $F(x) = \int_a^b f(x, y)dy$. We need to show

$$F'(x) = \int_a^b \frac{\partial}{\partial x} f(x, y)dy,$$

i.e.

$$\lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \int_a^b \frac{\partial}{\partial x} f(x, y)dy,$$

or

$$\lim_{h \to 0} \left[\frac{F(x + h) - F(x)}{h} - \int_a^b \frac{\partial}{\partial x} f(x, y)dy \right] = 0$$

Our strategy of achieving this is to use a **squeezing argument:** if we can find a function $\epsilon(h)$ satisfying

$$0 \le \left| \frac{F(x+h)-F(x)}{h} - \int_a^b \frac{\partial}{\partial x} f(x,y)dy \right| \le \epsilon(h)$$

and $\epsilon(h) \to 0$ as $h \to 0$, then we must have

$$\lim_{h\to 0} \left| \frac{F(x+h)-F(x)}{h} - \int_a^b \frac{\partial}{\partial x} f(x,y)dy \right| = 0$$

which implies that the limit of the quantity inside the absolute value sign is 0. A key step in this strategy is to rewrite and change the quantity inside the absolute value sign through using inequalities, to arrive at a simpler expression which can be used as $\epsilon(h)$. The criterion is that $\epsilon(h) \to 0$ should be evident.

Let us now start this process. We have

$$\begin{aligned}
& \left| \frac{F(x+h)-F(x)}{h} - \int_a^b \frac{\partial}{\partial x} f(x,y)dy \right| \\
= & \left| \frac{1}{h}\left[\int_a^b f(x+h,y)dy - \int_a^b f(x,y)dy\right] - \int_a^b \frac{\partial}{\partial x} f(x,y)dy \right| \\
= & \left| \int_a^b \left[\frac{f(x+h,y)-f(x,y)}{h} - \frac{\partial}{\partial x} f(x,y)\right] dy \right| \quad \text{(by properties of integrals)} \\
= & \left| \int_a^b \left[\frac{f_x(x,y)h + f_{xx}(x+\theta h, y)\frac{h^2}{2}}{h} - f_x(x,y)\right] dy \right| \quad \text{(by using Taylor's formula)} \\
= & \left| \int_a^b \frac{h}{2} f_{xx}(x+\theta h,y)dy \right| \\
\le & \frac{|h|}{2} \int_a^b |f_{xx}(x+\theta h,y)|dy \\
\le & \frac{|h|}{2} \int_a^b g(y)dy \quad \text{by condition (ii)} \\
= & K \cdot \frac{|h|}{2}
\end{aligned}$$

Clearly we can take $\epsilon(h) = K \cdot \frac{|h|}{2}$. This finishes the squeezing process and hence proved what we want. □

Example 1 Find $\displaystyle\frac{d}{dx}\int_1^\infty \frac{e^{-xy}}{y}dy \quad (x > 0)$.

Solution $\displaystyle f(x,y) = \frac{e^{-xy}}{y}, \; f_x(x,y) = -e^{-xy}, \quad f_{xx}(x,y) = ye^{-xy}$

For any $d > c > 0$, $f_{xx}(x,y) = ye^{-xy} \le ye^{-cy}$ when $c < x < d$ and $g(y) = ye^{-cy}$ is clearly integrable on $[1,\infty)$. Hence condition (ii) of Theorem 1 is satisfied. It is easily seen that condition (i) is also satisfied (please check this). Therefore we can use theorem 1 to conclude

$$\frac{d}{dx}\int_1^\infty \frac{e^{-xy}}{y}dy = \int_1^\infty \frac{\partial}{\partial x}\left(\frac{e^{-xy}}{y}\right)dy$$
$$= \int_1^\infty -e^{-xy}dy = \frac{e^{-xy}}{x}\Big|_1^\infty = \frac{-e^{-x}}{x}.$$

□

From the last part of the above calculation, we observe that if we change the integration limits to from 0 to ∞, then

$$\int_0^\infty e^{-xy}dy = \frac{e^{-xy}}{x}\Big|_0^\infty = \frac{1}{x}.$$

Therefore, by using Theorem 1 (please check that the conditions are satisfied),

$$\begin{aligned} \frac{-1}{x^2} &= \left(\frac{1}{x}\right)' = \frac{d}{dx}\int_0^\infty e^{-xy}dy = \int_0^\infty \frac{\partial}{\partial x}e^{-xy}dy = \int_0^\infty -ye^{-xy}dy \\ \frac{2}{x^3} &= \left(\frac{-1}{x^2}\right)' = \frac{d}{dx}\int_0^\infty -ye^{-xy}dy = \int_0^\infty \frac{\partial}{\partial x}(-ye^{-xy})dy = \int_0^\infty y^2e^{-xy}dy \end{aligned}$$

$$\cdots \quad \cdots$$

$$\frac{(-1)^n n!}{x^{n+1}} = \int_0^\infty (-1)^n y^n e^{-xy}dy$$

This last identity can be written as

$$\int_0^\infty y^n e^{-xy}dy = \frac{n!}{x^{n+1}},\ n = 1, 2\ldots$$

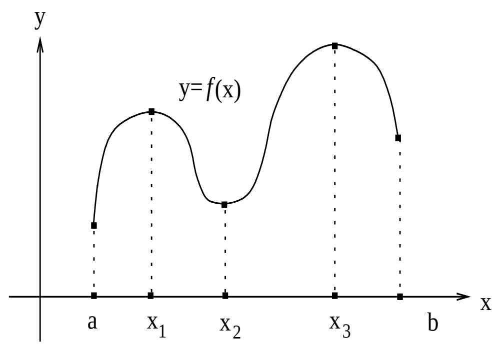
This turns out to be a useful formula. For example, if we let $x = m$, we obtain

$$\int_0^\infty y^n e^{-my}dy = \frac{n!}{m^{n+1}},\ n = 1, 2, \ldots, \quad m = 1, 2, \ldots$$

This formula may not be as easily proved by other methods (can you?)

Lecture 19 Maxima and Minima

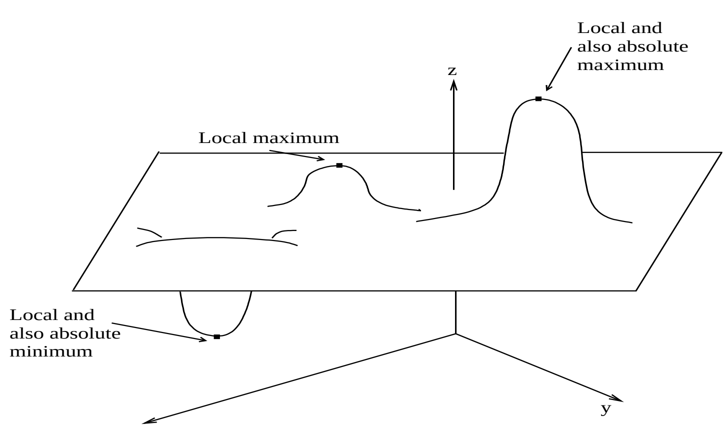
Recall that if the graph of the function $y = f(x)$ is as in the following diagram, then $f(x)$ has local maxima at $x = x_1, x = x_3$ and it has a local minimum at $x = x_2$. The absolute maximum in $[a, b]$ is achieved at $x = x_3$ and the absolute minimum is achieved at $x = a$.



These notions carry to multivariable functions naturally. We will study in detail the situation with two variable functions.

Definitions. We say $f(x, y)$ has a **relative maximum** (or **local maximum**) at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) near (x_0, y_0). If the inequality sign is reversed, then we say $f(x, y)$ has a **relative minimum** (or **local minimum**) at (x_0, y_0).

$f(x, y)$ is said to have an absolute maximum at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in the domain of $f(x, y)$; if this inequality is reversed, then we say $f(x, y)$ has an **absolute minimum** at (x_0, y_0).



Theorem 1 (Necessary condition) If $f(x, y)$ has a relative extremum (i.e., it has either a local max or local min) at (x_0, y_0), and if $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ exist, then

$$f_x(x_0, y_0) = 0, \quad f_y(x_0, y_0) = 0$$

Proof. Denote $G(x) = f(x, y_0)$ and $H(y) = f(x_0, y)$. Then $G(x)$ has a local extremum at $x = x_0$, $H(y)$ has a local extremum at $y = y_0$. Therefore, by the properties for single variable functions, $G'(x_0) = 0$, $H'(y_0) = 0$. Since $G'(x_0) = f_x(x_0, y_0), H'(y_0) = f_y(x_0, y_0)$, we obtain

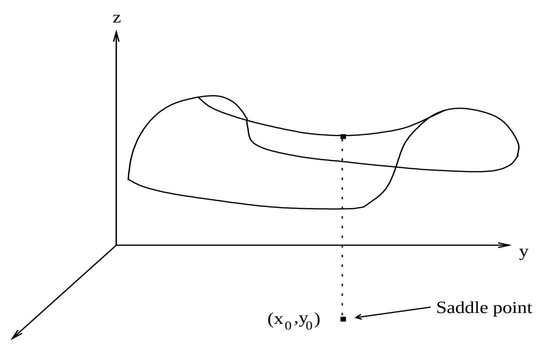
$$f_x(x_0, y_0) = 0, f_y(x_0, y_0) = 0.$$

□

Theorem 1 motivates the introduction of the following definition.

Definition. If $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, then (x_0, y_0) is called a **critical point** of $f(x, y)$.

Theorem 1 implies that a local extremum point must be a critical point (provided that the partial derivatives exist). The converse conclusion, i.e., a critical point must be a local extremum point, however, is not true in general. When a critical point is not a local extremum point, then it is called a **saddle point**. A typical saddle point is indicated in the following diagram.



Note that if (x_0, y_0) is a critical point of $z = f(x, y)$, then at (x_0, y_0, z_0) on the surface, the tangent plane has normal $\vec{n} = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle = \langle 0, 0, -1\rangle$, which is perpendicular to the xy-plane. Therefore the tangent plane is horizontal. In particular, the tangent plane at a saddle point is horizontal.

Just as in the case of single variable functions, to ensure that a critical point is a local extremum point for a two variable function, one needs extra conditions. This is the contents of the following theorem, whose proof we omit.

Theorem 2 (Sufficient conditions, also known as the second derivative test) Suppose that (x_0, y_0) is a critical point of $f(x, y)$, and $f(x, y)$ has continuous second order partial derivatives near (x_0, y_0). Denote

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0]^2.$$

Then

(a) $D > 0$ and $f_{xx}(x_0, y_0) > 0$ imply that $f(x, y)$ has a local minimum at (x_0, y_0);

(b) $D > 0$ and $f_{xx}(y_0, y_0) < 0$ imply that $f(x, y)$ has a local maximum at (x_0, y_0);

(c) $D < 0$ implies that (x_0, y_0) is a saddle point;

(d) $D = 0$ implies that no conclusion can be drawn unless higher order partial derivatives are used.

Proof. If (x_0, y_0) is a critical point of f then the second order Taylor polynomial is

$$f(x,y) \approx f(x_0,y_0)+\frac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2+\frac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2+f_{xy}(x_0,y_0)(x-x_0)(y-y_0).$$

The graph f is approximately a paraboloid of the form

$$z = D + A(x - x_0)^2 + 2B(x - x_0)(y - y_0) + C(y - y_0)^2$$

where $D = f(x_0, y_0)$, $A = \frac{1}{2}f_{xx}(x_0, y_0)$, $B = \frac{1}{2}f_{xy}(x_0, y_0)$, $\frac{1}{2}f_{yy}(x_0, y_0)$. We have a cup-like elliptic paraboloid, corresponding to a minimum, if $A > 0$ and $AC - B^2 > 0$, an upside-down elliptic paraboloid, corresponding to a maximum, if $A < 0$ and $AC - B^2 > 0$ and a saddle if $AC - B^2 < 0$. The remaining cases are inconclusive.

This theorem is very useful in classifying critical points of a given function $f(x, y)$, and is the main tool in finding local extremum points. Let us see how the theorem is used in some concrete problems below.

Example. Find all local extrema of $f(x, y) = 3xy - x^3 - y^3$.

Solution. This function is a polynomial in x and y. Therefore it has all the partial derivatives, and the partial derivatives are continuous (they are again polynomials in x and y).

By Theorem 1 we know all the extrema are critical points, and by Theorem 2, we can determine whether a critical point is local maximum or local minimum points, provided that case (d) does not occur.

Thus we can follow two main steps: Step 1, find all the critical points, and Step 2, classify the critical points obtained in Step 1 through using Theorem 2.

We know critical points are the solutions of the equation system

$$\begin{cases} f_x(x,y) &= 0 \\ f_y(x,y) &= 0 \end{cases}$$

i.e.

$$\begin{cases} 3y - 3x^2 &= 0 \qquad\qquad (1) \\ 3x - 3y^2 &= 0 \qquad\qquad (2) \end{cases}$$

¿From (1) we obtain $y = x^2$. Substituting this into (2), we obtain

$$3x - 3(x^2)^2 = 0, \quad \text{i.e.} \;\; x(1 - x^3) = 0$$

The solutions are $x = 0,\ x = 1$. When $x = 0,\ y = x^2 = 0$ and we obtain one critical point $(x, y) = (0, 0)$. When $x = 1,\ y = x^2 = 1$ and we obtain one more critical point $(x, y) = (1, 1)$. Since these are the only solutions to the system (1) - (2), we have exactly two critical points for the function. Step 1 is done.

To use Theorem 2, we calculate

$$f_{xx}(x, y) = -6x, f_{yy}(x, y) = -6y, f_{xy}(x, y) = 3.$$

Hence

$$D = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9$$

At $(0, 0)$, $D = -9 < 0$. By Theorem 2, this critical point is a saddle point.

At $(1, 1)$, $D = 36 - 9 > 0$ and $f_{xx} = -6 < 0$. By Theorem 2, (1, 1) is a local maximum point. This finishes step 2, and we conclude that the function $f(x, y)$ has one local maximum point at (1, 1), and there is no other extremum point. $\square$

Lecture 20 Extrema Over a Given Region

Very often, we need to find the absolute extrema of a given funciton $f(x, y)$ on a given region R which may only be part of the natural domain of $f(x, y)$. The following theorem guarantees that such extrema usually exist.

Theorem 1. If $f(x, y)$ is continuous on a closed and bounded region R, then f has both an absolute maximum and an absolute minimum on R.

Though this theorem looks evident, a rigorous proof is not trivial – it depends on the very notion of continuity and the theory on the completeness of real numbers, the later being outside the scope of this unit. Therefore we will not give the proof of this theorem here.

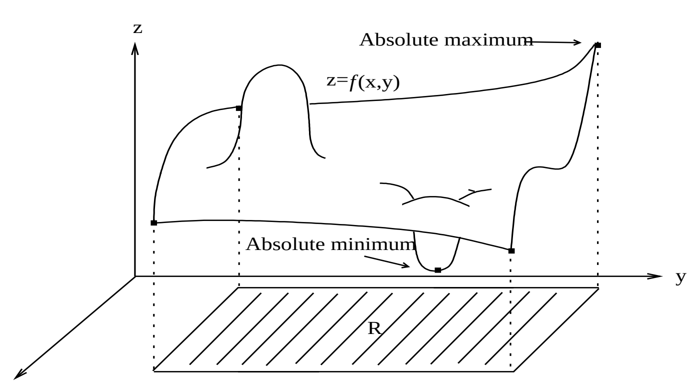
Nevertheless, let us see a few examples showing that each condition in the theorem is necessary.

Example 1. $f(x, y) = \frac{1}{x^2+y^2}$ is continuous on the bounded region $R = \{(x, y) : 0 < x^2 + y^2 \leq 1\}$, but it has no absolute maximum on R (when (x, y) approaches $(0, 0)$, $f(x, y)$ approached $+\infty$). The reason is that R is not closed.

Example 2 $f(x, y) = x^2 + y^2$ is continuous on the closed region $R = \{(x, y) : x^2+y^2 \geq 1\}$, but it has no absolute maximum on R. This is because R is unbounded.

Example 3. $f(x, y) = \begin{cases} \frac{1}{x^2+y^2} & \text{when}(x, y) \neq (0, 0) \\ 1 & \text{when } (x, y) = (0, 0) \end{cases}$ has no absolute maximum on the bounded closed region $R = \{(x, y) : x^2 + y^2 \leq 1\}$. This is because $f(x, y)$ is not continuous at $(0, 0)$.

The following diagram shows that when the conditions of Theorem 1 are satisfied, the extrema of $f(x, y)$ on R may occur at an interior point of R as well as a boundary point of R.



Theorem 2. If $f(x, y)$ has an absolute extremum at an interior point (x_0, y_0) of

R, then (x_0, y_0) is a critical point if the partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.

Proof. As such an extremum is also a local extremum, the conclusion follows form Theorem 1 in Lecture 18. □

The above theorem implies that absolute extrema of $f(x, y)$ on R can only occur at critical points or boundary points. This helps very much when we want to locate the absolute extrema. Indeed, we can follow the following three steps:

Finding absolute extrema of $f(x, y)$ over R:	
Step 1.	Find all the critical points that lie in the interior of R.
Step 2.	Find the boundary of R and the extrema of $f(x, y)$ over the boundary.
Step 3.	Evaluate $f(x, y)$ at all the points in Step 1, and compare them with the extrema obtained in Step 2. The largest is the absolute maximum, and the smallest is the absolute minimum.

Example Find the absolute maximum and minimum of $f(x, y) = x^2 + y^2 + 6xy - y$ on the closed triangular region R with vertices $(0, 0), (1, 0)$ and $(0, 1)$.

Solution. We follow the three steps listed above.

Step 1. Find critical points.

