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Chapter 1

Sets and Functions

1.1 Sets

The terminology of sets is used in mathematics in order to collect objects of a similar kind. The objects are called the \textbf{elements} of the set and the set is often described by listing its elements enclosed in curly brackets. For instance, the set of odd numbers between 0 and 10 can be listed as

$$\{1, 3, 5, 7, 9\}.$$ 

Most of the set we will consider are sets of numbers, but we may also use sets of geometrical points (on a line, a plane or in space) or other sets. Sets are usually denoted by a capital letter such as $A, B, C, \ldots$. The elements are usually denoted by lower case letters $a, b, c, \ldots$. The statement that $a$ is an element of the set $A$ is written as $a \in A$.

Some sets are so important that they have reserved letter symbols:

1. The \textbf{empty set}, that has no elements is denoted by $\emptyset$.
2. The set of counting numbers $\{1, 2, 3, \ldots\}$ is denoted by $\mathbb{N}$.
3. The set of integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is denoted by $\mathbb{Z}$.
4. The set of rational numbers (or fractions) is denoted by $\mathbb{Q}$.
5. The set of real numbers is denoted by $\mathbb{R}$. We will often represent a real number as a point on the number line. Recall that the number line is a straight line with marks for 0 a scale and a (positive) orientation. Then any positive real number corresponds to the point at a distance of its magnitude from 0 in the positive direction and a negative real number corresponds to the point at a distance of its magnitude from 0 in the negative direction. (See Figure 1.1 below.) Notice that there is only one arrow indicating the positive direction.
Having in mind this correspondence between numbers and points on the number line we will often refer to numbers as points.

A set can have finitely or infinitely many elements. The number sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are infinite sets. They cannot be described by listing their elements but we have to specify a rule for the elements. Sometimes we use dots to indicate a obvious rule: We described $\mathbb{N} = \{1, 2, 3, \ldots \}$ assuming that the set contains all numbers we get by consecutively adding 1, and $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots \}$ assuming that the set contains all numbers we get by consecutively adding or subtracting 1.

We can describe $\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}$. This means that the set of all rational numbers consists of fractions of the form $\frac{p}{q}$ where $p$ is an integer and $q$ is a counting number.

We describe the set of even integers as $\{2n : n \in \mathbb{Z}\}$

that is as numbers of the form $2n$ (that is 2 times $n$) where $n$ is any integer. We will use such notation later to describe the set of solutions of trigonometric equations.

We say that a set $A$ is a subset of a set $B$ and write $A \subseteq B$ if each element of $A$ is also an element of $B$.

**Example.** $A = \{2, 3, 5\}$ is a subset of $B = \{0, 1, 2, 3, 4, 5\}$ because 2, 3, 5 are elements of $B$.

The empty set is a subset of any set and any set is a subset of itself.

In this unit we will use special subsets of $\mathbb{R}$ called intervals. The interval $[a, b] = \{x : a \leq x \leq b\}$ is the set of all real numbers that are bigger or equal to a given number $a$ and smaller or equal to another given number $b$. Notice that the ends $a, b$ are elements of the set here. If the ends $a$ and $b$ are excluded from the set we write $(a, b) = \{x : a < x < b\}$

The notations like $(-\infty, b]$ and $(a, \infty)$ are often used as shorthands for the sets $\{x : x \leq b\}$ and $\{x : x > a\}$ respectively.

Finally, we can define new sets as the intersection or the union of given sets: The union $A \cup B$ of two sets $A$ and $B$ is the set whose elements are elements of $A$ or elements of $B$.

**Example.** 1. If $A = \{2, 3, 4, 5\}$ and $B = \{4, 5, 6\}$ then $A \cup B = \{2, 3, 4, 5, 6\}$.

2. Let $X = [\frac{3}{2}, \frac{10}{3}]$ and $Y = [3, 9]$. Then $X \cup Y = [\frac{3}{2}, 9]$.

\[1\text{In some texts reversed excluding brackets (e.g. ]a,b[) are used instead of the round parentheses for} \]
The **intersection** $A \cap B$ is defined as the set whose elements are simultaneously in $A$ and $B$.

**Example.** With $A, B, X, Y$ as above, the intersections are $A \cap B = \{4, 5\}$, $X \cap Y = [3, \frac{10}{3}]$.

1.2 Functions

Functions are the main subject of calculus. They model the dependence between different quantities. Sometimes it happens that we need to know a certain quantity that is not accessible through direct measuring or counting, but we know another related quantity. We may then try to express the desired quantity through the known one by a formula.

**Example.** You may want to know the number $N$ of a certain kind of screws contained in some box. You know the mass of one screw is 15.34 (gram). Now, rather than counting the screws you can determine the mass $M$ (in grams) and compute the number by the formula

$$N = \frac{M}{15.34}.$$  

In this example $M$ is called the independent variable and $N$ the dependent variable (it depends on $M$). Mostly we denote the independent variable by $x$ and the dependent intervals without the ends. This is to avoid confusion with the pairs of coordinates, which are also written in parentheses.
variable by \( y \). But sometimes, as in the example above, we use different letters that hint at the natural meaning of the variables.

Though this formula above makes sense for any numbers \( M \) and \( N \), the problem indicates that \( M \) is a reading of a scale with a certain range and precision and \( N \) is a non-negative integer.

