Question 1. Find the following indefinite integrals

(a) \( \int (6x^2 - 8x + \frac{3}{x}) \, dx, \, x \neq 0; \)

Solution.

\[
\int (6x^2 - 8x + \frac{3}{x}) \, dx = \int (6x^2) \, dx - \int (8x) \, dx + \int \frac{3}{x} \, dx =
\]

\[
6 \int x^2 \, dx - 8 \int x \, dx + 3 \int \frac{1}{x} \, dx.
\]

We use the formulae \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \) for \( n \neq -1 \), and \( \int \frac{1}{x} \, dx = \ln |x| + C \) for \( n = -1 \).

\[
6 \int x^2 \, dx - 8 \int x \, dx + 3 \int \frac{1}{x} \, dx = 6 \cdot \frac{x^{2+1}}{2+1} - 8 \cdot \frac{x^{1+1}}{1+1} + 3 \ln |x| + C =
\]

\[
= 2x^3 - 4x^2 + 3 \ln |x| + C.
\]

(b) \( \int 3\sqrt{x} \, dx, \, x \geq 0. \)

Solution.

\[
\int 3\sqrt{x} \, dx = 3 \int x^{\frac{1}{2}} \, dx,
\]

because \( \sqrt{x} = x^{\frac{1}{2}}. \) Use \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + C \) for \( n = \frac{1}{2} : \)

\[
3 \int x^{\frac{1}{2}} \, dx = 3 \cdot \frac{x^{1+1}}{1+1} + C = 3 \cdot \frac{x^2}{3/2} + C = 2x^{\frac{3}{2}} + C.
\]

Question 2. Find the following definite integrals

(a) \( \int_0^\frac{\pi}{2} \sin(2x) \, dx; \)

Solution. We use that \( \int \sin(nx) \, dx = -\frac{\cos(nx)}{n} + C. \) Hence

\[
\int_0^{\frac{\pi}{2}} \sin(2x) \, dx = \left[ -\frac{1}{2} \cos(2x) \right]_0^{\frac{\pi}{2}} = -\frac{1}{2} \cos(2 \times \frac{\pi}{2}) - \left( -\frac{1}{2} \cos(2 \times 0) \right) =
\]
\[-\frac{1}{2} \cos \pi + \frac{1}{2} \cos 0 = -\frac{1}{2} \times (-1) + \frac{1}{2} \times 1 = 1. \]

(b) \[\int_0^1 e^{-2x} \, dx.\]

**Solution.** Use that \[\int e^{nx} \, dx = \frac{e^{nx}}{n} + C.\] For \(n = -2\) we have

\[
\int_0^1 e^{-2x} \, dx = \left[\frac{e^{-2x}}{-2}\right]_0^1 = -\frac{e^{-2\times1}}{2} - \left(-\frac{e^{-2\times0}}{2}\right) = \frac{1 - e^{-2}}{2},
\]

because \(e^0 = 1.\)

**Question 3.** The vertical speed of a rocket \(t\) seconds after lift–off is \(t + 6t^2\) meters per second. How high is the rocket 2 minutes after lift–off?

**Solution.** First we notice that 2 minutes is 120 seconds.

\[h(t) = \int_0^{120} t + 6t^2 \, dt = \left[\frac{t^2}{2} + 6 \times \frac{t^3}{3}\right]_0^{120} = \left[\frac{t^2}{2} + 2t^3\right]_0^{120} = 3,463,200 \text{ m} = 3,463 \text{ km} 200 \text{ m}.

**Question 4.** Find the area between the graphs of the functions given by \(y = x^2\) and \(y = x + 2.\)

**Solutions.** We find the intersection points of these two curves: \(x^2 = x + 2.\)

\[x^2 - x - 2 = 0, \quad x_{1,2} = \frac{1 \pm \sqrt{1 - (-4 \times 2)}}{2} = \frac{1 \pm 3}{2} = 2, -1.\] The function \(y = x + 2\) is larger than \(y = x^2\) on the interval \((-1, 2).\) (See the picture below.)
The area between the curves is

\[
\int_{-1}^{2} (x + 2) - x^2 \, dx = \left[ \frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \\
\frac{2^2}{2} + 2 \times 2 - \frac{2^3}{3} - \left( -\frac{(-1)^2}{2} + 2 \times (-1) - \frac{(-1)^3}{3} \right) = 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3} = 4 \frac{1}{2}.
\]

**Question 5.** Find the area enclosed by the curve given by \( y = x^3 - x^2 \) and the \( x \)-axis.

**Solutions.** We find the intersection points of the curve \( y = x^3 - x^2 \) and the \( x \)-axis, which is given by the equation \( y = 0 \) solving \( x^3 - x^2 = 0 \). Then \( x^2(x - 1) = 0 \), which has two solutions, \( x_1 = 0 \) and \( x_2 = 1 \). The function \( y = x^3 - x^2 \) is smaller than zero on the interval \((0, 1)\) (see the sketch below),

hence

\[
A = \int_{0}^{1} 0 - (x^3 - x^2) \, dx = \left[ -\frac{x^4}{4} + \frac{x^3}{3} \right]_{0}^{1} = -\frac{1}{4} + \frac{1}{3} = \frac{1}{12}.
\]

**Question 6.** Find the average of the function \( f(t) = \sin t \) for \( \pi < t < 2\pi \).

**Solution.** We find the average of the function for the interval \((a,b)\) by the formula \( \frac{\int_{a}^{b} f(x) \, dx}{b - a} \). For \( f(t) = \sin t \), \( a = \pi \) and \( b = 2\pi \) we get

\[
\frac{\int_{\pi}^{2\pi} \sin t \, dt}{2\pi - \pi} = \frac{1}{\pi} \left[ -\cos t \right]_{\pi}^{2\pi} = -\frac{1}{\pi} (1 - (-1)) = -\frac{2}{\pi}.
\]