TUTORIAL 4 - Sample Solutions

Question 1. Find the derivatives of the functions given by

(a) \( y = 3x^4; \)

Solution. Use that \((Cf(x))' = Cf'(x)\) for any constant \(C:\)

\[ y' = (3x^4)' = 3x^4'. \]

Then use the rule \((x^n)' = nx^{n-1}:\)

\[ y' = 3x^4' = 3 \times 4x^3 = 12x^3. \]

(b) \( y = e^{101x} + 0.1; \)

Solution. The derivative of a sum of two functions is equal to sum of the
derivatives of these functions:

\[ y' = (e^{101x} + 0.1)' = (e^{101x})' + (0.1)'. \]

Now we use the formula \((e^{ax})' = ae^{ax}\) and get \((e^{101x})' = 101e^{101x}.\) The derivative
of a constant is equal to zero: \((0.1)' = 0.\) We get

\[ y' = (e^{101x})' + (0.1)' = 101e^{101x} + 0 = 101e^{101x}. \]

(c) \( y = x \cos x; \)

Solution. We use the product rule:

\[ y' = (x \cos x)' = x' \cos x + x(\cos x)' = \cos x - x \sin x. \]

(d) \( y = 5\frac{\sin x}{x}, \) \( x \neq 0; \)

Solution. Use the quotient rule:

\[ y' = \left(5\frac{\sin x}{x}\right)' = 5\frac{(\sin x)' \times x - \sin x \times (x)'}{x^2}. \]

As \( x' = 1 \) and \((\sin x)' = \cos x,\)

\[ y' = 5 \frac{x \cos x - \sin x}{x^2}. \]
(e) \(y = \ln(2 - x), \quad x > 2;\)

**Solution.** Use the chain rule:

\[ y' = (\ln(2 - x))' = \frac{1}{2-x} \times (2-x)', \]

as \((\ln x)' = \frac{1}{x} \). Now, \((2-x)' = 2' - (x)' = 0 - 1 = -1\), and

\[ y' = \frac{-1}{2-x} = \frac{1}{x-2}. \]

(f) \(y = \cos(x^4);\)

**Solution.** Use the chain rule:

\[ y' = (\cos(x^4))' = -\sin(x^4) \times (x^4)', \]

As \((x^4)' = 4x^3\),

\[ y' = -4x^3 \sin(x^4). \]

(e) \(y = (\cos x)^4.\)

**Solution.** Use the chain rule:

\[ y' = ((\cos x)^4)' = 4(\cos x)^3 \times (\cos x)', \]

As \((\cos x)' = -\sin x,\)

\[ y' = 4(\cos x)^3 \times (-\sin x) = -4 \sin x (\cos x)^3. \]

**Question 2.** Find the third derivative of the function \(f\) given by

\[ f(x) = 2x^6 - 4x^3 + 3x - 10x^{-2}. \]

**Solution.** At each step we use the formula \((x^n)' = nx^{n-1}.\)

\[ f'(x) = 2 \times 6x^{6-1} - 4 \times 3x^{3-1} + 3 - 10 \times (-2)x^{-2-1} = 12x^5 - 12x^2 + 3 + 20x^{-3}, \]

\[ f''(x) = (f'(x))' = 60x^4 - 24x + 0 + 20 \times (-3)x^{-3-1} = 60x^4 - 24x - 60x^{-4}, \]

\[ f'''(x) = (f''(x))' = 240x^3 - 24 + 240x^{-5}. \]
Question 3. Find an equation of the tangent to the curve given by

\[ y = x^2 + 4x + 4 \]

at the point where \( x = 1 \).

Solution. The slope of the tangent line is equal to the value of the derivative \( y' \) at the point \( x = 1 \). First find the derivative: \( y' = 2x + 4 \). Substituting \( x = 1 \) we find the slope: \( y'(1) = 2 \times 1 + 4 = 6 \).

The tangent passes the point with the \( x \)-coordinate \( x = 1 \). We find its \( y \)-coordinate from the equation \( y = x^2 + 4x + 4 \): \( y(1) = 1^2 + 4 \times 1 + 4 = 9 \).

The equation of the tangent is a linear equation \( y = 6x + b \) and passes through the point \((1, 9)\). Substituting the coordinates of this point in the linear equation, we find \( b \): \( 9 = 6 \times 1 + b \), then \( b = 9 - 6 = 3 \).