More formally, a **function** is a rule that to each number from a given set, called the *domain*, assigns another number from another (possibly different) set, called the *codomain*. You may think of a function as “black box” that processes some numerical input into some numerical output. Usually the input variable is denoted by “\( x \)” and called the *independent variable* or the *argument* of the function and the output variable is denoted by “\( y \)” and called the *dependent variable* or the *value* of the function. The function itself is denoted by letters like \( f, g, h, u, v \) etc. The set of all possible inputs is called the *domain* (denoted by upper case \( X \)) and the set of all possible outputs is called the *codomain* of the function (denoted by upper case \( Y \)). In most applications the codomain of a function does not matter. Unless we explicitly specify it we take the set of all real numbers as the standard codomain of all functions in this unit.

The set of actually attained values is called the *range* of the function (denoted by upper case \( Y \)). The range sometimes coincides with the codomain but sometimes is a smaller subset. We write

\[
y = f(x) \quad \text{or} \quad f: x \mapsto y \quad \text{or} \quad x \mapsto y
\]

to indicate that \( y \) is a function of \( x \). We write

\[
f: X \rightarrow Y \quad \text{or} \quad X \xrightarrow{f} Y
\]

to indicate that \( f \) is a function with domain \( X \) and codomain \( Y \). (Notice the difference of the two arrows used.)

The “rule” of the function \( f \) can be given by an algebraic formula, such as \( y = f(x) = x^2 \), or a table or a verbal or other description. Functions often occur from measurement data. The following table displays temperature measurements \( y \) in dependence of time \( x \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>14</td>
<td>15</td>
<td>14</td>
<td>25</td>
<td>28</td>
</tr>
</tbody>
</table>

Here the domain \( X = \{8, 9, 10, 11, 12\} \), the codomain are all possible thermometer readings, say \( Y = \{-20, -19, \ldots, 48, 49, 50\} \), while the range is \( R = \{14, 15, 25, 28\} \).

In this unit the rules of the functions are usually given by algebraic formulae. Instead of dealing with a possibly large array of data we can deal with a single formula, which models the data.

Typical domains of such functions are intervals \([a, b]\) or \((a, b]\). If the domain of a function is not specified and the function is given by a formula we usually assume that the domain
1.2. FUNCTIONS

is the set of all real numbers for which the formula makes sense and call this the natural domain.

**Example.** For \( y = f(x) = \frac{1}{x+1} \), the natural domain \( X \) is the set of all real numbers except \(-1\) because the fraction is well-defined unless the denominator is 0, which occurs exactly if \( x = -1 \).

In the following example the function is given by a formula that involves the letter “\( x \)”.

\[
y = f(x) = x^2 + 2x - 1.
\]

For any real number we substitute for \( x \) we can compute the corresponding value \( y \). The table below displays 6 pairs of \((x, y)\) computed by the rule \( f \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-3)</th>
<th>(-2)</th>
<th>(-1)</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>2</td>
<td>(-1)</td>
<td>(-2)</td>
<td>(-1)</td>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

Since the formula makes sense of any real number \( x \) the natural domain is the set of all real numbers \( \mathbb{R} \). We can take \( \mathbb{R} \) as codomain. We will determine the range of this function later in the section on quadratic functions.

**Basic Principle:** *For each number chosen in place of \( x \), the function \( f \) produces precisely one number in place of \( y \).*

This is clear in the example above - you can’t get two different values for \( y \) from the same value of \( x \). But notice that two different choices of \( x \) can sometimes give the same value for \( y \). For example, \( x = 1 \) and \( x = -3 \) both give the value \( y = 2 \) for the function above.

The pairs of numbers \((x, y = f(x))\) for a given function \( f \) can be interpreted as the coordinates of points in the plane with respect to a “coordinate system”. The horizontal axis usually represents the independent variable \( x \) and therefore is called the “\( x \)-axis”. The vertical axis usually represents the dependent variable \( y \) and is called the “\( y \)-axis”.

The set of all such points is referred to as the graph of \( y = f(x) \). Many properties of functions can be read directly from the graph and it is always a good idea to start with a plot when we deal with a function. E.g., one can see if the function is constant, negative, positive, increasing or decreasing.

On the other hand, a function can be given by its graph. Think of a plot produced by a seismograph ([http://en.wikipedia.org/wiki/File:Kinemetrics_seismograph.jpg](http://en.wikipedia.org/wiki/File:Kinemetrics_seismograph.jpg)).

The graph of \( y = x^2 + 2x - 1 \) is plotted in Figure 1.4 Here we have restricted the domain to the interval \([-3, 2]\).
1.3 Composition of functions, inverse functions and solving equations

Let $f: V \rightarrow Y$ and $g: X \rightarrow W$ be two functions such that the range of $g$ is a subset of the domain of $f$. Then we can define the composition of $f$ and $g$ by first applying $g$ to an input $x$ from $X$ and then applying $f$ to the output $g(x)$. This is well-defined because $g(x)$ is by our assumption in the domain $V$ of $f$.