We need to solve

$$\begin{cases} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{cases}$$

simultaneously.

$$f_x = 2x + 6y, \ f_y = 2y + 6x - 1$$

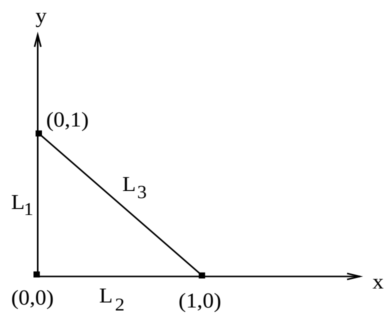
Solving

$$\begin{cases} 2x + 6y & = & 0 \\ 2y + 6x - 1 & = & 0 \end{cases}$$

we obtain one solution $x = \frac{3}{16}, y = -\frac{1}{16}$. Thus there is one critical point $(x, y) = \left(\frac{3}{16}, -\frac{1}{16}\right)$. A sketch of the region R shows this point is not in R. Therefore this point will not be counted.

Step 2. Find the boundary of R and the extrema of $f(x, y)$ on the boundary.

We need a sketch of the region R and the equations of its boundary. It is convenient in this case to divide the boundary of R into three parts L_1, L_2 and L_3, as indicated in the diagram.



On $L_1, x = 0$ and $f(x, y) = f(0, y) = y^2 - y, 0 \leq y \leq 1$. This is a single variable function $g(y) = y^2 - y$ over the interval $0 \leq y \leq 1$. Using first year calculus we can easily obtain that it attains maximum 0 at $y = 0$ and $y = 1$, minimum $-\frac{1}{4}$ at $y = \frac{1}{2}$.

On $L_2, y = 0$ and $f(x, y) = f(x, 0) = x^2, 0 \leq x \leq 1$. Clearly its maximum is 1 attained at $x = 1$, and minimum is 0 at $x = 0$.

On $L_3, y = 1 - x$ and $f(x, y) = f(x, 1 - x) = -4x^2 + 5x, 0 \leq x \leq 1$. Using first year calculus, we find the maximum is $\frac{25}{16}$ occurring at $x = \frac{5}{8}$, and minimum is 0 at $x = 0$.

Combining results on L_1, L_2 and L_3, we find that the maximum on the boundary is $\frac{25}{16}$, minimum on the boundary is $-\frac{1}{4}$.

Step 3. As there is no critical point in R, the absolute maximum on R is $\frac{25}{16}$ and absolute minimum on R is $-\frac{1}{4}$, both achieved on the boundary of R. $\square$.

Lecture 21 Lagrange Multipliers

We know from the last lecture that when we want to find the absolute extrema of $f(x,y)$ on a given bounded region R, we need to find the extrema of $f(x,y)$ on the boundary of R. If, for example, $f(x,y) = xy$ and $R = \{(x,y) : x^2 + y^2 \leq 1\}$, then the boundary of R is just the unit circle $C : x^2 + y^2 = 1$. To find the maximum of $f(x,y)$ on C is to find the maximum of $f(x,y)$ under the constraint $x^2 + y^2 = 1$. Such a maximum or minimum is called the **constrained maximum or minimum**.

In many practical situations, we need to solve such constrained extremum problems. Let us consider in some detail the constrained extremum problems for two and three variable functions.

Problem: Maximize or minimize $f(x,y)$ under the constraint $g(x,y) = 0$.

There is a general method, called the **Lagrange multiplies method**, which can usually be used effectively in solving this problem. It asserts that the extrema occur at such points (x_0, y_0) which satisfy

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0),$$

where λ is some unknown parameter, called the **Lagrange multiplier**.

We will not give the rigorous mathematical theory for the validity of this method. Instead, we will explain the use of this method through examples.

Example 1. At what point on the circle $x^2 + y^2 = 1$ does $f(x,y) = xy$ have a maximum? What is the maximum?

Solution. Denote $g(x,y) = x^2 + y^2 - 1$. Then we need to maximize $f(x,y)$ under the constraint $g(x,y) = 0$. By Lagrange multipliers method, maximum occurs at some (x,y) which solves

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{for some} \quad \lambda. \tag{11}$$

We have

$$\nabla f(x,y) = y\vec{i} + x\vec{j}, \quad \nabla g(x,y) = 2x\vec{i} + 2y\vec{j}$$

Therefore (1) is equivalent to

$$\left\{ \begin{array}{rcl} y & = & \lambda 2x \\ x & = & \lambda 2y \end{array} \right.$$

Remember we also require $g(x,y) = 0$. Therefore, we need to solve the system of equations

$$\begin{cases} y = \lambda 2x \;\cdots\cdots (2) \\ x = \lambda 2y \;\cdots\cdots (3) \\ x^2 + y^2 = 1 \;\cdots\cdots (4) \end{cases}$$

Substituting (2) to (3) we obtain

$$\begin{aligned} x = \lambda 2(\lambda 2x) &= 4x\lambda^2, \\ \text{i.e. } x(1-4\lambda^2) &= 0 \end{aligned}$$

Hence we have either (i) $x = 0$ or (ii) $1 - 4\lambda^2 = 0$, i.e. $\lambda = \pm\frac{1}{2}$. If case (i) happens, by (2), $y = 0$ and hence (4) can never be satisfied. This shows that (i) cannot occur. Thus we must have case (ii), i.e. $\lambda = \frac{1}{2}$ or $\lambda = -\frac{1}{2}$.

When $\lambda = \frac{1}{2}$, by (2), $y = x$. Substituting this into (4) we deduce $2x^2 = 1$, $x = \pm\frac{1}{\sqrt{2}}$. It gives in turn $y = \pm\frac{1}{\sqrt{2}}$. Therefore we obtain two pairs of solutions: $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), (x, y) = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$.

When $\lambda = -\frac{1}{2}$, by (2), $y = -x$. Substituting this into (4) and we obtain another two pairs of solutions: $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), (x, y) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

The Lagrange multiples method says the maximum occurs at some of these solutions. To determine at which solution, we calculate

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2},\ f\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{1}{2},\ f\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = -\frac{1}{2},\ f\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}.$$

Therefore, maximum is $\frac{1}{2}$, and it occurs at $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$. □

Lagrange multiples method works for three and n-variable functions as well. For three variable functions, it asserts that if an extremum of $f(x, y, z)$ under the constraint $g(x, y, z) = 0$ occurs at (x_0, y_0, z_0), then

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) \text{ for some parameter } \lambda.$$

Example 2. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest and furthest to the point $(2, 1, 1)$.

Solution. The distance from $(2, 1, 1)$ to an arbitrary point (x, y, z) is $d(x, y, z) = \sqrt{(x-2)^2 + (y-1)^2 + (z-1)^2}$. Therefore, our problem is equivalent to:

Minimize/maximise $d(x, y, z)$ under the constraint $x^2 + y^2 + z^2 = 1$.

This is equivalent to

$$\begin{array}{rl} \text{Minimize} & D(x,y,z) = (x-2)^2 + (y-1)^2 + (z-1)^2 \\ \text{under} & g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0 \end{array}$$

We will see later that working with the function $D(x,y,z)$ is much simpler than working with $d(x,y,z)$, though the minimization problems are equivalent.

Using Lagrange multipliers, we first solve

$$\begin{cases} \nabla D(x,y,z) = \lambda \quad \nabla g(x,y,z) \\ g(x,y,z) = 0 \end{cases}$$

$$\begin{array}{rcl} \text{Since } \nabla D(x,y,z) & = & 2(x-2)\vec{i} + 2(y-1)\vec{j} + 2(z-1)\vec{k} \text{ (compare with } \nabla d(x,y,z)) \\ \nabla g(x,y,z) & = & 2x\vec{i} + 2y\vec{j} + 2z\vec{k} \end{array}$$

We need to solve

$$\begin{cases} 2(x-2) & = & \lambda 2x & (5) \\ 2(y-1) & = & \lambda 2y & (6) \\ 2(z-1) & = & \lambda 2z & (7) \\ x^2+y^2+z^2 & = & 1 & (8) \end{cases}$$

$$(5) \Rightarrow x = \frac{2}{1-\lambda}, \ (6) \Rightarrow y = \frac{1}{1-\lambda}, \ (7) \Rightarrow z = \frac{1}{1-\lambda}$$

Substituting all these expressions into (8), we obtain

$$\left(\frac{2}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 + \left(\frac{1}{1-\lambda}\right)^2 = 1, \quad \text{i.e.}$$

$$(1-\lambda)^2 = 6, \quad \text{or} \quad 1-\lambda = \pm\sqrt{6}, \ \lambda = 1 \pm \sqrt{6}.$$

When $\lambda = 1-\sqrt{6}$, we obtain $x = \frac{2}{\sqrt{6}}, y = \frac{1}{\sqrt{6}}, z = \frac{1}{\sqrt{6}}$, or $(x,y,z) = \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$.

When $\lambda = 1+\sqrt{6}$, we obtain $(x,y,z) = \left(\frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$.

Now we calculate

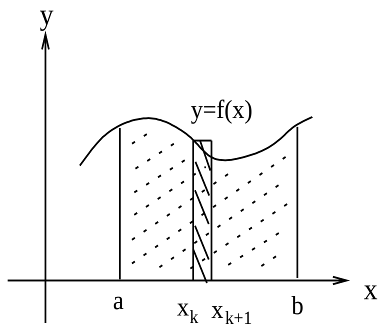
$$\begin{array}{rcl} D\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) & = & \left(\frac{2}{\sqrt{6}} - 2\right)^2 + \left(\frac{1}{\sqrt{6}} - 1\right)^2 + \left(\frac{1}{\sqrt{6}} - 1\right)^2 = 6\left(\frac{1}{\sqrt{6}} - 1\right)^2 \\ D\left(\frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right) & = & 6\left(\frac{1}{\sqrt{6}} + 1\right)^2. \end{array}$$

Therefore the minimum occurs at $\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ and the maximum occurs at $\left(\frac{-2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$.
□

Lecture 22 Double Integrals

Recall that the area under the curve $y = f(x)$ between $x = a$ and $x = b$ is given by

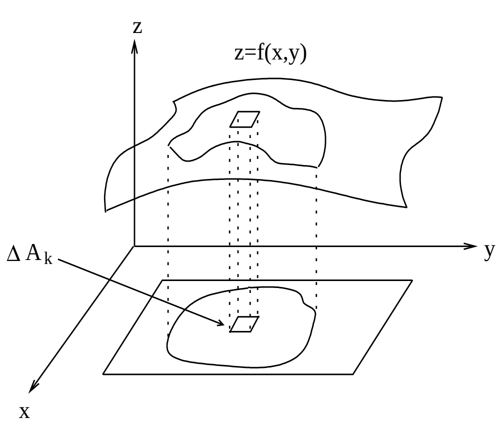
$$\int_a^b f(x)dx = \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k,$$



where for fixed n, $\sum_{k=1}^n f(x_k^*)\Delta x_k$ is the sum of the areas of n rectangles with base Δx_k and height $f(x_k^*), k = 1, \ldots, n$. $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$ divide the interval $[a, b]$ into n parts with $\Delta x_k = x_k - x_{k-1}$ converging to 0 (for all k) as $n \longrightarrow \infty, x_k^* \in [x_{k-1}, x_k]$. Thus, for fixed n, $\sum_{k=1}^n f(x_k^*)\Delta x_k$ gives an approximation of the area under the curve, and by passing to the limit $n \longrightarrow \infty$, we obtain the accurate area under the curve.

A similar consideration introduces double integrals. This time the question is to find the volume under a surface $z = f(x, y)$.

To be more accurate, we suppose $f(x, y)$ is positive everywhere, and we want to find the volume of the solid lying directly above a given region R in the xy–plane but under the surface $z = f(x, y)$.



We divide R into n small parts with area $\Delta A_k, k = 1, \ldots, n$ and choose an arbitrary point (x_k^*, y_k^*) in each part and form the cylinders with base ΔA_k and height $f(x_k^*, y_k^*)$. Then

$$\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

gives a good approximation of the volume under surface. If there is a limit for this quantity as $n \longrightarrow \infty$ (as before, we require $\Delta A_k \longrightarrow 0$ uniformly in k when $n \longrightarrow \infty$), then this limit is the accurate volume under surface, and we define this to be the double integral $\iint_R f(x,y)dA$, i.e.

$$\iint_R f(x,y)dA = \lim_{n\longrightarrow\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k.$$

The above limit can be used for any function $f(x,y)$ (not necessarily positive) and hence the double integral $\int\int_R f(x,y)dA$ is defined for any function for which the limit exists.

From the definition, it is easy to deduce the following properties.

(a) $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA, c$ is a constant.

(b) $\iint_R [f(x,y) \pm g(x,y)]dA = \iint_R f(x,y)dA \pm \iint_R g(x,y)dA$

(c) $\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$, where R is divided into two subregions R_1 and R_2.

Remember that though $\int_a^b f(x)dx$ is defined by a limit, in practice, $\int_a^b f(x)dx$ is calculated by the formula

$$\int_a^b f(x)dx = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$, i.e. $F'(x) = f(x)$.

Similarly, we don't want to calculate $\iint_R f(x,y)dA$ by its definition, i.e., by using the limiting process described in the definition. A practical method of calculating a double integral is by changing it into iterated integrals.

Theorem 1. Let R be the rectangle: $a \le x \le b, c \le y \le d$. If $f(x,y)$ is continuous on R, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dy\,dx = \int_c^d \int_a^b f(x,y)dx\,dy.$$

Here

$$\int_a^b \int_c^d f(x,y)dy\,dx = \int_a^b \left[\int_c^c f(x,y)dy\right] dx,$$

where we calculate the integral $\int_c^d f(x,y)dy$ (regarding x as a parameter) first, which gives a function of x, and then integrate this function of x from a to b.

$\int_c^d \int_a^b f(x,y)dx\,dy$ is done similarly but with the order of integration reversed (integrate for x first, for y second). These are called iterated integrals.

The proof of Theorem 1 uses the definition of integrals, and we will not give it here.

Example 1 Find $\iint_R y^2(x^2+1)dA$, where $R = \{(x,y) : 1 \le x \le 2, 0 \le y \le 1\}$.

Solution By Theorem 1,

$$\begin{aligned}
\iint_R y^2(x^2+1)dA &= \int_1^2 \int_0^1 y^2(x^2+1)dy\,dx \\
&= \int_1^2 \frac{y^3}{3}(x^2+1)\bigg|_0^1 dx = \int_1^2 \frac{1}{3}(x^2+1)dx \\
&= \frac{1}{3}\left(\frac{x^3}{3}+x\right)\bigg|_1^2 = \frac{10}{9}.
\end{aligned}$$

Or

$$\begin{aligned}
\iint_R y^2(x^2+1)dA &= \int_0^1 \int_1^2 y^2(x^2+1)dx\,dy \\
&= \int_0^1 y^2\left(\frac{x^3}{3}+x\right)\bigg|_1^2 dy \\
&= \int_0^1 y^2 \cdot \frac{10}{3}dy = \frac{10}{3}\cdot\frac{y^3}{3}\bigg|_0^1 = \frac{10}{9}.
\end{aligned}$$

□

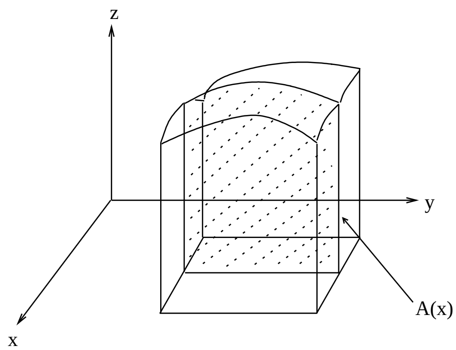
Theorem 1 can be explained geometrically in the following way. For fixed x, $z = f(x,y)$ is a curve, and

$$A(x) = \int_c^d f(x,y)dy$$

is the area under this curve. $\int_a^b A(x)dx$ then gives the volume of the solid under the surface and above the rectangle $a \le x \le b, c \le y \le d$.

There is a similar explanation for $B(y) = \int_a^b f(x,y)dx$ and $\int_c^d B(y)dy$.

Example 2. Find the volume of the solid that is bounded above by the surface $z = x^2 + y^2$ and below by the rectangle $R = \{(x,y) : 0 \le x \le 1, 0 \le y \le 1\}$.



Solution

$$\begin{aligned} V &= \iint_R (x^2+y^2)dA \\ &= \int_0^1 \int_0^1 (x^2+y^2)dx\,dy \\ &= \int_0^1 \left(\frac{x^3}{3}+y^2x\right)\Big|_0^1 dy \\ &= \int_0^1 (\frac{1}{3}+y^2)dy = \left(\frac{1}{3}y+\frac{y^2}{3}\right)\Big|_0^1 = \frac{2}{3}. \end{aligned}$$

$\square$

Example 3 Find $\iint_R y\sin(xy)dA, R = \left\{(x,y) : 0 \le x \le 1, \frac{\pi}{4} \le y \le \frac{\pi}{2}\right\}$.

Solution

$$\begin{aligned} \iint_R y\sin(xy)dA &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^1 y\sin(xy)dx\,dy \\ &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y\cdot\frac{-1}{y}\cos(xy)\Big|_0^1 dy = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(-\cos y+1)dy \\ &= (-\sin y+y)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{4}+\frac{\sqrt{2}-2}{2} = \frac{\pi-4+2\sqrt{2}}{4} \end{aligned}$$

$\square$.