The equation of the tangent to the given curve at the point where \( x = 1 \) is \( y = 6x + 3 \).

Question 4. We want to build a closed rectangular box with square base. We can pay for 6 square meters of material. What dimensions will result in a box with the largest possible volume?

Solution. If the size of the square base is \( x \times x \), and the height of the box is \( h \), then its volume is

\[ V = x^2h, \]

and the surface area is

\[ A = 2x^2 + 4xh. \]

For \( A = 6 \) we get \( 2x^2 + 4xh = 6 \). Solve this equation for \( h \):
\[ 4xh = 6 - 2x^2, \text{ i.e. } h = \frac{6 - 2x^2}{4x} = \frac{3 - x^2}{2x}. \]

Substitute this \( h \) in the equation for the volume and get

\[ V = x^2 \times \frac{3 - x^2}{2x} = \frac{1}{2}(3x - x^3). \]

Now we find for which \( x \) the function has its maximum.

First find the stationary points, i.e. the points where the derivative \( V'(x) \) is zero.

\[ V'(x) = \left( \frac{1}{2}(3x - x^3) \right)' = \frac{1}{2}(3 - 3x^2) = \frac{3}{2}(1 - x^2) = \frac{3}{2}(1 - x)(1 + x) = 0 \]

at \( x = 1 \) and \( x = -1 \). As a length, \( x \) can not be negative. Use the second derivative test to find out if the volume has maximum or minimum at \( x = 1 \).

\[ V''(x) = \left( \frac{3}{2}(1 - x^2) \right)' = \frac{3}{2}(-2x) = -3x. \]

\[ V''(1) = -3 \times 1 = -3 < 0, \]

i.e. the function \( V(x) \) takes its maximum value at \( x = 1 \). The height of the box \( h = \frac{3 - x^2}{2x} \) at \( x = 1 \) is \( h = \frac{3 - 1^2}{2 \times 1} = 1. \)

The dimensions of the box with the largest possible volume are \( 1m \times 1m \times 1m. \)

**Question 5.** Find and classify the stationary points of the functions \( f \) given by

(a) \( f(x) = x^4 + 2x^3 + 4; \)

**Solution.** Find the stationary points, i.e. the points where the derivative \( f'(x) \) is zero.

\[ f'(x) = 4x^3 + 6x^2 = 2x^2(2x + 3) = 0 \]

at \( x = 0 \) and \( x = -\frac{3}{2} \). We use the first-derivative test to classify the stationary points. See the picture below.

On the interval \((-\infty, -\frac{3}{2})\), the first derivative \( f'(x) \) is negative, i.e. \( f(x) \) is decreasing on this interval. We find that \( f'(x) > 0 \) on the interval \((-\frac{3}{2}, 0)\), i.e. \( f(x) \) is increasing on that interval. It means that \( f(x) \) has a minimum at \( x = -\frac{3}{2}. \)

For positive \( x \) we get \( f'(x) > 0 \), i.e. \( f'(x) \) stays increasing and \( f(x) \) has an inflexion point at \( x = 0. \)
(b) $f(x) = xe^x$;

Solution.

$$f'(x) = (xe^x)' = e^x + xe^x = e^x(1 + x) = 0$$

only at $x = -1$, as $e^x$ is positive for all real $x$.

We use the first derivative test. The first derivative is negative for $x < -1$ and positive for $x > -1$, i.e. the function $f$ decreases on the interval $x < -1$ and increases on the interval $x > -1$. Hence, $f(x)$ has its minimum at $x = -1$.

(c) $f(x) = \frac{1}{5} (\cos x)^2$.

Solution.

$$f'(x) = \frac{1}{5} \times 2 \cos x (\cos x)' = -\frac{2}{5} \cos x \sin x = 0$$

if $\cos x = 0$ or $\sin x = 0$.

- $\cos x = 0$ at $x = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$
- $\sin x = 0$ at $x = \pi k$, $k \in \mathbb{Z}$.

The union of these two sets is the set $x = \frac{\pi}{2} + \frac{\pi}{2} k$, $k \in \mathbb{Z}$. See the picture below.

