UPDATE ON LOGARITHMS

1. Introduction

The concept of logarithms in full depth will be subject of the first year units Math110 or Math101/Math102. Here we can only give some rough ideas and instructions how to use logarithms.

Let us recall some notation. If \( a \) is a positive number and \( b \) any (real) number then the expression \( a^b \) is called the \( b \)-th power of \( a \). If \( b \) was a positive integer, then \( a^b \) is the product of \( b \) copies of factors \( a \). The number \( a \) is called the base of the power and \( b \) is the exponent.

**Example:** \( 2^4 = 2 \cdot 2 \cdot 2 \cdot 2 \), 2 is the base, 4 is the exponent.

We know the following rule for powers: \( 2^x \cdot 2^y = 2^{x+y} \) or \( 10^x \cdot 10^y = 10^{x+y} \) or more generally

\[ a^x \cdot a^y = a^{x+y}. \]

This leads us to the observation:

If we want to multiply two powers with the same base then the result is a again a power with the same base whose exponent is the sum of the exponents of the factors.

Since addition is much simpler than multiplication (or at least it used to be, before pocket calculators were invented) one would like to replace multiplication of two numbers by the addition of their exponents with respect to a suitable base. This raises the question:

What numbers can be expressed in the form \( 10^x \) (or any other base \( a \) instead of 10)?

The answer is a fundamental fact which will be proved in Math101: Any positive number \( z \) can be expressed in a unique way as \( 10^x \) for a suitable \( x \) (this is true with any positive number \( a \neq 1 \) replacing the base 10).

For a given number \( z \) the exponent \( x \) such that \( 10^x = z \) is called the logarithm of \( z \). We write \( x = \log_{10} z \) or shorter \( x = \log z \).

**Example.** \( \log 10 = 1 \), since \( 10^1 = 10 \). \( \log 1000 = 3 \), since \( 10^3 = 1000 \). \( \log 0.01 = -2 \), since \( 10^{-2} = 0.01 \).
2. **Multiplication using logarithms**

We assume that all factors are positive. Suppose, we want to multiply 3456 \cdot 4455.

**First step** Find the logarithms of 3456 and 4455. With the help of a calculator we find (approximately) \( \log 3456 = 3.5385 \) and \( \log 4455 = 3.6488 \). In former times logarithms were looked up in tables. To do so, one expresses the numbers in the form \( z = r \cdot 10^n \) where \( 1 \leq r < 10 \) and \( n \) is an integer. Then

\[
\log z = \log r \cdot 10^n = \log r + \log 10^n = \log r + n.
\]

This is why traditional logarithm tables show only logarithms for numbers \( r \) between 1 and 10. Notice that their logarithms are numbers between 0 and 1. The number \( n \) is called characteristic of \( z \) and \( \log r \) is called mantissa. In our example 3 is the characteristic of 3456 and 0.5385 is the mantissa.

**Second step** Add the logarithms \( 3.5385 + 3.6488 = 7.1873 \).

**Third step** Find the “antilogarithm” of the sum 7.1873. Again, with the help of the calculator we find \( 10^{7.1873} = 15392175 \). The precise result would be 15396480 which matches with the approximate result up to 4 places. We used approximations of \( \log \) with a precision of 4 places.

2.1. **Division using logarithms.** Suppose, we want to divide 3456 : 4455.

**First step** Find the logarithms of 3456 and 4455. Again we need to look up the logarithms of dividend and divisor: \( \log 3456 = 3.5385 \) and \( \log 4455 = 3.6488 \).

**Second step** Subtract the logarithms \( 3.5385 - 3.6488 = -0.1103 \).

**Third step** Find the “antilogarithm” of the difference \(-0.1103 \). Using the calculator we obtain \( 10^{-0.1103} = 0.7757 \). In order to look up the antilogarithm in a table we need to rewrite \(-0.1103 = 0.8897 - 1 \) with characteristic \(-1\) and mantissa 0.8897. The antilog of the mantissa is 7.757. Characteristic \(-1\) tells us that we need to multiply with \( 10^{-1} = 0.1 \). Hence, we get the same result 0.7757.