Lecture 23 Integration of Double Integrals

We learned in the last lecture that if R is a rectangle: $R = \{(x,y) : a \le x \le b, c \le y \le d\}$, then the calculation of the double integral $\int\int_R f(x,y)dA$ can be reduced to calculating iterated integrals, namely

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dx\,dy = \int_a^b \int_c^d f(x,y)dy\,dx.$$

This technique can be extended to more general regions. If $g_1(x)$ and $g_2(x)$ are continuous functions and $g_1(x) \le g_2(x)$ for $a \le x \le b$, then the region

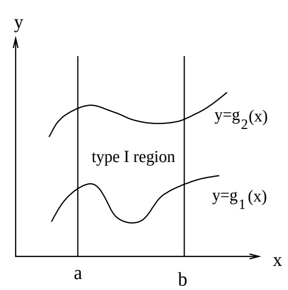
$$R = \{(x,y) : g_1(x) \le y \le g_2(x), a \le x \le b\}$$

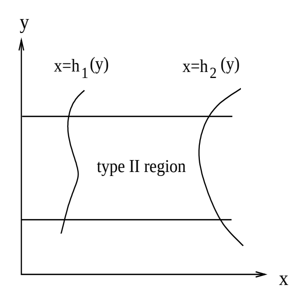
is called a **type I region**. In other words, a type I region is a region that is bounded by two vertical lines ($x = a$ and $x = b$) and two curves $y = g_1(x)$ and $y = g_2(x)$.

If $h_1(y)$ and $h_2(y)$ are continuous functions of y and $h_1(y) \le h_2(y)$ for $c \le y \le d$, then the region

$$R = \{(x,y) : h_1(y) \le x \le h_2(y), c \le y \le d\}$$

is called a **type II region**. It is bounded by two horizontal lines ($y = c$ and $y = d$) and two curves $x = h_1(y), x = h_2(y)$.





For type I or type II regions, the double integral $\int\int_R f(x,y)dA$ can also be calculated through iterated integrals, as the following theorem asserts.

Theorem 1

(a) If R is a type I region on which $f(x,y)$ is continuous, then

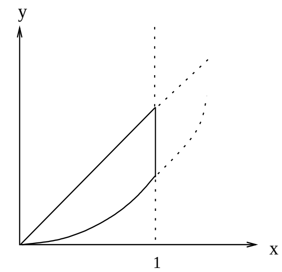
$$\iint_R f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$$

(b) If R is a type II region on which $f(x,y)$ is continuous, then

$$\iint_R f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy$$

Again we will not prove the theorem, but will see how it can be used in concrete examples.

Example 1. Find $\iint_R x^2ydA$ where R is the region enclosed by $y = x, y = \frac{1}{2}x^2$ and $x = 1$.



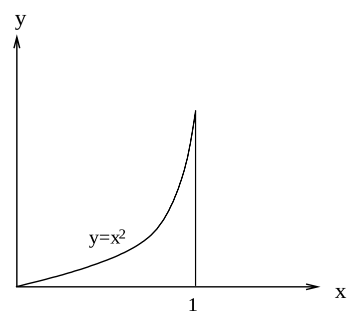
Solution By sketching the region, we see that it is a type I region, given by
$\frac{1}{2}x^2 \leq y \leq x,\ 0 \leq x \leq 1$
or $a = 0, b = 1, g_1(x) = \frac{1}{2}x^2, g_2(x) = x$
in the old notations. Therefore, by Theorem 1,

$$\iint_R x^2y\,dA = \int_0^1 \int_{\frac{1}{2}x^2}^{x} x^2y\,dy\,dx = \int_0^1 x^2\frac{y^2}{2}\bigg|_{\frac{1}{2}x^2}^{x} dx = \int_0^1 x^2\left[\frac{x^2}{2} - \frac{1}{2}\left(\frac{1}{2}x^2\right)^2\right] dx$$
$$= \left(\frac{1}{2}\frac{x^5}{5} - \frac{1}{8}\frac{x^7}{7}\right)\bigg|_0^1 = \frac{1}{10} - \frac{1}{56} = \frac{23}{280}.$$

Example 2. Evaluate $\int_0^1 \int_{\sqrt{y}}^1 e^{x^3} dx\,dy$ by changing the order of integration. □

Solution The integral $\int e^{x^3}dx$ is difficult to handle. However, the iterated integral can be changed back to a double integral, with region R given by

$\sqrt{y} \leq x \leq 1,\ 0 \leq y \leq 1.$
Sketching R, we realize that it is also a type I region:
$0 \leq y \leq x^2,\ 0 \leq x \leq 1.$



Therefore

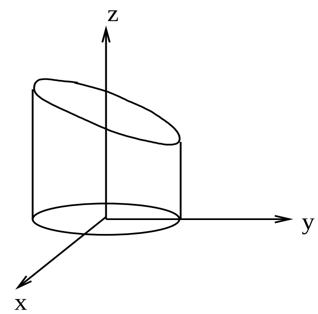
$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}dx, dy &= \iint_R e^{x^3}dA \\ &= \int_0^1 \int_0^{x^2} e^{x^3}dy\,dx \\ &= \int_0^1 e^{x^3}y\bigg|_0^{x^2} dx = \int_0^1 e^{x^3}x^2dx \\ &= \frac{1}{3}e^{x^3}\bigg|_0^1 = \frac{1}{3}(e-1)\end{aligned}$$

□

Example 2 above shows that changing the order of integration sometimes can make the integration much easier (or harder).

Example 3. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 1$, the xy-plane and the plane $z = 2 - y$.

Solution $V = \iint_R (2-y)dA$
where R is the disk $x^2 + y^2 \leq 1$,
which can be regarded as a
type I region:
$-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},\ -1 \leq x \leq 1$
or a type II region:
$-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2},\ -1 \leq y \leq 1.$



Therefore,

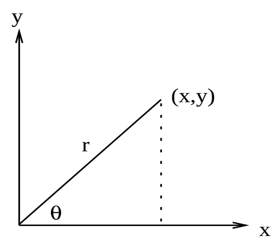
$$
\begin{aligned}
V &= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2-y)dy\,dx \\
&= \int_{-1}^{1} \left(2y - \frac{y^2}{2}\right)\Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\
&= \int_{-1}^{1} 4\sqrt{1-x^2}\,dx \\
&= 2\int_{0}^{1} 4\sqrt{1-x^2}\,dx \text{ (since } 4\sqrt{1-x^2} \text{ is an even function)} \\
&= 2\int_{0}^{\frac{\pi}{2}} 4\sqrt{1-\sin^2\theta}d\sin\theta \quad (x = \sin\theta) \\
&= 2\int_{0}^{\frac{\pi}{2}} 4\cos\theta \cdot \cos\theta d\theta \\
&= 8\int_{0}^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2} d\theta \\
&= 4\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{0}^{\frac{\pi}{2}} \\
&= 2\pi
\end{aligned}
$$

□

Lecture 24 Double Integrals in Polar Coordinates, Surface Area.

Recall that a point (x, y) with polar coordinates r and θ has the following relations between its coordinates:

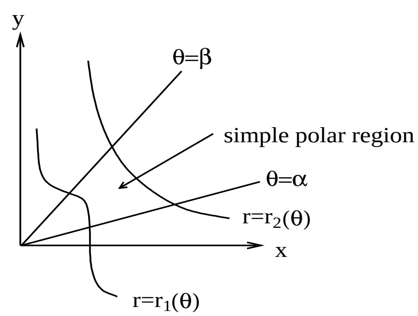
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



A region R given in polar coordinates in the way

$$R = \{(r, \theta) : r_1(\theta) \le r \le r_2(\theta),\ \alpha \le \theta \le \beta\}$$

is called **a simple polar region**.



Theorem 1 If R is a simple polar region on which $f(x, y) = f(r\cos\theta, r\sin\theta)$ is continuous, then

$$\iint_R f(x, y)dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r dr\, d\theta$$

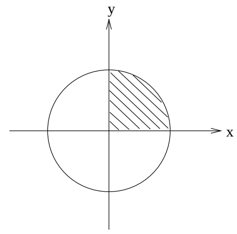
Thus we have one more class of regions for which a double integral can be calculated through an iterated integral.

Theorem 1 can be proved by the definition of double integrals. Instead of Riemann sums with rectangular area elements $dA = dx \cdot dy$ we have now Riemann sums with area elements of the form of sectors with radii r and $r + dr$ and angle $d\theta$, hence $dA = r\, dr\, d\theta$.

Example 1. Evaluate $\iint_R (1-x^2-y^2)dA$, where R is the part of the unit disk in the first quadrant of the xy-plane.

Solution. R is a simple polar region:

$0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 1.$



Therefore,

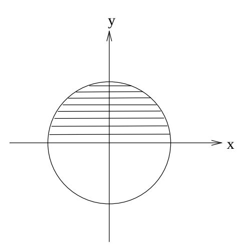
$$\iint_R (1-x^2-y^2)dA = \int_0^{\frac{\pi}{2}} \int_0^1 (1 - r^2\cos^2\theta - r^2\sin^2\theta) r dr\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} \int_0^1 (1-r^2) r dr\, d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 d\theta$$
$$= \int_0^{\frac{\pi}{2}} \frac{1}{4} d\theta = \frac{1}{4}\cdot\frac{\pi}{2} = \frac{\pi}{8}.$$

□

Example 2. Use polar coordinates to evaluate $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} x\sqrt{x^2+y^2}dy\,dx$.

Solution The region of integration is

$$R: \quad -1 \le x \le 1, \qquad 0 \le y \le \sqrt{1-x^2}$$



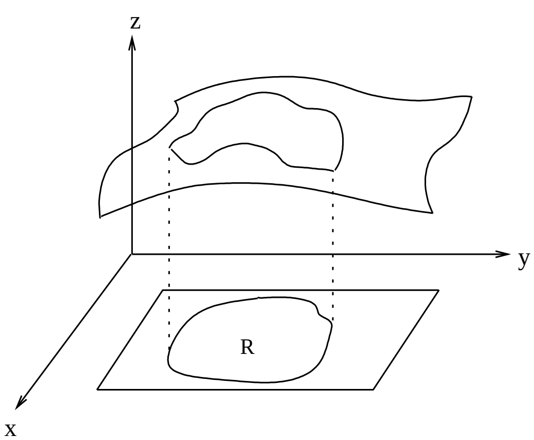
It is also a simple polar region, described by

$$R: 0 \le \theta \le \pi, \qquad 0 \le r \le 1$$

Therefore, by Theorem 1 of this lecture and Theorem 1 of Lecture 22, we have

$$\begin{aligned}
\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} x\sqrt{x^2+y^2}dy\,dx &= \iint_R x\sqrt{x^2+y^2}dA \\
&= \int_0^{\pi}\int_0^1 r\cos\theta\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}rdrd\theta \\
&= \int_0^{\pi}\int_0^1 r^3\cos\theta drd\theta \\
&= \int_0^{\pi} \frac{r^4}{4}\Big|_0^1 \cos\theta d\theta \\
&= \int_0^{\pi} \frac{1}{4}\cos\theta d\theta \\
&= \frac{1}{4}\sin\theta\Big|_0^{\pi} = 0
\end{aligned}$$

Apart from calculating the volume of a solid enclosed by some surfaces, double integrals can also be used to calculate the surface area S of a portion of a given surface $z = f(x,y)$ whose projection on the xy-plane is a certain region R.



The formula is

$$\boxed{S = \iint_R \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA}$$

Notice the similarity to the arc length formula for a curve $y = f(x)$ for $a \le x \le b$

$$\ell = \int_a^b \sqrt{[f']^2 + 1}\, dx.$$

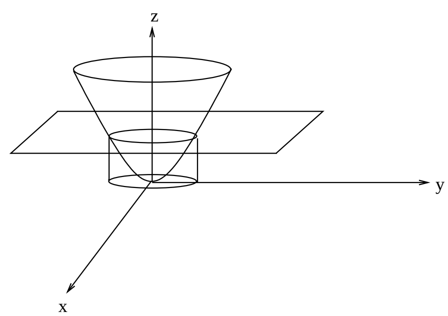
The area formula is plausible by the following argument. The area is approximately the sum of area elements on the surface that lie over rectangles with area

$dA = dx\,dy$. These curved area elements can be approximated by parallelograms in the corresponding tangent plane (think of fish scales attached to the surface) that project down to dA. Such a parallelogram over the vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ in the xy-plane is spanned by the vectors $\langle 1, 0, f_x\rangle$ and $\langle 0, 1, f_y\rangle$. Using the cross product we find its area

$$||\langle 1, 0, f_x\rangle \times \langle 0, 1, f_y\rangle|| = \sqrt{f_x^2 + f_y^2 + 1}.$$

Hence the area element $dA = dx\,dy$ is stretched by that factor, which results in the above formula.

Example 3 Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 2$.



Solution. A sketching of the paraboloid and the plane shows that R is the disk $x^2 + y^2 \le 2$. This is because the paraboloid $z = x^2 + y^2$ and the plane $z = 2$ intersect at $x^2 + y^2 = 2, z = 2$. R is a simple polar region: $0 \le \theta \le 2\pi, 0 \le r \le \sqrt{2}$.

Therefore,

$$\begin{aligned}
S &= \iint_R \sqrt{f_x^2 + f_y^2 + 1}dA = \iint_R \sqrt{(2x)^2 + (2y)^2 + 1}dA \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}rdrd\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{4r^2 + 1}rdrd\theta \\
&= \int_0^{2\pi} \frac{1}{12}(4r^2 + 1)^{3/2}\Big|_0^{\sqrt{2}} d\theta \\
&= \int_0^{2\pi} \frac{13}{6}d\theta = \frac{13}{6}\cdot 2\pi = \frac{13}{3}\pi.
\end{aligned}$$

$\square$.

Lecture 25 Triple Integrals

For a function $f(x)$, we have the integral $\int_a^b f(x)dx$, which may be called a single integral. For a two variable function $f(x, y)$, we defined the double integral $\iint_R f(x, y)dA$ over a region R in the xy–plane, which is the limit of $\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$ as $n \longrightarrow \infty$, where ΔA_k is the area of the k–th subregion of R, and (x_k^*, y_k^*) is an arbitrarily chosen point in it. This idea generalises naturally to three variable functions $f(x, y, z)$, and we define the triple integral $\iiint_G f(x, y, z)dV$ by a similar process. Here G is a bounded solid in the xyz–space. Move precisely, we assume that G is contained in a large box–like region B:

$$a_1 < x < a_2, \quad b_1 < y < b_2, \quad c_1 < z < c_2$$

and divide B into m subboxes by planes parallel to the coordinate planes. We require that as $m \longrightarrow \infty$, the size of each of the subboxes shrinks to 0. We then discard those subboxes that contain any points outside G and let n denote the number of the remaining subboxes. Let ΔV_k denote the volume of the kth remaining subbox. Let ΔV_k denote the volume of the kth remaining subbox and (x_k^*, y_k^*, z_k^*) an arbitrarily selected point in it. We can then form the sum (called the Riemann Sum)

$$S_n = \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

The limit $\lim_{n\longrightarrow\infty} S_n$, when it exists, is defined to be the triple integral

$$\iiint_G f(x, y, z)dV.$$

If $f(x, y, z)$ is continuous on G, and G is not too complicated, then it can be proved that the triple integral $\iiint_G f(x, y, z)dV$ always exists. If $f(x, y, z)$ is the varying density of the solid G, then $\iiint_G f(x, y, z)dV$ gives the mass of G.

Triple integrals have also the usual properties enjoyed by single and double integrals.

$\iiint_G cf(x, y, z)dV = c\iiint_G f(x, y, z)dV, c$ is a constant

$\iiint_G [f \pm g]dV = \iiint_G f dV \pm \iiint_G g\, dV$

$\iiint_G f dV = \iiint_{G_1} f\, dV + \iiint_{G_2} f\, dV,$
where G is the union of G_1 and G_2

$\iiint_G 1 dV =$ volume of G

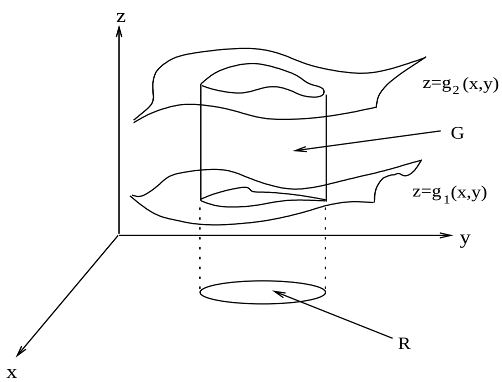
The evaluation of triple integrals is again through reducing it to iterated integrals when the region G has certain special peoperties.

A region G is called a **simple solid**, if it can be expressed by

$$G = \{(x, y, z) : g_1(x, y) \le z \le g_2(x, y),\ (x, y) \in R\}$$

where R is a bounded region in the xy–plane, and $g_1(x, y)$, $g_2(x, y)$ are continuous functions.

In other words, G is a simple solid if it consists of all the points directly above or below R and are between the surfaces $z = g_1(x, y)$ and $z = g_2(x, y)$. We call R the **projection of** G on the xy–plane.



Theorem 1. If G is a simple solid: $G = \{(x, y, z) : g_1(x, y) \le z \le g_2(x, y), (x, y) \in R\}$, and $f(x, y, z)$ is continuous on G, then

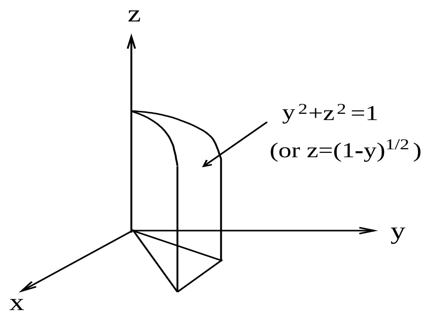
$$\iiint_G f(x, y, z)dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)dz\right] dA$$

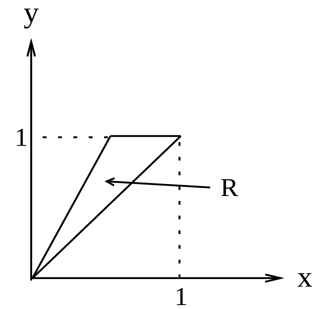
This theorem shows that, if G is a simple solid, then the calculation of the triple integral $\iiint_G f\, dV$ can be reduced to calculating the single integral $\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)dz$ first, where x, y are regarded as constants, and after this integral is done, say it is $F(x, y)$ (now x, y are regarded as variables), then calculating the double integral $\iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)dz\right] dA$, i.e. $\iint_R F(x, y)dA$. Of course, we need to change to iterated integrals to evaluate the double integral $\iint_R F(x, y)dA$.