$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x)).$$

**Example.** Let $w = g(x)$ be the temperature function given by the table

<table>
<thead>
<tr>
<th>$x$</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w$</td>
<td>14</td>
<td>15</td>
<td>14</td>
<td>25</td>
<td>28</td>
</tr>
</tbody>
</table>

and $y = f(v)$ be the function of volume in dependence of temperature given by the table

<table>
<thead>
<tr>
<th>$v$</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>23</th>
<th>24</th>
<th>25</th>
<th>26</th>
<th>27</th>
<th>28</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>9.1</td>
<td>9.2</td>
<td>9.3</td>
<td>9.4</td>
<td>9.5</td>
<td>9.6</td>
<td>9.7</td>
<td>9.8</td>
<td>9.9</td>
<td>10</td>
<td>10.1</td>
<td>10.2</td>
<td>10.3</td>
<td>10.4</td>
<td>10.6</td>
<td>10.6</td>
</tr>
</tbody>
</table>

Then the composite $f(g(x))$ is well-defined because each output from the first table occurs as input in the second table. The composite function expresses volume in dependence of
time and can be given by the table

<table>
<thead>
<tr>
<th>x</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>w</td>
<td>14</td>
<td>15</td>
<td>14</td>
<td>25</td>
<td>28</td>
</tr>
<tr>
<td>y</td>
<td>9.2</td>
<td>9.3</td>
<td>9.2</td>
<td>10.3</td>
<td>10.6</td>
</tr>
</tbody>
</table>

If \( f \) was given by another table

<table>
<thead>
<tr>
<th>v</th>
<th>14</th>
<th>18</th>
<th>22</th>
<th>26</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>9.2</td>
<td>9.6</td>
<td>10</td>
<td>10.4</td>
<td>10.8</td>
</tr>
</tbody>
</table>

we could not compose \( f \) and \( g \) because \( g(9) = 15 \) is not in the domain (i.e. the upper line of the table) of that function \( f \) and therefore \( f(g(9)) \) cannot be defined.

For some applications we need to determine the input of a function \( f \) for a given output. To stay with our previous example, we might need to determine the time when a certain temperature occurred. There are clearly two possible obstructions.

1. The number \( y \) might not be in the range of \( f \), e.g. \( y \) might not occur in the lower line of the table. Then there is no input \( x \) corresponding to \( y \). In our example of a temperature function the temperature 17 did not occur. there is no input time \( x \) for \( y = 17 \).
2. There might be more than one input that correspond to the same output \( y \). In our temperature example two times \( x = 8 \) and \( x = 10 \) correspond to the same temperature \( w = 14 \).

The concept of inverse function solves the problem of finding the inputs for given outputs. Such inverse function can only exist if the two obstructions explained above do not occur. We call a function \( f \) surjective or onto if the range of \( f \) coincides with the codomain, i.e. obstruction 1 is ruled out. We call a function \( f \) injective or one-to-one if different inputs produce different outputs, i.e. obstruction 2 is ruled out. Functions that are at the same time injective and surjective are called bijective and these are exactly the functions that possess an inverse. The inverse of \( f \) is usually denoted by \( f^{-1} \) and its domain equals to the codomain (=range) of \( f \) and its codomain (=range) equals to the domain of \( f \).

If a function is given by a table then its inverse can be obtained by swapping the upper and lower lines of the table (if the inverse exists.)

If a function \( f \) is given by a formula \( y = f(x) \) we would like to have a formula that expresses \( x \) in terms of \( y \). Sometimes this can be done by solving the ‘equation’ \( y = f(x) \) for \( x \). Sometimes we need to invent a new formula for the inverse function (if it exists.) To obey the convention that \( x \) is the input and \( y \) is the input we swap \( x \) and \( y \). We will discuss this in detail for the elementary functions in the next Chapter.

Inverse functions are very useful for solving equations. Usually an equation is given in the form

\[ f(x) = 0 \]
where \( f \) is a function described by a formula and we need to determine all numbers \( x \) such that the equation \( f(x) = 0 \) is satisfied. Such \( x \) is called a solution and we may combine all solutions into a set of solutions. The set of solutions can consist of one or more elements or it can be empty.

**Examples.** The equation \( 2x - 4 = 0 \) for \( f(x) = 2x - 4 \) has exactly one solution \( x = 2 \). The set of solutions is \( \{2\} \).
The equation \( x^2 - 4 = 0 \) has exactly two solutions, namely \( x = 2 \) and \( x = -2 \). The set of solutions is \( \{-2, 2\} \).
The equation \( x - x + 1 = 0 \) has no solutions. No matter what \( x \) we plug in we get a wrong statement \( 1 = 0 \). The set of solutions is \( \emptyset \).
The equation \( x^2 - (x + 1)(x - 1) - 1 = 0 \) has infinitely many solutions. In fact, it is satisfied by any real number. The solution set is \( \mathbb{R} \).

If we consider an equation \( f(x) = 0 \) (or \( f(x) = a \)) and we know that \( f \) has an inverse function \( f^{-1} \) then we can immediately conclude that \( f(x) = a \) has exactly one solution \( x = f^{-1}(a) \) if \( a \) is in the range of \( f \) and no solution if \( a \) is not in the range of \( f \).
Chapter 2

Elementary Functions

2.1 Linear Functions

Linear functions are the simplest functions. They model processes with constant growth or decay.