For the classification of these stationary points we use the first-derivative test:

On the interval $\left(0, \frac{\pi}{2}\right)$ $\cos x > 0$ and $\sin x > 0$, hence $f'(x) = -\frac{2}{5} \cos x \sin x < 0$,

On the interval $\left(\frac{\pi}{2}, \pi\right)$ $\cos x < 0$ and $\sin x > 0$, hence $f'(x) > 0$,

On the interval $\left(\pi, \frac{3\pi}{2}\right)$ $\cos x < 0$ and $\sin x < 0$, hence $f'(x) < 0$,

On the interval $\left(\frac{3\pi}{2}, 2\pi\right)$ $\cos x > 0$ and $\sin x > 0$, hence $f'(x) > 0$.

This pattern repeats as $\sin x$ and $\cos x$ are periodic functions with the period $2\pi$.

We conclude that the function $f(x) = \frac{1}{5} (\cos x)^2$ has its maxima at $x = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$ and minima at $x = \pi k$, $k \in \mathbb{Z}$.

**Question 6.** Find the absolute maximum and the minimum of the function

$$f(x) = -(2x - 1)^2 + 100$$

on the interval $0 \leq x \leq 5$.

To find the absolute minimum and maximum on the interval, we

1. find the stationary points on the given interval,
2. evaluate the values of the function $f(x)$ at these stationary points and at the end points of the interval,
3. the smallest of these values is the absolute minimum, the largest of these values is the absolute maximum.

1. We use chain rule to differentiate the first term:

$$f'(x) = -2(2x - 1)^{2-1} \times (2x - 1)' + 0 = -4(2x - 1).$$

The stationary point occurs if $f'(x) = -4(2x - 1) = 0$ at $x = \frac{1}{2}$. The second derivative $f''(x) = -8 < 0$. It follows that the stationary point is a local maximum and $f\left(\frac{1}{2}\right) = -(2 \times \frac{1}{2} - 1)^2 + 100 = 100$.

2. The values of the function at the end points of the interval $0 \leq x \leq 5$ are:

$$f(0) = -(2 \times 0 - 1)^2 + 100 = 99, \text{ and } f(5) = -(2 \times 5 - 1)^2 + 100 = 19.$$

3. The maximum of the function $f(x)$ on the given interval is 100 and occurs at $x = \frac{1}{2}$, and the minimum is 19 and occurs at $x = 5$. 

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Q. 5

(a) 

\[
\begin{array}{cccccc}
& f' < 0 & -3 & \frac{3}{2} & + & 0 & + \\
\downarrow & & f'' < 0 & & f'' > 0 & \\
& \text{minimum} & & & & \text{inflexion point}
\end{array}
\]

(b) 

\[
\begin{array}{cccccc}
& f' < 0 & -1 & \frac{1}{2} & + & f' > 0 & \\
\downarrow & & & f'' < 0 & & f'' > 0 & \\
& \text{minimum} & & & & \\
\end{array}
\]

(c) 

\[
\begin{array}{cccccc}
& \frac{3\pi}{2} & \frac{3\pi}{4} & \frac{\pi}{2} & 0 & \frac{\pi}{2} & \frac{3\pi}{2} \\
\downarrow & & + & - & + & - & + & - & + & - & \ldots
\end{array}
\]

... max min max min max ...
Question 6. Find the absolute maximum and the minimum of the function
\[ f(x) = -(2x - 1)^2 + 100 \]
on the interval 0 \leq x \leq 5.

To find the absolute minimum and maximum on the interval, we
1. find the stationary points on the given interval,
2. evaluate the values of the function \( f(x) \) at these stationary points and at the end points of the interval,
3. the smallest of these values is the absolute minimum, the largest of these values is the absolute maximum.

1. We use the chain rule to differentiate the first term:
\[
    f'(x) = -2(2x - 1)^{2-1} \times (2x - 1)' + 0 = -4(2x - 1).
\]
The stationary point occurs if \( f'(x) = -4(2x - 1) = 0 \) at \( x = \frac{1}{2} \). The second derivative \( f''(x) = -8 < 0 \). It follows that the stationary point is a local maximum and \( f\left(\frac{1}{2}\right) = -(2 \times \frac{1}{2} - 1)^2 + 100 = 100 \).

2. The values of the function at the end points of the interval 0 \leq x \leq 5 are: \( f(0) = -(2 \times 0 - 1)^2 + 100 = 99 \), and \( f(5) = -(2 \times 5 - 1)^2 + 100 = 19 \).

3. The maximum of the function \( f(x) \) on the given interval is 100 and occurs at \( x = \frac{1}{2} \), and the minimum is 19 and occurs at \( x = 5 \).