Example 1. Evaluate $\iiint_G zdV$, where G is the wedge in the first octant cut from the cylindrical solid $y^2 + z^2 \le 1$ by the planes $y = x$ and $y = 2x$.

Solution We sketch the graph of G and realize that it is a simple solid:

$$0 \le z \le \sqrt{1 - y^2},\ (x, y) \in R,$$





$$R = \{(x,y) : \frac{y}{2} \leq x \leq y,\ 0 \leq y \leq 1\}$$

Therefore, by Theorem 1,

$$\begin{aligned}
\iiint_G z\,dV &= \iint_R \left[\int_0^{\sqrt{1-y^2}} z\,dz\right] dA \\
&= \iint_R \frac{z^2}{2}\Bigg|_0^{\sqrt{1-y^2}} dA = \iint_R \frac{1}{2}(1-y^2)dA \\
&= \int_0^1 \int_{\frac{y}{2}}^{y} \frac{1}{2}(1-y^2)dx\,dy \\
&= \int_0^1 \frac{1}{2}(1-y^2)\frac{y}{2}dy = \frac{1}{4}\left(\frac{y^2}{2} - \frac{y^4}{4}\right)\Bigg|_0^1 = \frac{1}{16}
\end{aligned}$$

□

By rotating the roles of x, y and z in Theorem 1, we have the following variants of it.

Theorem 2.

(a) If R is a bounded region in the xz–plane, and

$$G = \{(x,y,z) : g_1(x,z) \leq y \leq g_2(x,z), (x,z) \in R\}$$

and f is continuous on G, then

$$\iiint_G f(x,y,z)dV = \iint_R \left[\int_{g_1(x,z)}^{g_2(x,z)} f(x,y,z)dy\right] dA$$

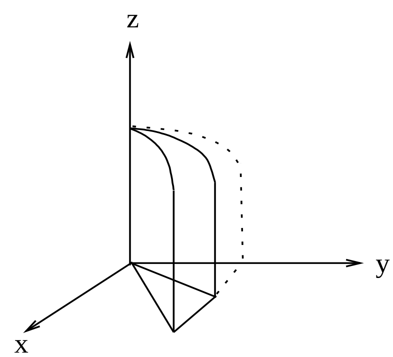
(b) If R is a bounded region in the yz–plane, and

$$G = \{(x,y,z) : g_1(y,z) \leq x \leq g_2(y,z), (y,z) \in R\}$$

and f is continuous on G, then

$$\iiint_G f(x,y,z)dV = \iint_R \left[\int_{g_1(y,z)}^{g_2(y,z)} f(x,y,z)dx\right] dA.$$

Example 2. Evaluate $\iiint_G zdV$ by integrating first with respect to x, where G is as in Example1.



Solution The projection of G on the yz–plane is (in polar coordinates)
$R' : 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 1$
G is between the plane
$x = y$ and $x = \frac{1}{2}y$ (i.e. $y = 2x$)
Therefore, by Theorem 2 part (b)

$$\begin{aligned}
\iiint_G zdV &= \iint_{R'} \left[\int_{\frac{1}{2}y}^{y} zdx\right] dA \\
&= \iint_{R'} z \cdot \frac{y}{2} dA \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{2}(r\sin\theta)(r\cos\theta)rdrd\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{2} \cdot \frac{r^4}{4}\bigg|_0^1 \sin\theta\cos\theta d\theta \\
&= \int_0^{\frac{\pi}{2}} \frac{1}{8} \cdot \frac{1}{2}\sin(2\theta)d\theta \\
&= \frac{1}{16}\frac{-\cos(2\theta)}{2}\bigg|_0^{\frac{\pi}{2}} = \frac{1}{16}
\end{aligned}$$

$\square$

Lecture 26 Change of Variables

Recall that, if $g(t) = f(x(t))x'(t)$, then

$$\int_a^b g(t)dt = \int_a^b f(x(t))x'(t)dt = \int_{x(a)}^{x(b)} f(x)dx.$$

This is called integration by substitution, a very useful technique of integration. For example,

$$\int_0^2 te^{t^2}dt \overset{(x=t^2)}{=} \int_0^4 e^x \frac{1}{2}dx = \frac{1}{2}e^x\Big|_0^4 = \frac{1}{2}(e^4-1)$$

In double and triple integrals, there are similar techniques, which are the topics of this lecture.

1. Change of Variables in Double Integrals

As the shape of the region R is important in changing the double integral $\int\int_R f(x,y)dA$ into iterated integrals (recall what is type I region, what is type II region), it is better to view the change of variables $x = x(u,v), y = y(u,v)$ as a transformation, which maps a point (u,v) in the uv–plane to a point (x,y) in the xy–plane. Let S be a region in the uv–plane, and D a region in the xy–plane. We say the transformation

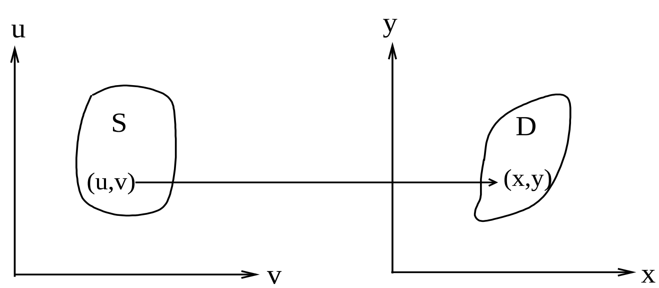
$$\begin{cases} x &= x(u,v) \\ y &= y(u,v) \end{cases}$$

is $1-1$ (one-to-one) from S onto D if the following are satisfied.

(i) Every point in S gets mapped to a point in D;

(ii) Every point in D is the image of a point in S;

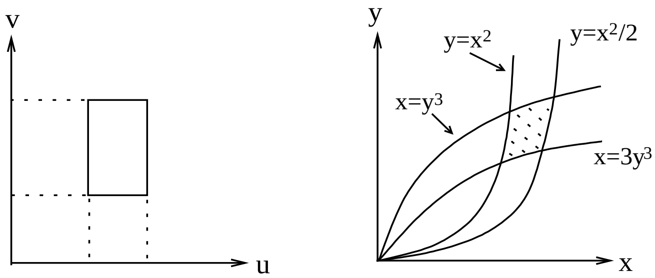
(iii) Different points in S get mapped to different points in D.



Example 1. The transformation

$$x = u^{-\frac{3}{5}}v^{-\frac{1}{5}},\ y = u^{-\frac{1}{5}}v^{-\frac{2}{5}}$$

transforms $S = \{(u,v) : \frac{1}{2} \le u \le 1, 1 \le v \le 3\}$ onto the set $D = \{(x,y) : \frac{1}{2} \le \frac{y}{x^2} \le 1, 1 \le \frac{x}{y^3} \le 3\}$



In fact, from $x = u^{-\frac{3}{5}}v^{-\frac{1}{5}}$ and $y = u^{-\frac{1}{5}}v^{-\frac{2}{5}}$ we deduce

$$\frac{y}{x^2} = u^{-\frac{1}{5}}v^{-\frac{3}{5}}u^{\frac{6}{5}}v^{\frac{3}{5}} = u.$$

and

$$\frac{x}{y^3} = u^{-\frac{3}{5}}v^{-\frac{1}{5}}u^{\frac{3}{5}}v^{\frac{6}{5}} = v.$$

Thus,

$$S : \frac{1}{2} \le u \le 1, \quad 1 \le v \le 3$$

is transformed to

$$\frac{1}{2} \le \frac{y}{x^2} \le 1, \quad 1 \le \frac{x}{y^3} \le 3$$

or

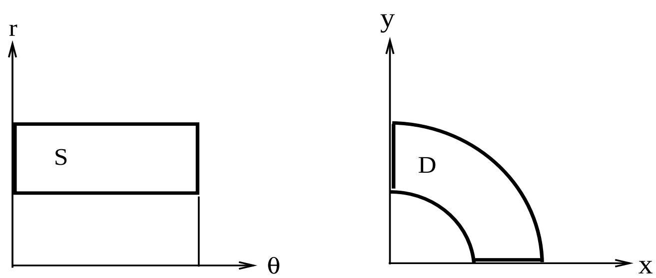
$$\frac{x^2}{2} \le y \le \frac{1}{x^2}, \quad y^3 \le x \le 3y^3$$

□

Example 2. The transformation

$$\begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases}$$

maps $S = \{(r,\theta) : 1 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}\}$ onto the part of the annulus $1 \le x^2+y^2 \le 4$ lying in the first quadrant.



This fact comes directly from the geometric interpretation of the transform, as (r,θ) is the polar coordinates in the xy–plane.

Let us now come back to double integrals. If we use $dA = dx\,dy$ to denote the area element in the xy–plane , and $d\tilde{A} = dudv$ that in the uv–plane, then

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| d\tilde{A}, \text{ or } dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv,$$

where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

is called the **Jacobian** of $x = x(u,v)$ and $y = y(u,v)$. The change of variable formula for double integral is

$$\boxed{\iint_D f(x,y)dx\,dy = \iint_S f(x(u,v),y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

Notice that

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

denotes the *absolute value* of the Jacobian. The proof of this formula is based on the fact that the rectangle with sides du and dv is approximately mapped to a parallelogram with sides $\langle \frac{\partial x}{\partial u}du, \frac{\partial y}{\partial u}du\rangle$ and $\langle \frac{\partial x}{\partial v}dv, \frac{\partial y}{\partial v}dv\rangle$ in the xy plane by the linear mapping

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}.$$

Hence the area of the rectangle $dx\,dy$ corresponds to the area of the parallelogram

$$\left|\det\begin{pmatrix} \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv \end{pmatrix}\right| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix}\right| du\,dv.$$

Example 3. For $x = r\cos\theta, y = r\sin\theta$,

$$\begin{aligned} \frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\ &= r\cos^2\theta + r\sin^2\theta = r. \end{aligned}$$

Hence we have

$$\iint_D f(x,y)dx\,dy = \iint_S f(r\cos\theta, r\sin\theta)r dr\,d\theta.$$

Example 4. Find the area of the region D given by

$$x^2 \le y \le 2x^2,\ y^3 \le x \le 3y^3$$

Solution. Let $u = \frac{y}{x^2}, v = \frac{x}{y^3}$. Then D is mapped to $S = \{(u,v) : 1 \le u \le 2, 1 \le v \le 3\}$ under the transformation

$$\begin{cases} u &= \frac{y}{x^2} \\ v &= \frac{x}{y^3} \end{cases}$$

To calculate $\frac{\partial(x,y)}{\partial(u,v)}$, we can first solve for x and y in terms of u and v to obtain (see Example 1)

$$x = u^{-\frac{3}{5}} v^{-\frac{1}{5}}, \ y = u^{-\frac{1}{5}} v^{-\frac{2}{5}}.$$

Hence

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= \begin{vmatrix} -\frac{3}{5}u^{-\frac{8}{5}}v^{-\frac{1}{5}} & -\frac{1}{5}u^{-\frac{3}{5}}v^{-\frac{6}{5}} \\ -\frac{1}{5}u^{-\frac{6}{5}}v^{-\frac{2}{5}} & -\frac{2}{5}u^{-\frac{1}{5}}v^{-\frac{7}{5}} \end{vmatrix} \\ &= \frac{1}{5}u^{-\frac{9}{5}}v^{-\frac{8}{5}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Area of } D &= \iint_D 1 dx\, dy = \iint_S 1 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \\ &= \iint_S \left| \frac{1}{5}u^{-\frac{9}{5}}v^{-\frac{8}{5}} \right| du\, dv \\ &= \int_1^3 \int_1^2 \frac{1}{5}u^{-\frac{9}{5}}v^{-\frac{8}{5}} du\, dv \\ &= \int_1^3 \frac{1}{5} \left(\frac{u^{-\frac{4}{5}}}{-\frac{4}{5}} \right) \Bigg|_1^2 v^{-\frac{8}{5}} dv \\ &= \int_1^3 \frac{1 - 2^{-\frac{4}{5}}}{4} v^{-\frac{8}{5}} dv \\ &= \frac{1}{4} \left(1 - 2^{-\frac{4}{5}} \right) \frac{v^{-\frac{3}{5}}}{-\frac{3}{5}} \Bigg|_1^3 \\ &= \frac{5}{12} \left(1 - 2^{-\frac{4}{5}} \right) \left(1 - 3^{-\frac{3}{5}} \right). \end{aligned}$$

$\square$

2. Change of Variables in Triple Integrals.

If the change of variables

$$x = x(u, v, w),\ y = y(u, v, w),\ z = z(u, v, w)$$

gives a $1-1$ transformation that maps a region S in the uvw–space onto a region D in the xyz–space, then

$$\boxed{\begin{array}{l} dV (= dxdydz) = \left|\dfrac{\partial(x, y, z)}{\partial(u, v, w)}\right| dudvdw \\ \text{and} \\ \displaystyle\iiint_D f(x, y, z)dxdydz = \iiint_S g(x(u, v, w), y(\ldots), z(\ldots)) \left|\dfrac{\partial(x, y, z)}{\partial(u, v, w)}\right| dudvdw \end{array}}$$

Here the Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ is given by the determinant

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Again the reason is that the cube with edges dx, dy, dz corresponds approximately to a parallelepiped with edges $\langle \frac{\partial x}{\partial u}du, \frac{\partial y}{\partial u}du, \frac{\partial z}{\partial u}du\rangle$, $\langle \frac{\partial x}{\partial v}dv, \frac{\partial y}{\partial v}dv, \frac{\partial z}{\partial v}dv\rangle$ and $\langle \frac{\partial x}{\partial w}dw, \frac{\partial y}{\partial w}dw, \frac{\partial z}{\partial w}dw\rangle$ with volume

$$\left|\det \begin{pmatrix} \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du & \frac{\partial z}{\partial u}du \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv & \frac{\partial z}{\partial v}dv \\ \frac{\partial x}{\partial w}dw & \frac{\partial y}{\partial w}dw & \frac{\partial z}{\partial w}dw \end{pmatrix}\right| = \left|\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial z} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix}\right| du\, dv\, dw.$$

Example 5. If $x = r\cos\theta,\ y = r\sin\theta,\ z = z$, then

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Example 6. If $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho\cos\phi$, then

$$
\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + \rho\sin\phi \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi \\
&= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) \\
&= \rho^2\sin\phi.
\end{aligned}
$$

$\square$

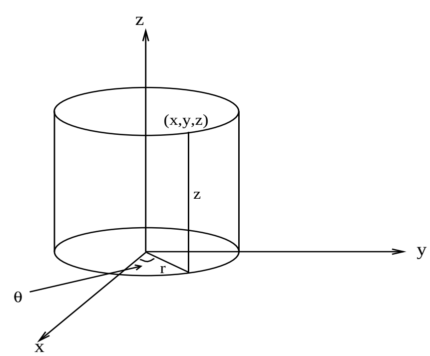
Lecture 27 Triple Integrals in Cylindrical and Spherical Coordinates

From the previous Lecture we know that for the change of variables $x = r\cos\theta, y = r\sin\theta, z = z$, we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r, dxdydz = rdrd\theta dz \quad \text{and}$$

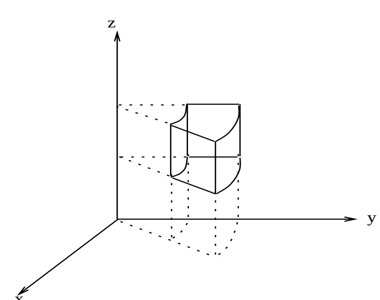
$$\boxed{\iiint_D f(x,y,z)dxdydz = \iiint_S f(r\cos\theta, r\sin\theta, z)rdrd\theta dz}$$

Instead of viewing (r,θ,z) as coordinates of some $r\theta z$–space, it is more convenient to view (r,θ,z) by their geometric meanings in the xyz–space, as indicated by the following diagram. Since a cylinder is involved in the diagram, (r,θ,z) are called the **cylindrical coordinates** of a point (x,y,z) in the xyz–space.



Example 1. The following diagram shows a solid in the xyz–space whose cylindrical coordinates satisfy:

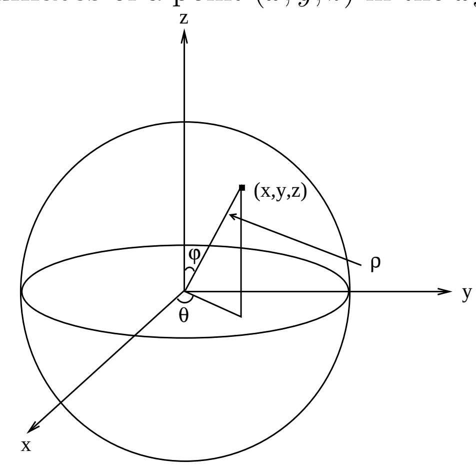
$$\frac{\pi}{4} \le \theta \le \frac{\pi}{2},\ 1 \le r \le 2,\ 1 \le z \le 2.$$



The change of variables

$$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$$

gives the geometric meanings of ρ, ϕ, and θ as shown in the following diagram, and (ρ, ϕ, θ) are called the **spherical coordinates** of a point (x, y, z) in the xyz–space.