Example. 1. Johnny has 2350$ in his savings account and he is able to save 200$ each month. After \( x \) months he will have \( y = 200x + 2350 \) dollars in his saving account.
2. Alice drives with a constant speed of \( 120 \, \text{km/h} \). She has just passed a sign stating 63\( \text{km} \) to Armidale. Then the function \( y = -120x + 63 \) expresses the distance to Armidale in dependence of time in hours (assuming that she does not get caught for speeding).

The general formula of a linear function has the form

\[
y = mx + b,
\]

where \( m \) and \( b \) are constant parameters. After the model is chosen the parameters \( m, b \) cannot be changed, while the independent and dependent variables \( x \) and \( y \) may assume different values in the domain and range respectively.

We show below that the graphs of linear functions are straight lines, whence the name. The parameters \( m \) and \( b \) determine the position and slope of the straight line.

Since a straight line is completely determined by any two of its points we can draw the graph of a linear equation by joining two of its points (and perhaps checking for a third point to make sure our computations were correct.)

For example if

\[
y = 2x - 1
\]

then for \( x = 0 \) we compute \( y = 2 \cdot 0 - 1 = -1 \). The argument \( x = 0 \) is a good choice for easy computation and it always yields the \textbf{y-intercept} \( b \) as the corresponding \( y \)-value.
For a second point we may choose $x = 1$ (again for easy computation) and find the corresponding $y$-value $y = 2 \cdot 1 - 1 = 1$. Now join the points together to get a graph of $y$ as a function of $x$ as in Figure 2.1 and check if a third point, e.g. $x = -1$, $y = -3$ is on the line.

Figure 2.1: Graph of $y = 2x - 1$

The **slope** of a straight line is the ratio of the change in $y$ values to the change in $x$ values for any points lying on the line:

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}.$$  

We use here the greek upper case Delta (for difference) to denote the change in the respective variables. This is a common notation which we will see again in differential calculus.

If $(x_1, y_1)$ and $(x_2, y_2)$ are two points on the line, then

$$\Delta y = y_2 - y_1$$

and

$$\Delta x = x_2 - x_1$$

so

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$
Since \( y_1 = mx_1 + b \) and \( y_2 = mx_2 + b \)

\[
\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m.
\]

So the slope of the straight line \( y = mx + b \) is \( m \).

The geometric meaning of the slope is the following: Consider the right triangle with vertices \((x_1, y_1), (x_2, y_2), (x_2, y_1)\). The slope is the ratio of the length of the vertical, opposite leg and the length of the horizontal adjacent leg, with a positive sign if the line slopes up and a negative sign, if the line slopes down. The slope is the tan of the angle at the vertex \((x_1, y_1)\). The slope is always the same, no matter which points \((x_1, y_1), (x_2, y_2)\) we choose. The resulting triangles are similar and the ratio of the lengths of the legs and the angles do not change.

For example, \( y = 2x - 1 \) has slope 2. We can check this by taking a pair of points on the line, e.g., \((x_1, y_1) = (0, -1)\) and \((x_2, y_2) = (1, 1)\) and calculating

\[
\text{slope} = \frac{1 - (-1)}{1 - 0} = \frac{2}{1} = 2.
\]

If the slope of a line is positive it slopes from lower left to upper right, if the slope is negative it slopes from lower right to upper left. The greater the slope, the steeper the straight line appears. See Figure 2.2.

It is often useful to know where a straight lines crosses the \( x \) and \( y \) axes. Points on the \( x \) axis have \( y = 0 \) and points on the \( y \) axis have \( x = 0 \). The corresponding \( x \)- and \( y \)-coordinates are called \textit{x-intercept} and \textit{y-intercept}, respectively.

To find the \( x \)-intercept we set \( y = 0 \) in the equation \( y = mx + b \)

\[
0 = mx + b
\]

subtract \( b \) from both sides

\[
mx = -b
\]

and divide both sides by \( m \) (if \( m \neq 0 \))

\[
x = -\frac{b}{m}.
\]

So the line \( y = mx + b \) cuts the \( x \) axis where \( x = -b/m \). If \( m = 0 \) the line is parallel to the \( x \)-axis and either does not intersect or coincides with the \( x \)-axis.
CHAPTER 2. ELEMENTARY FUNCTIONS

To find the $y$-intercept set $x = 0$ in the equation of the line:

$$y = m \cdot 0 + b = b.$$ 

So the line $y = mx + b$ cuts the $y$-axis at the point $y = b$. For example, the line $y = 2x - 1$ cuts the $x$ axis at

$$x = -\frac{1}{2} = \frac{1}{2}$$

and the $y$ axis at

$$y = -1$$

(see Figure 2.1).