3. **Powers and roots using logarithms**

We recall the rule

\[
(a^b)^c = a^{bc}.
\]

It follows for \( a = 10 \) and \( b = \log z \)

\[
(10^{\log z})^c = 10^{(\log z)\cdot c}
\]
or

\[ z^c = 10^{\log z \cdot c} \]

or

\[ \log z^c = c \log z. \]

This helps to compute \( 2.7^{1.5} \). In fact, according to the above rule, the logarithm of \( 2.7^{1.5} \) is the product of the logarithm of the base \( 2.79 (= z) \) and the exponent \( 1.5 (= c) \). Hence we may proceed as follows:

**First step** Find the logarithm of the base \( \log 2.7 = 0.4313 \).

**Second step** Multiply the logarithm with the exponent \( 1.5 \cdot 0.4313 = 0.6470 \).

**Third step** Find the antilogarithm \( 10^{0.6470} = 4.4361 \).

In order to compute a root \( \sqrt[n]{a} \) we remember that \( \sqrt[n]{a} = a^{\frac{1}{n}} \) and proceed as above.

**Example.** For \( \sqrt[3]{35} \) we compute \( \log 35 \approx 1.544 \). Then multiply the result by \( \frac{1}{3} \) (i.e. divide it by 3) to get 0.515. This is the logarithm of the result, hence the antilog of 0.515, which is 3.27 is the approximate cubic root of 35.

### 4. Logarithms for other bases

Logarithms to the base 10 are best adapted to the decimal number system. We can read off the characteristic immediately and look up the mantissa in a table of logarithms for numbers between 1 and 10. There are two other bases which are important, namely 2 and even more important an irrational number which is approximately 2.718 and which is usually denoted by the letter e (in honour to the mathematician Leonhard Euler). Base 2 is best adapted to the binary number system. Hence it is used in computing. The number e is one of the most remarkable numbers in mathematics and will be discussed in Math110 or Math101/Math102.

In fact, it is easy to switch from one base to the other as you see from the example below:

**Example:** Compute \( \log_2 z \) using only logarithms to the base 10. We need to find \( x \) such that

\[ z = 2^x. \]

According to (1)

\[ \log_{10} z = \log_{10} 2^x = x \log_{10} 2. \]

Hence

\[ x = \log_2 z = \frac{\log_{10} z}{\log_{10} 2}. \]
4. UPDATE ON LOGARITHMS

This works with any positive base $a \neq 1$ instead of 2:

$$\log_a z = \frac{\log_{10} z}{\log_{10} a}. \quad (2)$$

5. POWERS AND LOGARITHMS

$\log_a z$ inverts the function $a^x$ in the sense that $\log_a a^x = x$ but also $a^{\log_a z} = z$. Equations involving powers can be rewritten as equations with logarithms and vice versa. We give examples:

**Example:** $9^2 = 81$ can be rewritten as $\log_9 81 = 2$. $9^{1/2} = 3$ can be rewritten as $\log_9 3 = \frac{1}{2}$. $9^{-1} = \frac{1}{9}$ can be rewritten as $\log_9 \frac{1}{9} = -1$.

Remember: The base $a$ becomes the subscript in $\log_a$, the exponent $b$ becomes the $\log_a a^b$ and the whole power $a^b$ is what $\log_a$ is applied to.

6. EQUATIONS INVOLVING POWERS AND LOGARITHMS

We solve some examples similar to the assignments.

1. Solve $5^x = 10$. We need to apply $\log_5$ to both sides in order to get hold of $x$:

   $$\log_5 5^x = x = \log_5 10 = \frac{\log 10}{\log 5} = \frac{1}{\log 5} = 1.43$$

2. Solve $3 e^{-2x} = 8$. First divide by 3 in order to isolate the power that contains the unknown $x$ as exponent. This yields

   $$e^{-2x} = \frac{8}{3}$$

   Now apply $\log_e$ to both sides:

   $$-2x = \log_e \frac{8}{3}$$

   or, after dividing by $-2$

   $$x = \frac{1}{2} \log_e \frac{8}{3} = -\frac{1}{2} \ln \frac{8}{3}.$$ 

   Notice that $\log_e$ can be evaluated using the ln-key of your calculator. ln is synonymous to $\log_e$. Recall that $e$ is a concrete number approximately equal to 2.718. Another way to compute the right hand side using the log(= log_{10})-key would be

   $$x = -\frac{1}{2} \log \frac{8}{3} = -\frac{\log \frac{8}{3}}{2 \log e} = \frac{\log 3 - \log 8}{2 \log e}.$$