From Lecture 26, we know

$$\boxed{\iiint_D f(x,y,z)dxdydz = \iiint_S f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\ \sin\phi\,d\rho\,d\phi\,d\theta}$$

Example 2. Find the volume of the solid bounded above by the sphere $x^2+y^2+z^2 = 4$ and below by the cone $z = \sqrt{x^2+y^2}$.

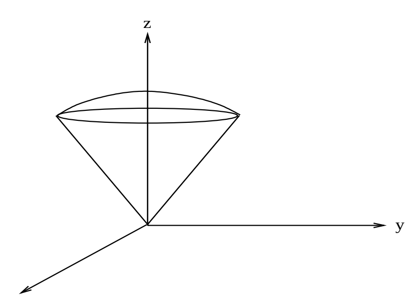
Solution The volume is

$$V = \iiint_G 1\,dV$$

where G denotes the given solid.

From a sketch of the solid, as shown below, we find that it can be described in spherical coordinates by

$$S : 0 \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{4}, 0 \le \rho \le 2.$$



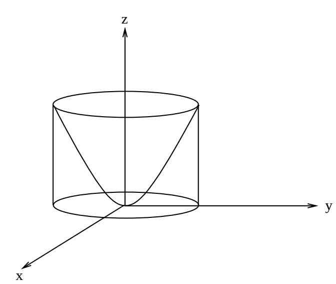
Therefore,

$$
\begin{aligned}
V &= \iiint_G 1dV = \iiint_S 1\cdot\rho^2\sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^2 \rho^2\sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left.\frac{\rho^3}{3}\right|_0^2 \sin\phi d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{8}{3}\sin\phi d\phi d\theta \\
&= \int_0^{2\pi} \left.\frac{8}{3}(-\cos\phi)\right|_0^{\frac{\pi}{4}} d\theta \\
&= \int_0^{2\pi} \frac{4}{3}\left(1-\frac{\sqrt{2}}{2}\right) d\theta \\
&= \frac{8-4\sqrt{2}}{3}\pi
\end{aligned}
$$

□

Example 3 Evaluate $\iiint_G z\sqrt{x^2+y^2}dV$ where G is the solid enclosed by the cylinder $x^2+y^2=1$, the xy–plane and the paraboloid $z=x^2+y^2$.

Solution By sketching the solid G, we see that it can be described by cylindrical coordinates as follows:



$$0\le\theta\le 2\pi, 0\le r\le 1, 0\le z\le r^2$$

Denote $S=\{(r,\theta,z): 0\le\theta\le 2\pi, 0\le r\le 1, 0\le z\le r^2\}$. We have

$$
\begin{aligned}
&\iiint_G z\sqrt{x^2+y^2}dV = \iiint_S z\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}r dr\,d\theta\,dz \\
&= \int_0^{2\pi}\int_0^1\int_0^{r^2} zr^2 dz dr d\theta \\
&= \int_0^{2\pi}\int_0^1 \left.\frac{z^2}{2}\right|_0^{r^2} r^2 dr d\theta = \int_0^{2\pi}\int_0^1 \frac{r^6}{2} dr d\theta \\
&= \int_0^{2\pi} \left.\frac{r^7}{14}\right|_0^1 d\theta = \int_0^{2\pi}\frac{1}{14}d\theta = \frac{\pi}{7}
\end{aligned}
$$

$\square$

Remark. Note that the region G in Example 3 is a simple solid, with its projection on the xy–plane $R = \{(x, y) : x^2 + y^2 \leq 1\}$, and is between the xy–plane $z = 0$ and the paraboloid $z = x^2 + y^2$. Therefore,

$$\begin{aligned} \iiint_G z\sqrt{x^2+y^2}dV &= \iint_R \left[\int_0^{x^2+y^2} z\sqrt{x^2+y^2}dz\right] dA \\ = \iint_R \frac{z^2}{2}\bigg|_0^{x^2+y^2} \sqrt{x^2+y^2}dA &= \iint_R \frac{1}{2}(x^2+y^2)^{\frac{5}{2}} dA \end{aligned}$$

Now we use polar coordinates to calculate the double integral to obtain

$$\begin{aligned} \iint_R \frac{1}{2}(x^2+y^2)^{\frac{5}{2}} dA &= \int_0^{2\pi}\int_0^1 \frac{1}{2}r^5 \cdot r dr d\theta \\ = \int_0^{2\pi} \frac{r^7}{14}\bigg|_0^1 d\theta &= \int_0^{2\pi} \frac{1}{14}d\theta = \frac{\pi}{7}. \end{aligned}$$

As a rule, it is always the case that when the cylindrical coordinates can be used directly, one can also do the calculation by reducing it to a double integral first, and then use polar coordinates to calculate the double integral.

Lecture 28 Line Integrals

The remaining part of this unit is devoted to integral along curved lines and surfaces. We start with line integrals. In fact there are two kinds of line integrals used for different applications. Let C be a smooth curve in 2-space given by $\vec{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$. We have encountred already one kind of line integrals over C when we computed the arc length of C

$$\ell = \int_C ds = \int_a^b \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt.$$

A slightly more sophisticated version of this is needed to compute the mass of a thin bent rod of the form C with a density distribution $\rho(t)$. The corresponding integral is

$$\ell = \int_C \rho(x,y) ds = \int_a^b \rho(t)\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt.$$

In this situation we integrate a *function* along a curve.

Another kind of integral is needed for the following problem: Imagine that an object is moved along the curve C against a force vector function $\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle$. We are interested in the total work W performed (which is the same as the amount of energy spent).

For a straight line C: $x = \alpha t, y = \beta t$ $(a \le t \le b)$ the vector of replacement is $(b-a)\langle \alpha, \beta\rangle$. If a constant force $\vec{F} = \langle f, g\rangle$ is applied then the work done is the dot product $\vec{F} \cdot \langle \alpha, \beta\rangle (b-a)$. On a curved line with changing force the work done is approximated by the Riemann sum with partition $a = t_0 < t_1 < \cdots < t_N = b$.

$$W \approx \sum_{n=1}^{N} \vec{F}(x(t_n), y(t_n)) \cdot \left\langle \frac{dx}{dt}(t_n), \frac{dy}{dt}(t_n) \right\rangle (t_n - t_{n-1}),$$

which tends to the integral

$$W = \int_a^b \vec{F}(x(t), y(t)) \cdot \left\langle \frac{dx}{dt}(t), \frac{dy}{dt}(t)\right\rangle dt = \int_C \vec{F}(x,y) \cdot d\vec{r},$$

where $d\vec{r} = \dfrac{d\vec{r}}{dt} dt$. This can be rewritten as an integral of the first kind

$$W = \int_C \vec{F}(x,y) \cdot \vec{\tau}(x,y)\, ds,$$

where $\tau = \frac{\vec{r}'}{||\vec{r}'||}$ is the unit tangent vector of C and $ds = ||\vec{r}'||dt$ is the arc length element. In other words we integrate a vector field along the curve C by turning it into a function using the dot product with the unit tangent vector of C at each point of C.

Notice that $d\vec{r} = \dfrac{d\vec{r}}{dt}dt$ does not depend on the actual parametrisation of C. If u was another parameter then, by chain rule,

$$d\vec{r} = \frac{d\vec{r}}{du}du = \frac{d\vec{r}}{dt}\frac{dt}{du}du = \frac{d\vec{r}}{dt}dt.$$

This justifies the notation

$$W = \int_C \vec{F}(x,y) \cdot d\vec{r} = \int_C f\,dx + g\,dy.$$

The expression

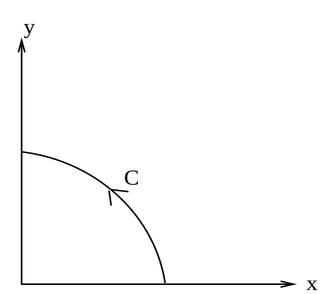
$$f(x,y)\,dx + g(x,y)\,dy$$

is called a 1-form, thus we integrate a 1-form along a curve C. We have encountred 1-forms before as total differentials

$$dF = F_x\,dx + F_y\,dx.$$

Any total differential is a 1-form but not any 1-form is a total differential. We will investigate this relation in the next Lecture.

Example 1. Find $\int_C xy^2dx - y^3dy$ over the circular arc $C : x = \cos t, y = \sin t, 0 \leq t \leq \frac{\pi}{2}$.



Solution. Using the parametrisation $x = \cos t$, $y = \sin t$ we find $dx = -\sin t\,dt$

and $dy = \cos t\, dt$. It follows

$$\begin{aligned}
\int_C xy^2dx - y^3dy &= \int_0^{\frac{\pi}{2}} (\cos t)(\sin t)^2(-\sin t)dt - \int_0^{\frac{\pi}{2}} (\sin t)^3 \cos t dt \\
&= \int_0^{\frac{\pi}{2}} \left[-\cos t \cdot \sin^3 t - \cos t \sin^3 t\right] dt \\
&= -2\int_0^{\frac{\pi}{2}} \sin^3 t \cos t dt \\
&= -2\int_0^{\frac{\pi}{2}} \sin^3 t d\sin t \\
&= -2 \cdot \frac{\sin^4 t}{4}\Big|_0^{\frac{\pi}{2}} = -\frac{1}{2}.
\end{aligned}$$

□

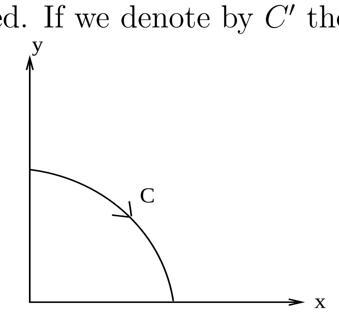
The following properties about line integrals can be proved easily by the definition (please have a try):

(1) Line integrals do not change under different parametrizations of the curve C as long as the parametrizations have the same orientation of the curve. If the orientation is reversed, then the sign of the line integral is changed.

(2) If C consists of finitely many smooth curves $C_1, \ldots, C_n$, joined end to end, then

$$\int_C f\, dx + g dy = \sum_{k=1}^{n} \int_{C_k} f\, dx + g\, dy.$$

Example 2. If we parametrize the circular arc C in Example 1 by $x = t, y = \sqrt{1-t^2}, 0 \le t \le 1$, then the orientation is reversed. If we denote by C' the same arc but with the orientation determined by this new parametrization, then, by property (1) above, we should have $\int_{C'} xy^2dx - y^3dy = -\int_C xy^2dx - y^3cy = \frac{1}{2}$. Indeed, calculating directly,



$$
\begin{aligned}
&\int_{C'} xy^2dx - y^3cy = \int_0^1 \left[t(\sqrt{1-t^2})^2(t)' - (\sqrt{1-t^2})^3(\sqrt{1-t^2})'\right] dt \\
= &\int_0^1 \left[t(1-t^2) - (\sqrt{1-t^2})^3 \frac{-t}{\sqrt{1-t^2}}\right] dt \\
= &\int_0^1 \left[t(1-t^2) + t(1-t^2)\right] dt \\
= &\ 2\left(\frac{t^2}{2} - \frac{t^4}{4}\right)\Big|_0^1 = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}.
\end{aligned}
$$

□

The above discussion of line integrals in 2–spaces extends naturally to 3–spaces. If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$ is the vector equation of a curve C in 3–space, and $\vec{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z)\rangle$ is a continuous vector function in a region containing C, then we define

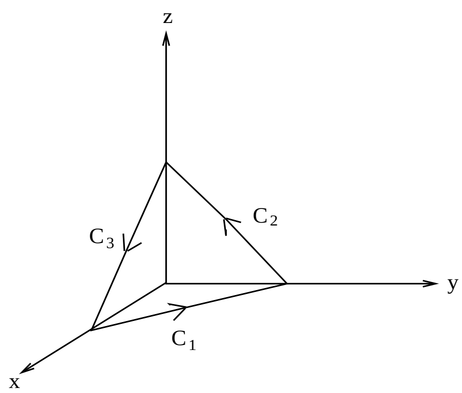
$$
\begin{aligned}
&\int_C \vec{F}(x, y, z) \cdot d\vec{r} = \int_C f(x, y, z)\, dx + g(x, y, z)\, dy + h(x, y, z)\, dz \\
= &\int_a^b [f(x(t), y(t), z(t))x'(t) + g(x(t), y(t), z(t))y'(t) + h(x(t), y(t), z(t))z'(t)]dt
\end{aligned}
$$

The same properties also hold for 3–space line integrals.

Example 3 Calculate $\int_C xy\, dx + z\, dy + (xy + z)dz$, where C is the boundary of the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, oriented in this order.

Solution C consists of three line segments C_1, C_2 and C_3 with parametrizations given by:

$$
\begin{aligned}
C_1: &\quad x = 1 - t, &\quad y = t, &\quad z = 0, &\quad 0 \le t \le 1 \\
C_2: &\quad x = 0, &\quad y = 1 - t, &\quad z = t, &\quad 0 \le t \le 1 \\
C_3: &\quad x = t, &\quad y = 0, &\quad z = 1 - t, &\quad 0 \le t \le 1
\end{aligned}
$$



Hence,

$$
\begin{aligned}
& \int_C xy\,dx + z\,dy + (xy+z)dz \\
= & \sum_{k=1}^{3} \int_{C_k} xy\,dx + z\,dy + (xy+z)dz \\
= & \int_0^1 \{(1-t)(t)(1-t)' + 0 \cdot (t)' + [(1-t)t + 0] \cdot (0)'\}\,dt \\
+ & \int_0^1 \{(0)(1-t)(0)' + t(1-t)' + [(0)(1-t) + t]\,(t)'\}\,dt \\
+ & \int_0^1 \{(t)(0)(t)' + (1-t)(0)' + [(t)(0) + (1-t)]\,(1-t)'\}\,dt \\
= & \int_0^1 [-(1-t)t - (1-t)]\,dt \\
= & \int_0^1 (t^2-1)dt = \left(\frac{t^3}{3} - t\right)\Bigg|_0^1 = -\frac{2}{3}
\end{aligned}
$$

$\square$

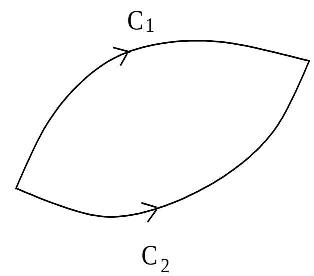
Lecture 29 Line Integrals Independent of Path

We know that the line integral

$$\int_C f(x,y)dx + g(x,y)dy = \int_C \vec{F}(x,y)\cdot d\vec{r}$$

does not depend on the particular parametrization of the curve C as long as the parametrization does not change the orientatin of C. In this lecture, we look at conditions under which the line integral does not even depend on the curve as long as the two end points are not changed.

For example, if C_1 and C_2 are as shown in the diagram, where the two end points of C_1 and C_2 are the same, then a line integral independent of path should have the same value over C_1 and C_2.



Theorem 1 (The Fundamental Theorem of Line Integrals)

Suppose $\vec{F}(x,y) = f(x,y)\vec{i}+g(x,y)\vec{j}$, where f and g are continuous in some open region containing the two points (x_0,y_0) and (x_1,y_1). If there exists some function $\phi(x,y)$ such that $\vec{F}(x,y) = \nabla\phi(x,y)$ at each point in this region,then for any smooth curve C starting at (x_0,y_0), ending at (x_1,y_1), and lying entirely inside the region, we have

$$\int_C \vec{F}(x,y)\cdot d\vec{r} = \phi(x_1,y_1) - \phi(x_0,y_0)$$

Note that in Theorem 1 the line integral is determined by ϕ and (x_1,y_1), (x_0,y_0) only, and is independent of the curve C. Therefore, we say the line integral is **independent of path**. Moreover, when $\vec{F} = \nabla\phi$ for some ϕ, we say $\vec{F}$ is **conservative**, and ϕ is a **potential** for $\vec{F}$. Notice that if ϕ is a potential for $\vec{F}$, then so is $\phi + c$, where c is an arbitrary constant.

Proof of Theorem 1. Let $x = x(t),\ y = y(t),\ a \le t \le b$ be a parametrization of a smooth curve C starting at (x_0,y_0) and ending at (x_1,y_1), i.e., $(x(a),\ y(a)) = (x_0,\ y_0),\ (x(b),\ y(b)) = (x_1,y_1)$. Moreover, C lies entirely inside the region where $\vec{F}(x,y) = \nabla\phi(x,y)$. Then, as $\vec{F}(x,y) = f(x,y)\vec{i}+g(x,y)\vec{j}$ and $\nabla\phi(x,y) = \phi_x(x,y)\vec{i}+\phi_y(x,y)\vec{j}$, we have

$$f(x,y) = \phi_x(x,y),\ g(x,y) = \phi_y(x,y)$$

and

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b [f(x(t),y(t))x'(t) + g(x(t),\ y(t))y'(t)]dt$$

$$\begin{aligned}
&= \int_a^b [\phi_x(x(t), y(t))x'(t) + \phi_y(x(t), y(t))y'(t)]dt \\
&= \int_a^b \left[\frac{d}{dt}\phi(x(t), y(t))\right] dt \\
&= \phi(x(t), y(t))\Big|_a^b = \phi(x(b), y(b)) - \phi(x(a), y(a)) \\
&= \phi(x_1, y_1) - \phi(x_0, y_0)
\end{aligned}$$

□

Corollary. If C is closed and $\vec{F}$ is conservative in a region containing C, then

$$\int_C \vec{F} \cdot d\vec{r} = 0.$$

Example 1. Show that $\vec{F}(x, y) = 2xy^3\vec{i} + (1 + 3x^2y^2)\vec{j}$ is conservative.