We have seen how to work out the slope and intercepts of a straight line from its equation. In general we can work out anything we want to know about a line from its equation. So if a straight line is described to us in some other way, for example its slope and $y$-intercept, it will often be useful to find its equation, as from that we can derive any other information needed. A common way lines occur in experimental science is as an approximate fit to a set of data (either a fit by eye or a computer generated fit) and if we have such a fit, plotted on graph paper for example, we can find some points on the line. An important point to remember is that if we know two points on the line we know the whole line uniquely, i.e., there is always only one distinct line that can pass through two given points in the plane.
Example

Suppose we know two points \((x_1, y_1)\) and \((x_2, y_2)\) on the line. Then

\[
m = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.
\]

To find \(b\) take either of the points \((x_1, y_1)\) or \((x_2, y_2)\) and put its coordinates into the equation of the line. For example,

\[
y_1 = mx_1 + b
\]

which gives

\[
b = -mx_1 + y_1
\]

from which we can find \(b\), since we know \(m\) already. Thus we proceed in two steps:

1. Use the two points to find the slope of the line.
2. Substitute one of the points into the equation for the line to find the \(y\)-intercept.

Example. Take the two points \((x_1, y_1) = (-1, -3)\) and \((x_2, y_2) = (1, 1)\) and compute the equation of the straight line through these points. The formula for the slope is

\[
m = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-3)}{1 - (-1)} = \frac{4}{2} = 2
\]

and substituting the first point into the equation for the line

\[
y_1 = mx_1 + b
\]

\[
-3 = 2(-1) + b,
\]

so \(b = -1\) and the equation of the line is

\[
y = 2x - 1.
\]

Another common situation is where we are given a point \((x_1, y_1)\) on the line and the slope \(m\) of the line. To find the equation of the line we use the fact that if \((x, y)\) is any point on the line then

\[
slope = m = \frac{y - y_1}{x - x_1}
\]

so

\[
m(x - x_1) = y - y_1
\]

or

\[
y = mx - mx_1 + y_1.
\]
The procedure here is to use the information to write down an equation for the slope of the line and to rearrange this equation into the standard form for the equation of a straight line.

**Example.** Find the equation of the line through the point \((2, -3)\) having slope \(-2\). The formula for the slope is
\[
slope = m = \frac{y - y_1}{x - x_1}
\]
and substituting values for \(m, x_1\) and \(y_1\) gives
\[
-2 = \frac{y - (-3)}{x - 2}
\]
from which it follows that
\[
-2(x - 2) = y + 3
\]
or
\[
y = -2x + 4 + (-3) = -2x + 1
\]
which is the equation of the line.

**Example.** A person on a strict diet plans to breakfast on cornflakes, milk, and a boiled egg. After allowing for the egg, his diet permits an additional 300 calories for this meal. If 1 ounce of milk contains 20 calories and 1 ounce of cornflakes (plus sugar) contains 160 calories, what is the relation between the number of ounces of milk and cornflakes that can be consumed?

Let \(x\) be the number of ounces of milk to be consumed, and \(y\) be the number of ounces of cornflakes. Then
\[
20x + 160y = 300,
\]
or, on dividing through by 160 and rearranging
\[
y = -\frac{1}{8}x + \frac{15}{8},
\]
which is the relation between the number of ounces of milk and cornflakes that can be consumed. Note that since we cannot have negative quantities consumed, we require that \(x \geq 0\) and \(y \geq 0\). The graph of \(y = -1/8x + 15/8\) is shown in Figure 2.3.

If \(x = 0\), then \(y = 15/8\); and if \(y = 0\), then \(x = 15\). Although the straight line extends indefinitely, we are only interested in points on the line between \((0, 15/8)\) and \((15, 0)\). Any point on the line between these two points gives a solution to our problem (e.g. if \(x = 5\), then \(y = 5/4\); so one possibility is to consume 5 ounces of milk and 1\(\frac{1}{4}\) ounces of cornflakes).
Inverse of linear functions and linear equations

Before we investigate injectivity of linear functions we introduce another important concept. A function $f$ is called **strictly increasing** on a subset $M$ of its domain if for two inputs $a, b$ from $M$ such that $a < b$ it follows that $f(a) < f(b)$.

A function $f$ is called **strictly decreasing** on a subset $M$ of its domain if for two inputs $a, b$ from $M$ such that $a < b$ it follows that $f(a) > f(b)$.

It is easy to detect strict increase and decrease of a function from its graph sloping up or down. Strictly increasing (or decreasing functions) are always injective because for two different inputs $a, b$ where $a < b$ we know that the outputs are $f(a) < f(b)$ (or $f(a) > f(b)$) hence they are different.

A linear function $y = mx + b$ is strictly increasing if $m > 0$, strictly decreasing if $m < 0$ and constant if $m = 0$. If $m \neq 0$ the inverse function exists:

1. Solve $y = mx + b$ for $x$.

   \[
   y - b = mx \\
   \frac{y - b}{m} = x \\
   x = \frac{1}{m}y - \frac{b}{m}.
   \]

2. Denote the new independent variable by $x$ and the dependent variable by $y$.

   \[
   y = \frac{1}{m}x - \frac{b}{m}.
   \]

The inverse function of a linear function with $m \neq 0$ is again a linear function with slope $\frac{1}{m}$ and $y$-intercept $-\frac{b}{m}$. Notice that the graph of the inverse function is a reflection of the original graph with respect to the bisector $y = x$. This is due to the fact that swapping the roles of $x$ and $y$ can be done by exactly this reflection.
By computing the inverse of a linear function we have solved an arbitrary linear equation

\[ mx + b = 0. \]

If \( m \neq 0 \) then there is exactly one solution \( x = -\frac{b}{m} \). If \( m = 0 \) and \( b \neq 0 \) there is no solution, because \( 0x + b = b = 0 \) can never be true. If \( m = 0 \) and \( b = 0 \) any real \( x \) is a solution, because \( 0x + 0 = 0 \) is always true.

If we want to find the point of intersection of two straight lines given by their equations we need to solve a system of two simultaneous equations. Suppose the two lines are given by their equations

\[
\begin{align*}
y &= mx + b \\
y &= nx + c
\end{align*}
\]

The coordinates \((x, y)\) of the point of intersection have to satisfy both equations. We can solve the simultaneous system by equating the two expressions for \( y \)

\[ mx + b = nx + c, \]

which is a single linear equation on one variable \( x \). We solve

\[
\begin{align*}
mx - nx &= c - b \\
(m - n)x &= c - b \\
x &= \frac{c - b}{m - n}
\end{align*}
\]

which give the \( x \)-coordinate if \( m \neq n \). If \( m = n \) the lines are parallel and either coincide (all of their points are common) or do not intersect (no solution). To find \( y \) we substitute \( x = \frac{c - b}{m - n} \) in either of the two line equations:

\[
y = m \frac{c - b}{m - n} + b = \frac{mc - nb}{m - n}. \]

**Example.** Compute the points of intersection of the lines \( y = 2x - 1 \) and \( y = x + 3 \). We equate the two expressions for \( y \) to get

\[
\begin{align*}
2x - 1 &= x + 3 \\
x &= 4 \\
y &= 4 + 3 = 7
\end{align*}
\]

We have used the second equation to determine \( y \) because it is simpler. We may check our solution by plugging it into the first equation: \( 7 = 2 \times 4 - 1 = 7 \), which is a correct statement.

Alternatively, we could have used the formulae from above:

\[
\begin{align*}
x &= \frac{c - b}{m - n} = \frac{3 - (-1)}{2 - 1} = \frac{4}{1} = 4, \\
y &= \frac{mc - nb}{m - n} = \frac{2 \times 3 - 1 \times (-1)}{2 - 1} = \frac{7}{1} = 7.
\end{align*}
\]
2.2 The absolute value function

A simple function but very important function is the absolute value function that assigns
a non-negative number to itself and a negative number to its opposite:

\[ f(x) = |x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
\end{cases} \]

This is a function that is not given by a single formula but by one formula, namely \( f(x) = -x \), for \( x < 0 \) and another formula, namely \( f(x) = x \), for \( x \geq 0 \). The graph of the absolute
value function is shown in Figure 2.4 below.

![Graph of y = |x|](image)

To solve equations or inequalities with \( |x| \) we have to take into account the two options.
E.g. the equation

\[ |x| = 4 \]

has two solutions \( x = 4 \) and \( x = -4 \).

The inequality

\[ |x| < 4 \]

means that \( x < 4 \) (if \( x \) is non-negative) and \( x > -4 \) (if \( x \) is positive). This combines into
saying the \( x \) belongs to the interval \( ] -4, 4 [ \) (excluding the ends). We often write the two
inequalities as

\[ -4 < x < 4 . \]

2.3 Translations

Here translation means moving or shifting the graphs of functions.
Vertical translations

Consider an equation
\[ y = f(x) \]  \hspace{1cm} (2.1)
and compare this to the equation
\[ y = f(x) + d. \]  \hspace{1cm} (2.2)

Suppose \( d \) is some fixed positive number. For a given value of \( x \) the corresponding value of \( y \) in equation (2.2) is \( d \) greater than the value of \( y \) in equation (2.1).

Changing \( f(x) \) to \( f(x) + d \) shifts the graph of the function upwards a distance \( d \).

(See Figure 2.5)

Figure 2.5: Vertical translation of a function.

Alternatively, we could think in the following way: rewrite our translated equation in the form
\[ y - d = f(x). \]

The \( x \)-values and the formula for \( f(x) \) stay the same, but the corresponding \( y \)-values are all pushed down by a constant amount \( d \). Relative to these new values the graph of \( f(x) \) appears to have been pushed up by the same amount in relation to its original position. To shift the graph downwards just take \( d \) negative.
2.4. QUADRATIC FUNCTIONS

Horizontal translations

Replacing \( x \) by \( x + l \) in equation 2.1 gives

\[
y = f(x + l). \tag{2.3}
\]

Suppose \( l \) is some fixed positive number. In this case the \( y \)-values and the formula for \( f(x) \) stay the same, but all the \( x \)-values have been pushed to the right by the constant amount \( l \). Relative to these new values the graph of \( f(x) \) appears to have been pushed to the left by the same amount in relation to its original position.

Changing \( x \) to \( x + l \) shifts the graph to the left a distance \( l \).

(See Figure 2.6)

![Figure 2.6: Horizontal translation of a function.](image)

To shift to the right take \( l \) negative.

2.4 Quadratic Functions

A function of the form \( y = ax^2 + bx + c \), where \( a \), \( b \) and \( c \) are any numbers and the highest power of \( x \) occurring is \( x^2 \), is called a quadratic function. We have \( a \neq 0 \), since otherwise the function is linear. The graph of a general quadratic function can be obtained from the special case \( y = ax^2 \) by translations.
**Special case** \( y = ax^2 \)

Here \( b = 0 \) and \( c = 0 \).