Proof. We need to show that there exists a function $\phi(x, y)$ such that $\nabla\phi(x, y) = \vec{F}(x, y)$, i.e.

$$\frac{\partial\phi}{\partial x} = 2xy^3, \ \frac{\partial\phi}{\partial y} = 1 + 3x^2y^2.$$

From $\dfrac{\partial\phi}{\partial x} = 2xy^3$ we obtain

$$\begin{aligned}
\phi(x, y) &= \int 2xy^3 dx + C(y) \\
&= x^2y^3 + C(y)
\end{aligned}$$

where $C(y)$ is some unknown function of y.

But we have $\dfrac{\partial\phi}{\partial y} = 1 + 3x^2y^2$. Using $\phi(x, y) = x^2y^3 + C(y)$, we deduce,

$$\frac{\partial\phi}{\partial y} = 3x^2y^2 + C'(y)$$

Therefore we should have $C'(y) = 1$, or $C(y) = y + C$ where C is an arbitrary constant. Thus

$$\phi(x, y) = x^2y^3 + y + C.$$

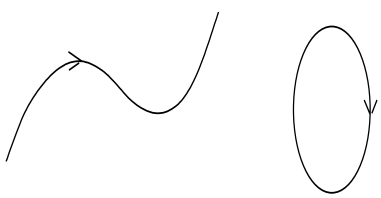
One easily checks that indeed we have

$$\begin{aligned}
\nabla\phi(x, y) &= \nabla(x^2y^3 + y + c) = 2xy^3\vec{i} + (3x^2y^2 + 1)\vec{j} \\
&= \vec{F}(x, y).
\end{aligned}$$

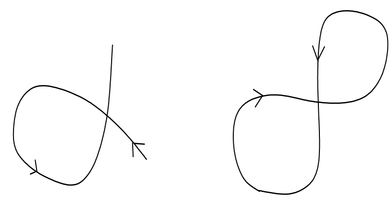
$\square$

The method used in the above example to check whether a given vector valued function $\vec{F}(x, y)$ is conservative is usually difficult to use, especially if the function $\vec{F}$ is complicated, like $\vec{F}(x, y) = \sin(xy)e^x\vec{i} + e^{\cos(xy)}\vec{j}$. In the following, we are to find an easier method. For this purpose, we need a few new notions.

Simple curve: A plane curve $\vec{r} = \vec{r}(t)(a \leq t \leq b)$ is said **simple** if it does not intersect itself between the end points.

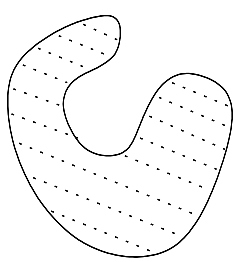


Simple Curves

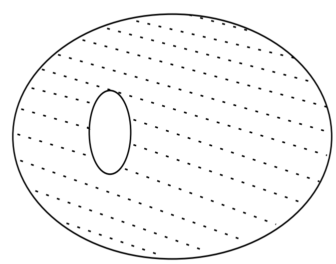


Non-simple Curves

Simply Connected Region: A plane region whose boundary consists of one simple closed curve is called a **simply connected region**. The entire xy-plane is regarded as a simply connected region (it has no boundary).



Simply Connected



Not Simply Connected

Theorem 3. Let $\vec{F}(x, y) = f(x, y)\vec{i} + g(x, y)\vec{j}$, where f and g have continuous partial derivatives in an open simply connected region. Then $\vec{F}$ is conservative in that region if and only if $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$ in that region.

Proof. We only prove the theorem for the simple case that the region is a rectangle $a < x < b$, $c < y < d$.

To show the necessity, we assume that $\vec{F}$ is conservative, i.e. $\vec{F} = \nabla\phi$ for some ϕ for all (x, y) in the region.

Then $f = \dfrac{\partial \phi}{\partial x}$, $\dfrac{\partial f}{\partial y} = \dfrac{\partial^2 \phi}{\partial y \partial x}$, $g = \dfrac{\partial \phi}{\partial y}$, $\dfrac{\partial g}{\partial x} = \dfrac{\partial^2 \phi}{\partial x \partial y}$

By a theorem on the mixed partial derivatives (please find this theorem), we have

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}.$$

Hence

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}.$$

Notice that for this part of the proof no assumptions on the region have been used.

Now we show the sufficiency. Suppose that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

Choose an arbitrary point (x_0, y_0) in the rectangular region, and define

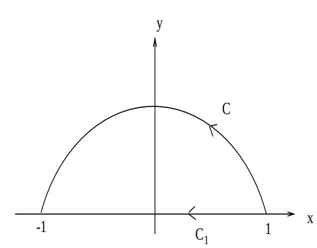
$$\phi(x, y) = \int_{x_0}^{x} f(t, y)dt + \int_{y_0}^{y} g(x_0, s)ds$$

Then

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= f(x, y) \\
\frac{\partial \phi}{\partial y} &= \int_{x_0}^{x} \frac{\partial}{\partial y} f(t, y)dt + g(x_0, y) \\
&= \int_{x_0}^{x} \frac{\partial}{\partial x} g(t, y)dt + g(x_0, y) \quad \left(\text{using } \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}\right) \\
&= g(t, y)\Big|_{x_0}^{x} + g(x_0, y) \\
&= g(x, y) - g(x_0, y) + g(x_0, y) \\
&= g(x, y) \\
\text{Hence} \quad \nabla \phi &= \vec{F}.
\end{aligned}$$

□

Example 2. A particle moves over the semicircle $C : \vec{F}(t) = \cos t\vec{i} + \sin t\vec{j},\ 0 \leq t \leq \pi$ while subject to the force $\vec{F}(x, y) = e^y\vec{i} + xe^y\vec{j}$. Find the work done.



Solution. We have $f(x, y) = e^y$, $g(x, y) = xe^y$ and

$$\frac{\partial f}{\partial y} = e^y = \frac{\partial g}{\partial x}$$

Therefore, the work done $W = \int_C \vec{F} \cdot d\vec{r}$ does not depend on the path.

Let $C_1 : x = 1 - t,\ y = 0,\ 0 \leq t \leq 2$ be the line segment joining $(1, 0)$ and $(-1, 0)$. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r}$$

$$\begin{aligned} &= \int_0^2 \left[e^0 \cdot (1-t)' + (1-t)e^0 \cdot (0)'\right] dt \\ &= \int_0^2 -1 dt = -2. \end{aligned}$$

□

Note. In Example 2, if we have used the original path C for the calculation, we would need to find

$$\begin{aligned} &\int_0^\pi \left[e^{\sin t}(\cos t)' + \cos t e^{\sin t}(\sin t)'\right] dt \\ &= \int_0^\pi e^{\sin t}(-\sin t + \cos^2 t) dt, \end{aligned}$$

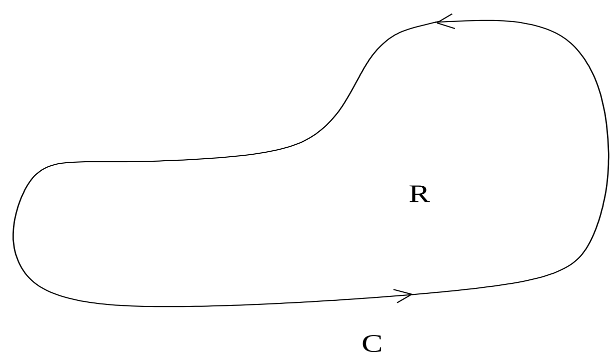
which is very difficult to integrate.

Lecture 30 Green's Theorem

Recall from Lecture 29 that if $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$ on a simply connected domain, then $\vec{F} = f\vec{i} + g\vec{j}$ is conservative, and therefore for any simple closed curve C (lying entirely in the region where $\vec{F}$ is conservative), $\displaystyle\int_C f(x,y)dx + g(x,y)dy = 0$. We will see in this lecture that this fact also follows from a more general result, known as Green's Theorem, which establishes an important relationship between line integrals and double integrals.

Theorem 1 (Green's Theorem). Let R be a simply connected plane region whose boundary is a simple, closed, piecewise smooth curve C oriented counterclockwise. If $f(x,y)$ and $g(x,y)$ have continuous first order partial derivatives on some set containing R, then

$$\int_C f(x,y)dx + g(x,y)dy = \iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

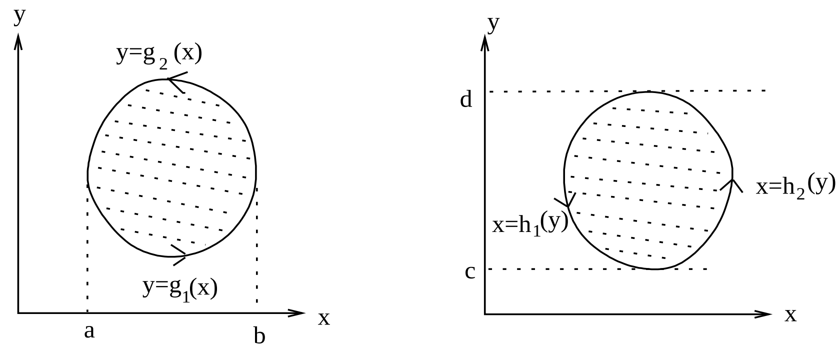


Proof. We only prove the theorem for the case that R is both a type I and type II region. So R can be described by both

$$g_1(x) \le y \le g_2(x),\ a \le x \le b \qquad \text{(type I region)}$$

and

$$h_1(y) \le x \le h_2(y),\ c \le y \le d \qquad \text{(type II region)}$$



We have,

$$\iint\limits_R \frac{\partial g}{\partial x} dA = \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial g}{\partial x} dx dy = \int_c^d \left[g(h_2(y), y) - g(h_1(y), y) \right] dy$$

$$= \int_c^d \left[g(h_2(t), t) - g(h_1(t), t) \right] dt.$$

Let C_1 be the part of C parametrized by

$$x = h_1(t),\ \ y = t,\ \ c \leq t \leq d$$

Then

$$\int_{C_1} g(x, y) dy = \int_c^d g(h_1(t), t)(t)' dt = \int_c^d g(h_1(t), t) dt$$

Let C_2 be the remaining part of C which is parametrized by

$$x = h_2(t), y = t,\ \ c \leq t \leq d.$$

Then

$$\int_{C_2} g(x, y) dy = \int_c^d g(h_2(t), t)(t)' dt = \int_c^d g(h_2(t), t) dt.$$

If we denote by $-C_1$ the curve C_1 but with the orientation reversed, then $C = (-C_1) \cup C_2$ (the union of $-C_1$ and C_2), and hence

$$\begin{aligned}
\int_C g(x, y) dy &= \int_{-C_1} g(x, y) dy + \int_{C_2} g(x, y) dy \\
&= -\int_c^d g(h_1(t), t) dt + \int_c^d g(h_2(t), t) dt \\
&= \int_c^d \left[g(h_2(t), t) - g(h_1(t), t) \right] dt \\
&= \iint\limits_R \frac{\partial g}{\partial x} dA.
\end{aligned}$$

In a similar fashion,

$$\begin{aligned}\iint\limits_R \frac{\partial f}{\partial y} dA &= \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial f}{\partial y} dy\ dx \\ &= \int_a^b \left[f(x, g_2(x)) - f(x, g_1(x)\right] dx \\ &= \int_a^b \left[f(t, g_2(t)) - f(t, g_1(t))\right] dt\end{aligned}$$

On the other hand, let $C^1 = \{(x, y) : x = t, y = g_1(t), a \leq t \leq b\}$ and $C^2 = \{(x, y) : x = t, y = g_2(t), a \leq t \leq b\}$

Then

$$\begin{aligned}\int_{C^1} f(x, y)dx &= \int_a^b f(t, g_1(t))dt \\ \int_{C^2} f(x, y)dx &= \int_a^b f(t, g_2(t))dt\end{aligned}$$

and $C = (C') \cup (-C^2)$. Hence

$$\begin{aligned}\int_C f(x, y)dx &= \int_{C^1} f(x, y)dx + \int_{-C^2} f(x, y)dx \\ &= \int_a^b f(t, g_1(t))dt - \int_a^b f(t, g_2(t))dt \\ &= -\int_a^b \left[f(t, g_2(t)) - f(t, g_1(t)\right] dt \\ &= -\iint\limits_R \frac{\partial f}{\partial y} dA\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\int_C f(x, y)dx + g(x, y)dy &= \iint\limits_R \frac{\partial g}{\partial x} dA - \iint\limits_R \frac{\partial f}{\partial y} dA \\ &= \iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA.\end{aligned}$$

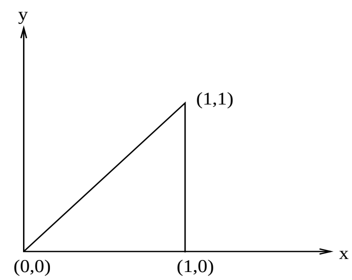
□

Example 1 Use Green's Theorem to evaluate $\int_C xy^2 dx + 2x^2 y\ dy$, where C is the boundary of the triangle with vertices $(0, 0), (1, 0)$ and $(1, 1)$, oriented in that order.

Solution. R can be described by

$0 \leq x \leq 1,\ 0 \leq y \leq x.$

Therefore, by Green's theorem,



$$\begin{aligned}
&\int_C xy^2dx + 2x^2ydy = \iint\limits_R \left[\frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(xy^2)\right] dA \\
&= \iint\limits_R (4xy - 2xy)dA = \iint\limits_R 2xydA \\
&= \int_0^1 \int_0^x 2xydydx = \int_0^1 xy^2 \mid_0^x dx \\
&= \int_0^1 x^3dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}.
\end{aligned}$$

□

Example 2. Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution Area $= \iint\limits_R 1\ dA.$

Let C denote the ellipse oriented counterclockwise. We deduce from Green's Theorem that

$$\begin{aligned}
\int_C x\ dy &= \int_C x\ dy + 0\ dx = \iint\limits_R \left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(0)\right] dA = \iint\limits_R 1\ dA \\
\int_C y\ dx &= \int_C y\ dx + 0\ dy = \iint\limits_R \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(y)\right] dA = -\iint\limits_R 1\ dA
\end{aligned}$$

Therefore

$$\text{Area} = \int_C x\ dy = -\int_C y\ dx$$

If we use the parametrization $x = a\cos t,\ y = b\sin t,\ 0 \leq t \leq 2\pi$ for C, then

$$
\begin{aligned}
\text{Area} &= \int_0^{2\pi} a\cos t(b\sin t)' dt \\
&= \int_0^{2\pi} ab\cos^2 t\ dt \\
&= \int_0^{2\pi} ab\frac{1+\cos 2t}{2}\ dt \\
&= \frac{ab}{2}\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi} \\
&= \frac{ab}{2}\cdot 2\pi = ab\pi.
\end{aligned}
$$

□

Note that in Example 1, we used a double integral to calculate a line integral, while in Example 2, a double integral was calculated by a line integral. Please try both integrals directly and see which method is simpler.

There is a nice way to formulate Green's theorem using the language of 1-forms and 2-forms. We have introduced 1-forms a expressions of the form $f(x,y)\,dx + g(x,y)\,dy$ (in 2-space) or $f(x,y,z)\,dx + g(x,y,z)\,dy + h(x,y,z)\,dz$ (in 3-space).

2-forms are expressions of the form $\phi(x,y)\,dxdy$ (in 2-space) or $\phi(x,y,z)\,dxdy + \psi(x,y,z)\,dzdx + \chi(x,y,z)\,dydz$. The 'products' dxdy etc. are not commutative but follow the rule $dydx = -dxdy$. This is often emphasised by the notation $dx \wedge dy$. In particular, $dx \wedge dx = -dx \wedge dx = 0$

The 'differential operator' d applied to a 1-form gives

$$
\begin{aligned}
d(f\,dx + g\,dy) &= (f_x dx + f_y dy) \wedge dx + (g_x dx + g_y dy) \wedge dy = (-f_y + g_x) dx \wedge dy \\
d(f\,dx + g\,dy + h\,dz) &= (-f_y + g_x) dx \wedge dy + (f_z - h_x) dz \wedge dx + (-g_z + h_y) dy \wedge dz
\end{aligned}
$$

Now, the necessary condition for the vector field $\vec{F} = \langle f, g \rangle$ being conservative can be expressed in terms of the corresponding 1-form $f\,dx + g\,dy$ by

$$d(f\,dx + g\,dy) = (-f_y + g_x) dx \wedge dy = 0.$$

Green's theorem can now be stated as

$$\int_C f\,dx + g\,dy = \int_R d(f\,dx + g\,dy) = \int_R (-f_y + g_x) dx \wedge dy,$$

where C is the boundary of R. In other words, the integral of a 1-form over the boundary of a simply connected domain equals the integral of the d differential of the form over the domain itself.

Lecture 31 Surface Integrals

Recall that the surface area of the part of surface $z = f(x,y)$ lying directly above or below the region R in the xy-plane is given by

$$\text{Area} = \iint\limits_R \sqrt{f_x^2 + f_y^2 + 1}dA.$$

If we want to find the mass of a curved lamina with equation $z = f(x,y),\ (x,y) \in R$ and density $\delta(x,y,z)$, then we need to calculate the integral

$$\begin{aligned}&\iint\limits_R \delta(x,y,z)\sqrt{f_x^2 + f_y^2 + 1}dA \\ &= \iint\limits_R \delta(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}dA\end{aligned}$$

which gives the mass.

In general, let σ be a surface $z = f(x,y)$ and R the projection of σ on the xy-plane. If $f(x,y)$ has continuous first order partial derivatives on R and $g(x,y,z)$ is continuous on σ, then the **surface integral** is defined by

$$\boxed{\iint\limits_\sigma g(x,y,z)dS = \iint\limits_R g(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\ dA}$$

If σ is given by $y = f(x,z)$ and R is the projection of σ on the xz-plane, then similarly,

$$\boxed{\iint\limits_\sigma g(x,y,z)dS = \iint\limits_R g(x,f(x,z),z)\sqrt{f_x^2 + f_z^2 + 1}\ dA}$$

If σ is given by $x = f(y,z)$ and R is the projection of σ on the yz-plane, then

$$\boxed{\iint\limits_\sigma g(x,y,z)dS = \iint\limits_R g(f(y,z),y,z)\sqrt{f_y^2 + f_z^2 + 1}\ dA}$$

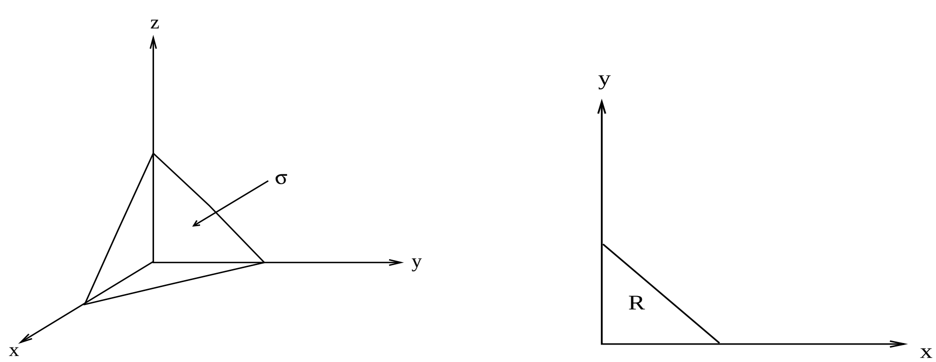
The following properties of surface integrals follow directly from the definition:

$$\iint_\sigma cg(x,y,z)\ dS = c\iint_\sigma g(x,y,z)\ dS, c \text{ is a constant}$$
$$\iint_\sigma (g_1 \pm g_2)\ dS = \iint_\sigma g_1\ dS \pm \iint_\sigma g_2\ dS$$
$$\iint_\sigma g\ dS = \iint_{\sigma_1} g\ dS + \iint_{\sigma_2} g\ dS$$
where σ consists of σ_1 and σ_2.

Example 1. Evaluate the surface integral $\iint_\sigma xy\ dS$, where σ is the part of the plane $x+y+z=1$ that lies in the first octant.

Solution. A sketch of σ shows that its projection on the xy-plane can be expressed by

$$R: 0 \le x \le 1, \quad 0 \le y \le 1-x$$



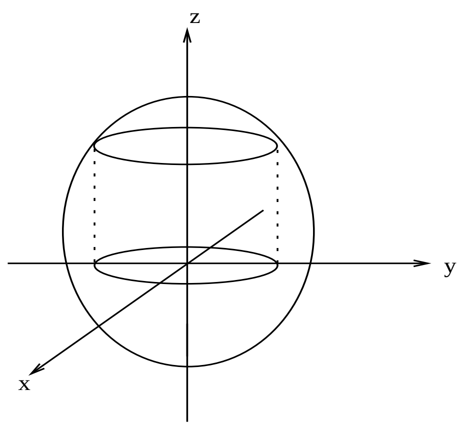
Moreover, the equation of σ can be written as $z = 1-x-y$. Therefore,

$$\begin{aligned}
\iint_\sigma xy\ dS &= \iint_\sigma xy\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\ dA \\
&= \iint_R xy\sqrt{(-1)^2+(-1)^2+1}\ dA = \iint_R \sqrt{3}xy\ dA \\
&= \int_0^1 \int_0^{1-x} \sqrt{3}xy\ dy\ dx = \int_0^1 \sqrt{3}x\frac{y^2}{2}\Big|_0^{1-x} dx \\
&= \int_0^1 \sqrt{3}x\frac{(1-x)^2}{2}\ dx \\
&= \frac{\sqrt{3}}{24}
\end{aligned}$$

□

Example 2. Evaluate the surface integral $\iint_\sigma (x^2+y^2)z\,dS$, where σ is the portion of the sphere $x^2+y^2+z^2=4$ above the plane $z=1$.

Solution. The plane $z=1$ and the sphere $x^2+y^2+z^2=4$ intersect at $x^2+y^2=3$, $z=1$, which is a circle 1 unit above the xy-plane, with centre on the z-axis, and radius $\sqrt{3}$.



The equation of σ can be rewritten as $z=\sqrt{4-x^2-y^2}$ and the projection of σ on the xy-plane is the disk $x^2+y^2\le 3$, or

$$R: 0\le\theta\le 2\pi,\quad 0\le r\le\sqrt{3}\quad\text{(in polar coordinates)}.$$

Therefore,

$$\iint_\sigma (x^2+y^2)z\,dS = \iint_R (x^2+y^2)\left(\sqrt{4-x^2-y^2}\right)\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\,dA$$

$$= \iint_R (x^2+y^2)\sqrt{4-x^2-y^2}\sqrt{\left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2+1}\,dA$$

$$= \iint_R (x^2+y^2)\sqrt{4-x^2-y^2}\sqrt{\frac{4}{4-x^2-y^2}}\,dA$$

$$= \iint_R 2(x^2+y^2)\,dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} 2r^2\ rd\ rd\ \theta$$

$$= \int_0^{2\pi} 2\frac{r^4}{4}\Big|_0^{\sqrt{3}}\,d\theta = \int_0^{2\pi}\frac{9}{2}\,d\theta = 9\pi.$$

□

Lecture 32 Surface Integrals of Vector Functions

Surface integrals also occur in calculations of the volume of fluid which passes through a certain surface σ.

Suppose

$$\vec{F}(x, y, z) = f(x, y, z)\vec{i} + g(x, y, z)\vec{j} + h(x, y, z)\vec{k}$$

is the velocity of the fluid at a point (x, y, z), and $\vec{n} = \vec{n}(x, y, z)$ is the unit normal vector of σ at (x, y, z). Then the volume of fluid that passes through σ per unit time is given by

$$V = \iint_{\sigma} \vec{F} \cdot \vec{n}\, dS$$

If the surface σ is given by $G(x, y, z) = 0$, then we know

$$\frac{\nabla G(x, y, z)}{||\nabla G(x, y, z)||}$$

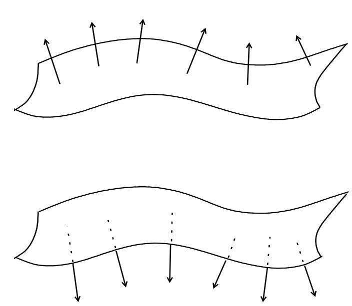
is a unit normal vector of σ at (x, y, z). Clearly,

$$\frac{-\nabla G(x, y, z)}{||\nabla G(x, y, z)||}$$

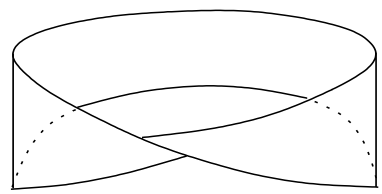
is also a unit normal vector.

A surface is called **orientable** if a unit vector can be constructed at each point of the surface such that the vectors vary continuously as we traverse any curve on the surface.

A surface is said to be **oriented** if such a set of unit normal vectors is constructed. For example, if the surface is the unit sphere, then it is orientable, with one orientation has all the unit normal vectors pointing outward the sphere, and the other has all the unit normal vectors pointing inward the sphere. Usually, an orientable surface has exactly two orientations.



There is a famous surface which is not orientable, i.e. the **Möbius strip**, as shown in the following diagram.



To help us describing orientations of surfaces, we say the vector $\vec{u} = a\vec{i} + b\vec{j} + c\vec{k}$ is pointing upward if $c > 0$, rightward if $b > 0$ and forward if $a > 0$. This describes the situation if the xyz-coordinate system is drawn (as usual) so that the z-axis points upward, y-axis points rightward and x-axis points forward.

Example 1 Find all the upward unit normals for $z = x^2 + y^2$.

Solution Let $G(x, y, z) = x^2 + y^2 - z$. Then $\nabla G(x, y, z) = 2x\vec{i} + 2y\vec{j} - \vec{k}$

$$\frac{\nabla G}{||\nabla G||} = \frac{2x\vec{i} + 2y\vec{j} - \vec{k}}{\sqrt{4x^2 + 4y^2 + 1}} = \frac{2x}{\sqrt{4x^2 + 4y^2 + 1}}\vec{i} + \frac{2y}{\sqrt{4x^2 + 4y^2 + 1}}\vec{j} - \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}\vec{k}$$

We know $\frac{\nabla G}{||\nabla G||}$ is a unit normal for $G = 0$, i.e. $z = x^2 + y^2$. Since the last component of $\frac{\nabla G}{||\nabla G||}$ is $-\frac{1}{\sqrt{4x^2 + 4y^2 + 1}} < 0$, it points downward. However,

$$\frac{-\nabla G}{||\nabla G||} = \frac{-2x}{\sqrt{4x^2 + 4y^2 + 1}}\vec{i} + \frac{-2y}{\sqrt{4x^2 + 4y^2 + 1}}\vec{j} + \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}\vec{k}$$

is also a unit normal vector, and it points upward. As x, y varies, it gives all the upward unit normals for the surface. $\square$

As each component of $\frac{-\nabla G}{||\nabla G||}$ in Example 1 is a continuous function of (x, y), these unit vectors vary continuously with (x, y). Hence $\frac{-\nabla G}{||\nabla G||}$ gives an orientation of the surface. The other orientation is given by $\frac{\nabla G}{||\nabla G||}$, the downward unit vectors.

Example 2. Suppose that σ is the portion of the surface (paraboloid) $z = 1 - x^2 - y^2$

above the xy-plane. Let σ be oriented by upward normals and let $\vec{F}(x,y,z) = y\vec{i} + x\vec{j} + z\vec{k}$. Evaluate $\displaystyle\iint_\sigma \vec{F}\cdot\vec{n}\, dS$.

Solution. The equation of the surface can be rewritten as

$$G(x,y,z) = z - 1 + x^2 + y^2 = 0$$

$$\begin{aligned} \nabla G &= 2x\vec{i} + 2y\vec{j} + \vec{k}, \\ \frac{\nabla G}{||\nabla G||} &= \frac{2x}{\sqrt{4x^2+4y^2+1}}\vec{i} + \frac{2y}{\sqrt{4x^2+4y^2+1}}\vec{j} + \frac{1}{\sqrt{4x^2+4y^2+1}}\vec{k} \end{aligned}$$

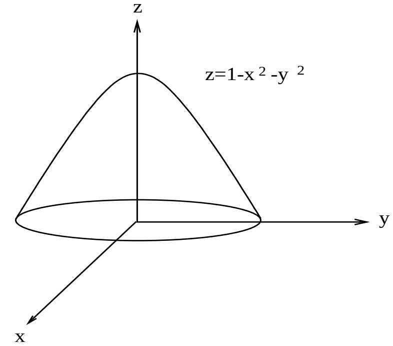
As $\dfrac{1}{\sqrt{4x^2+4y^2+1}} > 0$, the normal vector $\dfrac{\nabla G}{||\nabla G||}$ points upward.

Hence,

$$\begin{aligned} \vec{n} &= \frac{\nabla G}{||\nabla G||} = \frac{2x}{\sqrt{4x^2+4g^2+1}}\vec{i} + \frac{2y}{\sqrt{4x^2+4y^2+1}}\vec{j} + \frac{1}{\sqrt{4x^2+4y^2+1}}\vec{k} \\ \vec{F}\cdot\vec{n} &= \frac{2xy}{\sqrt{4x^2+4y^2+1}} + \frac{2xy}{\sqrt{4x^2+y^2+1}} + \frac{z}{\sqrt{4x^2+4y^2+1}} \\ &= \frac{4xy+z}{\sqrt{4x^2+4y^2+1}} \end{aligned}$$

A sketch of σ shows its projection on the xy-plane is the disk $x^2+y^2 \le 1$, or

$$R: 0 \le \theta \le 2\pi, \quad 0 \le r \le 1 \quad \text{in polar coordinates.}$$



Therefore,

$$\begin{aligned}
\iint\limits_{\sigma} \vec{F}\cdot\vec{n}\ ds &= \iint\limits_{\sigma} \frac{4xy+z}{\sqrt{4x^2+4y^2+1}}\ dS \\
&= \iint\limits_{R} \frac{4xy+(1-x^2-y^2)}{\sqrt{4x^2+4y^2+1}} \sqrt{\left[\frac{\partial}{\partial x}(1-x^2-y^2)\right]^2+\left[\frac{\partial}{\partial y}(1-x^2-y^2)\right]^2+1}\ dA \\
&= \iint\limits_{R} \frac{4xy+1-x^2-y^2}{\sqrt{4x^2+4y^2+1}}\sqrt{(2x)^2+(2y)^2+1}\ dA \\
&= \iint\limits_{R} (4xy+1-x^2-y^2)\ dA \\
&= \int_0^{2\pi}\int_0^1 [4(r\cos\theta)(r\sin\theta)+1-r^2]r\ dr\ d\theta \\
&= \int_0^{2\pi} \left(r^4\cos\theta\sin\theta+\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^1\ d\theta \\
&= \int_0^{2\pi} \left(\cos\theta\sin\theta+\frac{1}{4}\right) d\theta \\
&= \left(\frac{\sin^2\theta}{2}+\frac{\theta}{4}\right)\Big|_0^{2\pi} = \frac{\pi}{2}.
\end{aligned}$$

□

We may look at the integral

$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS$$

also in the following way: First assume that $\vec{F}=\langle 0,0,h\rangle$ and σ is given as a graph $z=\phi(x,y)$. Then $\vec{n}=\dfrac{\langle -\phi_x,-\phi_y,1\rangle}{\sqrt{\phi_x^2+\phi_y^2+1}}$ and

$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS = \iint_{\sigma_z} h(x,y,\phi(x,y))\frac{1}{\sqrt{\phi_x^2+\phi_y^2+1}}\sqrt{\phi_x^2+\phi_y^2+1}dxdy = \iint_{\sigma_z} h\,dxdy,$$

where σ_z is the projection of σ to the xy-plane. Similarly, for $g=h=0$ and $f=h=0$ we get

$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS = \iint_{\sigma_x} f\,dydz,$$
$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS = \iint_{\sigma_y} g\,dzdx.$$

Combining this we obtain a formula for general $\vec{F}$

$$\iint_\sigma \vec{F}\cdot\vec{n}\,dS = \iint_\sigma f\,dy\wedge dz + g\,dz\wedge dx + h\,dx\wedge dy.$$

We recognise that

$$f\,dy \wedge dz + g\,dz \wedge dx + h\,dx \wedge dy$$

as a 2-form and use $\wedge$ for the product of the differentials to emphasise the non-commutativity of this product.

Lecture 33 The Divergence Theorem

Recall that Green's Theorem relates a line integral with a double integral where the integrand of the double integral consists of certain partial derivatives of the integrands in the line integral, and the curve in the line integral is the boundary of the region in the double integral:

$$\int_C f(x,y)dx + g(x,y)dy = \iint\limits_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\, dA$$

Notice that Green's formula can also be written as

$$\int_C \langle f, g\rangle \cdot \vec{n}\, ds = \int_C \langle -g, f\rangle \cdot d\vec{r} = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)\, dA,$$

where $\vec{n}$ is the outer unit normal to C and $\vec{n}\, ds = \langle dy, -dx\rangle$.

There is a similar relationship between surface integrals and triple integrals, known as the Divergence theorem.

Given a vector-valued function $\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$, the **divergence** of $\vec{F}$, denoted by $\operatorname{div}\vec{F}(x,y,z)$, is defined by the following formula,

$$\operatorname{div}\vec{F}(x,y,z) = f_x(x,y,z) + g_y(x,y,z) + h_z(x,y,z),$$

or,

$$\boxed{\operatorname{div}\vec{F} = \tfrac{\partial f}{\partial x} + \tfrac{\partial g}{\partial y} + \tfrac{\partial h}{\partial z}}$$

Theorem 1 (The Divergence Theorem). Let G be a solid whose boundary is an orientable surface σ and is oriented with all the unit normals pointing outward of G. If

$$\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k}$$

has all its component functions f, g, h with continuous first order partial derivatives on some open set containing G, then

$$\iint_\sigma \vec{F} \cdot \vec{n} dS = \iiint_G \operatorname{div} \vec{F} dV$$

When G is simple, a proof of this theorem can be found in the reference books, which is in the spirit of the proof of Green's Theorem. However, for a general G, the proof is difficult.

Apart from its theoretical importance, the divergence theorem sometimes provides a way to simplify the calculation of surface or triple integrals.

Again, the divergence theorem can be nicely expressed in terms of differential forms: The surface integral at the left hand side equals

$$\iint_\sigma f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy.$$

Now,

$$d(f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy) = (f_x + g_y + h_z) dx \wedge dy \wedge dz = \operatorname{div} \vec{F} dV.$$

Hence,

$$\iint_\sigma f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy = \iiint_G d(f\, dy \wedge dz + g\, dz \wedge dx + h\, dx \wedge dy),$$

i.e., similar to Green's theorem, the integral of a 2-form over the boundary of a domain equals the volume integral of the d derivative of the 2-form over the domain itself.

Example 1 Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and σ the surface of a solid G oriented by outward unit normals. Show that

$$\operatorname{vol}(G) = \frac{1}{3} \iint_\sigma \vec{r} \cdot \vec{n} dS$$

where vol (G) denotes the volume of G.