(i) \( a > 0 \). Then \( y = 0 \) when \( x = 0 \) and \( y > 0 \) when \( x \neq 0 \) (since all squares of non-zero numbers are positive). So the minimum value of \( y \) is 0 and occurs at \( x = 0 \). The graph is **concave up** with lowest point \((0, 0)\).

(ii) \( a < 0 \). Then \( y = 0 \) when \( x = 0 \) and \( y < 0 \) when \( x \neq 0 \). So the maximum value of \( y \) is 0 and occurs at \( x = 0 \). The graph is **concave down** with highest point \((0, 0)\).

Some graphs are shown in Figure 2.7.

![Figure 2.7: Graphs of the form \( y = ax^2 \).](image)

**General case** \( y = ax^2 + bx + c \)

Start with \( y = ax^2 \) and shift left by \( l \):

\[
 y = a(x+l)^2 \\
  = a(x^2 + 2lx + l^2) \\
  = ax^2 + 2alx + al^2.
\]


If we match the coefficient at $x$ with $b$ we find that

$$b = 2al$$

hence $l = \frac{b}{2a}$.

Now the constant term is

$$al^2 = a \left( \frac{b}{2a} \right)^2 = \frac{b^2}{4a}.$$ 

By a vertical translation down with a suitable constant $d$

$$y = a(x + l)^2 + d = a \left( x + \frac{b}{2a} \right)^2 + d = ax^2 + bx + \frac{b^2}{4a} - d$$

we can achieve that the constant term matches $c$:

$$c = \frac{b^2}{4a} - d$$

hence

$$d = \frac{b^2}{4a} - c.$$ 

The graph of $y = ax^2 + bx + c$ is obtained by translating the graph of $y = ax^2$ left by $b/2a$ and down by $(b^2/4a) - c$.

This process is called **completing the square**. The graph of any quadratic function is called a “parabola”.

**Axis of symmetry and Vertex**

The graphs of the functions $y = ax^2$ are symmetric about the $y$-axis, that is the line $x = 0$. Since the general quadratic $y = ax^2 + bx + c$ is obtained from $y = ax^2$ by shifting left by an amount $b/2a$, its graph is symmetric about the vertical line $x = -b/2a$. This line is the “axis of symmetry” of the parabola.

**Note:** The axis of symmetry is a unique line associated with the graph of a quadratic function, but this line does not belong to the graph.

The “vertex”, or turning point, of the parabola is the point where the axis of symmetry intersects the parabola. It is also the place where the locus of points on the graph achieves either the maximum or minimum $y$-value (more about this below). For the basic function $y = ax^2$ the vertex occurs at the point $(0, 0)$. After horizontal and vertical translation have yielded a parabola of the general kind, we see that the coordinates of the vertex have been shifted to

$$x_0 = \frac{-b}{2a} ; \, y_0 = c - \frac{b^2}{4a}.$$
This is the general formula for the vertex coordinates of any parabola described by the function \( y = ax^2 + bx + c \). The vertex is the most important point on the parabola, so it should be plotted carefully.

(i) \( a > 0 \). The minimum value of \( y \) is \( c - (b^2/4a) \) and occurs at \( x = -b/2a \). The graph is concave up.

(ii) \( a < 0 \). The maximum value of \( y \) is \( c - (b^2/4a) \) and occurs at \( x = -b/2a \). The graph is concave down.

Some graphs are shown in Figure 2.8.

![Figure 2.8: Some quadratics](image)

**Inverse function and quadratic equations**

The quadratic function \( f(x) = x^2 \) is not injective on its natural domain \( \mathbb{R} \) because for each input \( x \) the input \(-x\) produces the same output \( x^2 = (-x)^2 \), e.g. \( 2^2 = (-2)^2 = 4 \). This can be fixed by restricting the domain to the non-negative real numbers \( \mathbb{R}^+ \). Now the graph of \( f(x) = x^2 \) with domain \( \mathbb{R}^+ \) indicates that the function is strictly increasing and the range is the set of non-negative real numbers \( \mathbb{R}^+ \). Hence the restricted function has an inverse, which is called the square root function:

\[
\begin{align*}
f^{-1} &\colon x \rightarrow \sqrt{x} \\
f^{-1} &\colon \mathbb{R}^+ \rightarrow \mathbb{R}
\end{align*}
\]
Notice that the square root function is only defined for non-negative numbers and it assumes only non-negative values.

If we had restricted the domain of \( f(x) = x^2 \) to the non-positive numbers, \( f \) would have been decreasing, hence injective, and the range would be again \( \mathbb{R}^+ \). This function has an inverse which is defined on \( \mathbb{R}^+ \) but produces non-positive outputs. This inverse function is \(-\sqrt{x}\). We see that the function \( f(x) = x^2 \) has two inverse functions, depending on whether we restrict it to the non-negative or non-positive domain. We need to take this into account when we solve quadratic equations

\[
x^2 = a.
\]

We can only solve this equation if \( a \) is in the range of the quadratic function, i.e. \( a \geq 0 \). In this case we have two solutions, namely \( x = \sqrt{a} \) and \( x = -\sqrt{a} \). (For \( a = 0 \) these two solutions coincide.)