Proof. By the divergence theorem,

$$\iint_\sigma \vec{r} \cdot \vec{n} dS = \iiint_G \operatorname{div} \vec{r}\, dV$$

But clearly $\operatorname{div} \vec{r} = 3$. Hence

$$\iint_\sigma \vec{r} \cdot \vec{n} dS = \iiint_G 3 dV = 3\operatorname{vol}(G)$$

i.e.

$$\text{vol}(G) = \frac{1}{3} \iint_{\sigma} \vec{r} \cdot \vec{n} dS.$$

□

Remark. A variant of the above proof shows that if $\vec{r}_1 = x\vec{i}+0\vec{j}+0\vec{k}$, $\vec{r}_2 = 0\vec{i}+y\vec{j}+0\vec{k}$ and $\vec{r}_3 = 0\vec{i}+0\vec{j}+z\vec{k}$, then

$$\text{vol}(G) = \iint_{\sigma} \vec{r}_m \cdot \vec{n} dS, \ m = 1, 2, 3$$

Example 2. Find $\iint_{\sigma} \vec{F} \cdot \vec{n} dS$ where $\vec{F} = y\vec{i} + x\vec{j} + z\vec{k}$ and σ is the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward unit normals.

Solution. We want to use the divergence theorem to transform the surface integral to a triple integral. We first compute the divergence of $\vec{F}$:

$$\text{div}\, \vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z) = 1$$

The unit sphere σ is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leq 1$. Thus, by the divergence theorem,

$$\iint_{\sigma} \vec{f} \cdot \vec{n} ds = \iiint_{B} \text{div}\, \vec{F} dV = \iiint_{B} 1 dV = \text{vol}(B) = \frac{4\pi}{3}.$$

□

Example 3 Evaluate $\displaystyle\iint_{\sigma} \vec{F} \cdot \vec{n} dS$, where

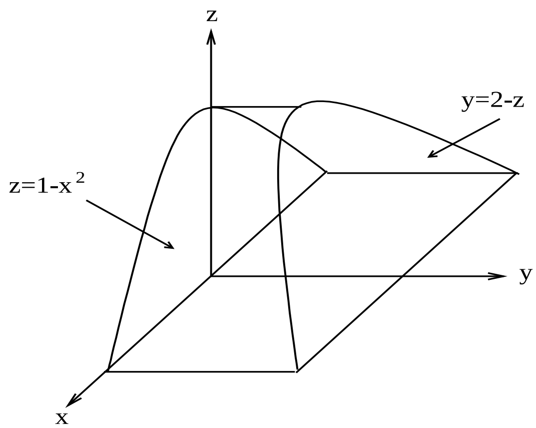
$$\vec{F}(x, y, z) = xy\vec{i} + (y^2 + e^{xz^2})\vec{j} + \sin(xy)\vec{k}$$

and σ is the surface of the region G bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$, and σ is oriented by outward normals.

Solution. A sketch of G shows that σ consists of four pieces of smooth surfaces, and a direct calculation of the surface integral would involve calculating four surface integrals corresponding to each piece. Therefore, we use the divergence theorem to reduce it to a triple integral.

Let R be the projection of G on the xz-plane. Then G can be expressed by

$$0 \leq y \leq 2 - z, \quad (x, z) \in R,$$



and R can be expressed by

$$0 \le z \le 1 - x^2, \quad -1 \le x \le 1$$

Therefore,

$$\begin{aligned}
& \iint_{\sigma} \vec{F} \cdot \vec{n} \; dS = \iiint_{G} \; \operatorname{div} \vec{F} \; dV \\
= & \iiint_{G} (y + 2y + 0) \; dV = \iiint_{G} \; 3y \; dV \\
= & \iint_{R} \int_0^{2-z} \; 3y \; dy \; dA \\
= & \iint_{R} \frac{3}{2} y^2 \Big|_0^{2-z} \; dA \\
= & \iint_{R} \frac{3}{2} (2 - z)^2 \; dA \\
= & \int_{-1}^{1} \int_0^{1-x^2} \frac{3}{2} (2 - z)^2 \; dz \; dx \\
= & \int_{-1}^{1} -\frac{1}{2} (2 - z)^3 \Big|_0^{1-x^2} \; dx \\
= & \int_{-1}^{1} \frac{1}{2} \left[2^3 - (1 + x^2)^3 \right] \; dx \\
= & \int_{-1}^{1} \frac{1}{2} (7 - x^6 - 3x^4 - 3x^2) \; dx \\
= & \frac{184}{35}
\end{aligned}$$

□

Lecture 34 Stokes' Theorem

Recall that Green's Theorem relates line integrals with double integrals, and the Divergence theorem relates surface integrals with triple integrals. The following theorem, known as Stokes' Theorem, will give a relationship between line integrals and surface integrals.

To state Stokes' theorem, we need one new notion. If

$$\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k},$$

then the **curl of $\vec{F}$** is defined by

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\vec{i} - \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\vec{j} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k}$$

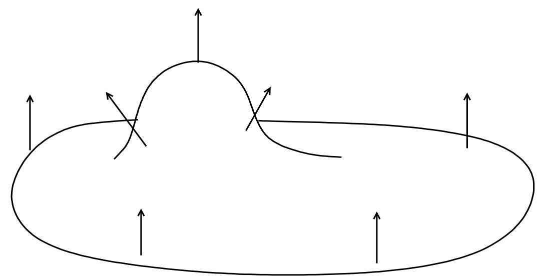
Note that, in the above notation, the determinant is used to help with remembering the formula, it does not give a number, instead, it gives a vector. (Compare with the notation in the definition of cross product). Also, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ are never numbers or functions.

Theorem 1 (Stokes' Theorem) Let σ be an oriented surface, bounded by a simple curve C. If the components of $\vec{F} = f\vec{i} + g\vec{j} + h\vec{k}$ have continuous first order partial derivatives on some open set containing σ, then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_\sigma (\text{curl } \vec{F}) \cdot \vec{n}\ dS,$$

where the line integral is taken in the **positive direction** of C, and the positive direction of C is determined in the following way:

If one moves the unit normal vector of σ (that determines the orientation of σ) along C in the positive direction, then the surface σ lies on the left side of the moving unit normal vector.



While we do not prove Stokes' theorem whose proof is very involved, we will see through examples how it can be used.

Again, Stoke's theorem can be reformulated in terms of differential forms: The integral at the left hand side is

$$\int_C f\,dx + g\,dy + h\,dz.$$

Now

$$d(f\,dx + g\,dy + h\,dz) = (-f_y + g_x)dx \wedge dy + (-g_z + h_y)dy \wedge dz + (f_z - h_x)dz \wedge dx,$$

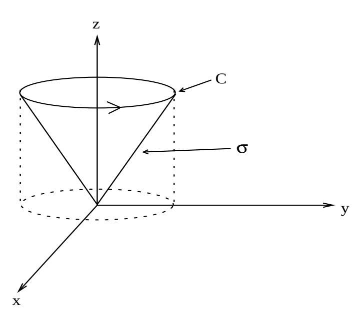
and therefore

$$\iint_\sigma (\text{curl } \vec{F}) \cdot \vec{n}\ dS = \iint_\sigma d(f\,dx + g\,dy + h\,dz).$$

Again, the theorem states that the integral of a 1-form over the boundary of a surface equals to the d-derivative of the 1-form over the surface itself.

Example 1. Verify Stoke's theorem by computing $\displaystyle\int_C \vec{F} \cdot d\vec{r}$ and $\displaystyle\iint_\sigma (\text{curl } \vec{F}) \cdot \vec{n}\ dS$, where $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$, σ is the portion of the cone $z = \sqrt{x^2+y^2}$ below the plane $z = 1$, and oriented by upward unit normals.

Solution. By sketching σ we see that the boundary of σ is
$C : x^2 + y^2 = 1,\ z = 1,$
i.e., the unit circle one unit above the xy-plane.



The following parametrization of $C : \vec{r} = \cos\theta\vec{i} + \sin\theta\vec{j} + \vec{k},\ 0 \le \theta \le 2\pi$ is in the positive direction of C. Therefore,

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \left[(\cos\theta)^2(\cos\theta)' + (\sin\theta)^2(\sin\theta)' + (1)^2(1)'\right]\ d\theta \\ &= \int_0^{2\pi} \left[(\cos\theta)^2(\cos\theta)' + (\sin\theta)^2(\sin\theta)'\right]\ d\theta \\ &= \left(\frac{\cos^3\theta}{3} + \frac{\sin^3\theta}{3}\right)\Big|_0^{2\pi} = 0.\end{aligned}$$

On the other hand,

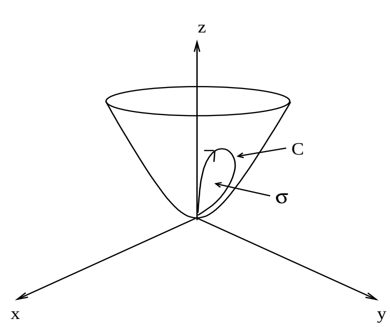
$$\begin{aligned}\text{curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial x}z^2 - \frac{\partial}{\partial z}y^2\right)\vec{i} + \left(\frac{\partial}{\partial z}x^2 - \frac{\partial}{\partial x}z^2\right)\vec{j} + \left(\frac{\partial}{\partial x}y^2 - \frac{\partial}{\partial y}x^2\right)\vec{k} \\ &= \vec{0}\end{aligned}$$

Hence

$$\begin{aligned}(\text{curl}\vec{F})\cdot\vec{n} &= \vec{0}\cdot\vec{n} = 0 \\ \iint_\sigma (\text{curl } \vec{F})\cdot\vec{n}\ dS &= \iint_\sigma 0\ dS = 0 = \int_C \vec{F}\cdot d\vec{r}.\end{aligned}$$

□

Example 2. Use Stokes' theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is the intersection of the paraboloid $z = x^2 + y^2$ and the plane $z = y$ with counterclockwise orientation when looked down the positive z-axis, where $\vec{F} = xy\vec{i} + x^2\vec{j} + z^2\vec{k}$.



Solution. Let σ be the portion of the paraboloid cut off by the plane $z = y$, and let σ be oriented by upward unit normals.
Then by Stokes' Theorem,
$\int_C \vec{F}\cdot d\vec{r} = \iint_\sigma (\text{curl } \vec{F})\cdot\vec{n}\ ds.$

$$\begin{aligned}
\text{curl}\vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & x^2 & z^2 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(x^2)\right)\vec{i} + \left(\frac{\partial}{\partial z}(xy) - \frac{\partial}{\partial x}(z^2)\right)\vec{j} + \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy)\right)\vec{k} \\
&= (2x - x)\vec{k} = x\vec{k}
\end{aligned}$$

Let $G(x,y,z) = z - x^2 - y^2 = 0$ denote the equation of the paraboloid. Then

$$\nabla G(x,y,z) = -2x\vec{i} - 2y\vec{j} + \vec{k}$$

gives an upward normal.

Hence

$$\vec{n} = \frac{\nabla G}{||\nabla G||} = \frac{-2x}{\sqrt{4x^2+4y^2+1}}\vec{i} + \frac{-2y}{\sqrt{4x^2+4y^2+1}}\vec{j} + \frac{1}{\sqrt{4x^2+4y^2+1}}\vec{k},$$

and

$$(\text{curl } \vec{F}) \cdot \vec{n} = \frac{x}{\sqrt{4x^2+4y^2+1}}.$$

The projection of σ on the xy-plane is the region enclosed by the projection of C on the xy-plane, the latter has the equation $y = x^2 + y^2$, i.e. $x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$. Thus the projection of σ on the xy-plane is the disk $R : x^2 + \left(y - \frac{1}{2}\right)^2 \le \frac{1}{4}$. We can regard R as a type II region expressed by

$$R : -\sqrt{\frac{1}{4} - \left(y - \frac{1}{2}\right)^2} \le x \le \sqrt{\frac{1}{4} - \left(y - \frac{1}{2}\right)^2}, 0 \le y \le 1.$$

Therefore,

$$\begin{aligned}
&\iint_\sigma (\text{curl } \vec{F}) \cdot \vec{n}\, dS = \iint_\sigma \frac{x}{\sqrt{4x^2+4y^2+1}}\, dS \\
= &\iint_R \frac{x}{\sqrt{4x^2+4y^2+1}} \sqrt{f_x^2 + f_y^2 + 1}\, dA \ (z = f(x,y) = x^2 + y^2 \text{ is the equation for } \sigma) \\
= &\iint_R x\, dA = \int_0^1 \int_{-\sqrt{\frac{1}{4}-(y-\frac{1}{2})^2}}^{\sqrt{\frac{1}{4}-(y-\frac{1}{2})^2}} x\, dx\, dy \\
= &\int_0^1 \frac{x^2}{2}\Big|_{-\sqrt{\frac{1}{4}-(y-\frac{1}{2})^2}}^{\sqrt{\frac{1}{4}-(y-\frac{1}{2})^2}} dy = \int_0^1 0\, dy = 0
\end{aligned}$$

$\square$

Lecture 35 Applications

Given a vector valued function

$$\vec{F}(x,y,z) = f(x,y,z)\vec{i} + g(x,y,z)\vec{j} + h(x,y,z)\vec{k},$$

in order to state the Divergence theorem and Stokes' theorem, we introduced functions

$$\operatorname{div} \vec{F} = f_x + g_y + h_z$$

and

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix}$$

It is important to understand the physical meaning of div $\vec{F}$ and curl $\vec{F}$. In fact, the identities consisting of the Divergence and Stokes' Theorems were first obtained from physical considerations.

If $\vec{F}(x,y,z)$ stands for the velocity of certain fluid, then the physical meaning of the surface integral $\iint\limits_{\sigma} \vec{F} \cdot \vec{n}\, dS$ is the **flux of the flow** $\vec{F}$ across σ, which represents the net volume of fluid that passes through σ per unit of time.

By the Divergence Theorem,

$$\iint\limits_{\sigma} \vec{F} \cdot \vec{n}\, dS = \iiint\limits_{G} \operatorname{div} \vec{F}\, dV$$

Hence $\iiint\limits_{G} \operatorname{div} \vec{F}\, dV$ also represents the flux of the flow $\vec{F}$ across the surface of G (i.e. σ).

Recall that if the density of a solid G is $\rho(x,y,z)$, then

$$\text{Mass of } G = \iiint\limits_{G} \rho(x,y,z)\, dv$$

Therefore, in a similar fashion,

$$\text{Flux of } \vec{F} = \iiint\limits_{G} \operatorname{div} \vec{F}\, dv$$

and div $\vec{F}$ is the **flux density of $\vec{F}$.**

curl$\vec{F}(x,y,z)$ is more difficult to explain, and we just mention that it measures how the flow $\vec{F}$ rotates near (x,y,z).

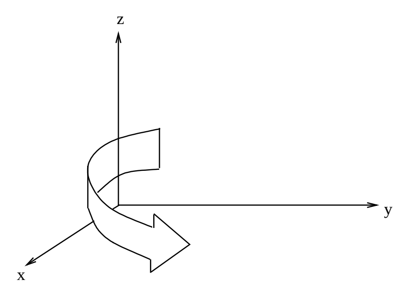
Example 1. The flow of the vector field

$$\vec{F}(x,y,z) = -y\vec{i} + x\vec{j} + 0\vec{k}$$

is $x = A\cos(t+t_0), y = A\sin(t+t_0), z = B$, where A, B, t_0 are some constants. (Verify that $\vec{F}$ is tangent to the flow lines.)

Geometrically, the flow is a rotation about the z-axis.

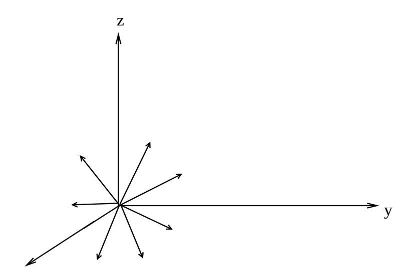
$$\begin{aligned} \text{curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} \\ &= 2\,\vec{k} \end{aligned}$$



and div $\vec{F} = 0$. □

Example 2. The flow of $\vec{F} = x\,\vec{i} + y\vec{j} + z\vec{k}$ is described by $x = A\,\mathrm{e}^t, y = B\,\mathrm{e}^t, z = C\,\mathrm{e}^t$, where A, B, C are some constants. It does not rotate, but diverges in all directions.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0},$$



and div$\vec{F} = 3$. □

Example 3. Let $\vec{F}$ be as in Example 1. Find the flux of $\vec{F}$ across the unit sphere $\sigma : x^2 + y^2 + z^2 = 1$ oriented by outward unit normals.

Solution. By the Divergence Theorem, if B denotes the unit ball: $x^2 + y^2 + z^2 \leq 1$, then

$$\begin{aligned} \text{flux} &= \iint\limits_{\sigma} \vec{F}\cdot\vec{n}\; dS = \iiint\limits_{B} \text{div}\,\vec{F}\; dv \\ &= \iiint\limits_{B} 0\; dv = 0. \end{aligned}$$

$\square$

Example 4. Let $\vec{F}$ be as in Example 2. Find the flux of $\vec{F}$ across the sphere $\sigma : x^2 + y^2 + z^2 = r^2$, oriented by outward unit normals.

Solution.

$$\begin{aligned}
\text{flux} &= \iint\limits_{\sigma} \vec{F} \cdot \vec{n}\ dS = \iiint\limits_{B_r} \operatorname{div} \vec{F}\ dV \\
&= \iiint\limits_{B_r} (a+b+c)\ dV = (a+b+c) \iiint\limits_{B_r} 1\ dV \\
&= (a+b+c)\text{vol}(B_r) = (a+b+c)\frac{4}{3}\pi r^3,
\end{aligned}$$

where B_r denotes the ball: $x^2 + y^2 + z^2 \leq r$. $\square$