The completion of the square from above helps us to solve general quadratic equations

\[
ax^2 + bx + c = 0.
\]

In exactly the same way we may rewrite this equation as

\[
a(x + l)^2 - d = a \left( x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a} = 0.
\]

Using the square root function we can now solve the latter equation

\[
a \left( x + \frac{b}{2a} \right)^2 + c - \frac{b^2}{4a} = 0
\]

\[
a \left( x + \frac{b}{2a} \right)^2 = \frac{b^2}{4a} - c
\]

\[
\left( x + \frac{b}{2a} \right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}
\]

\[
\left( x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2}
\]

We can apply the function \( \pm \sqrt{\cdot} \) to both sides if \( b^2 - 4ac \geq 0 \). Otherwise there is no solution. This yields

\[
x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}
\]

\[
x + \frac{b}{2a} = \frac{\pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}
\]

\[
x = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}
\]
You may memorise this formula (or look it up each time you need it) or you may use the completion of the squares method to solve quadratic equations

**Intercepts**

The first application of the quadratic formula is to compute the x-intercepts of a quadratic function \( f(x) = ax^2 + bx + c \):

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

There are three cases to consider:

- (i) \( b^2 - 4ac < 0 \). Here the quantity inside the square root is negative and there are no roots. This means that the graph lies entirely above or below the x-axis.

- (ii) \( b^2 - 4ac = 0 \). In this case there is one root \( x = -b/2a \). The graph just touches the x-axis and the root lies on the axis of symmetry of the quadratic.

- (iii) \( b^2 - 4ac > 0 \). In this case there are two roots.

Figure 2.9 shows the roots of some quadratics.

---

**Figure 2.9**: Roots of some quadratics

\[
\begin{align*}
y &= x^2 - 2x + 7 \\
y &= x^2 - 2x + 1 \\
y &= x^2 - 2x - 8
\end{align*}
\]
2.4. QUADRATIC FUNCTIONS

Sketching the graph of a quadratic

The procedure to sketch the graph of a quadratic function

\[ y = ax^2 + bx + c \]

is as follows:

1. Look at the sign of the leading term \( ax^2 \) to determine whether the graph is concave up or concave down.

2. Compute the vertex coordinates using the formula above, and plot this point carefully on your graph.

3. (Optional) Sketch in the axis of symmetry corresponding to the vertical line \( x = -\frac{b}{2a} \). Note that the graph must be symmetric about this line and that the maximum or minimum value also occurs on this line.

4. Find the roots

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

and (if there are any roots) plot them on your graph.

Example

A farmer has 200m of fencing with which to enclose a rectangular field. On one side of the field he can make use of a fence which already exists. What is the maximum area he can enclose?

Let the sides be of length \( x \) and \( y \) as shown in Figure 2.10, the side of length \( x \) being parallel to the already existing fence. Let the enclosed area be \( A \). Then the total length of fencing is

\[ x + 2y = 200 \]
and the enclosed area is 
\[ A = xy. \]

Solving the first equation for \( y \) gives 
\[ y = 100 - \frac{x}{2}, \]
and substituting into the formula for the area gives 
\[
A = x\left(100 - \frac{x}{2}\right)
= -\frac{x^2}{2} + 100x
\]

This is a quadratic equation for the area with \( a = -1/2 \), \( b = 100 \) and \( c = 0 \). Using the expression given earlier for the maximum of quadratic with \( a < 0 \) we find 
\[
A_{\text{max}} = c - \frac{b^2}{4a} = -\frac{10000}{-2} = 5000 \text{ square metres},
\]
which occurs when 
\[
x = \frac{-b}{2a} = 100 \text{ metres}.
\]

**Example**

Later in this unit we will study the area between two curves. To do so we need to compute the points of intersection of two curves. If the curves are given as graphs of two functions \( y = f(x) \) and \( y = g(x) \) then the coordinates \((x, y)\) must satisfy the two equations simultaneously, as in the case of the intersection of two straight lines. We can solve this system of two simultaneous equations in two steps:
1. Solve the equation \( f(x) = g(x) \) for \( x \).
2. Plug the solutions for \( x \) from step 1 into either of the functions to obtain \( y \).
3. (optional) Check your result by plugging the solutions \((x, y)\) into the other function.

**Example.** Let \( y = f(x) = 2x^2 + x + 1 \) and \( y = g(x) = x^2 + 2x + 3 \)

Then step 1 gives 
\[
2x^2 + x + 1 = x^2 + 2x + 3,
\]
which simplifies to 
\[
x^2 - x - 2 = 0.
\]
Now the quadratic equation formula gives 
\[
\begin{align*}
x &= \frac{1 \pm \sqrt{1 + 8}}{2} \\
&= \frac{1 + \sqrt{9}}{2} \text{ or } \frac{1 - \sqrt{9}}{2} \\
&= 2 \text{ or } -1.
\end{align*}
\]
Substituting these values in the equation \( y = f(x) \) gives

\[
y = 2 \times 2^2 + 2 + 1 = 11
\]

or

\[
y = 2 \times (-1)^2 + (-1) + 1 = 2.
\]

So there are two points of intersection \((x, y)\) of the graphs of \( f \) and \( g \): \((x, y) = (2, 11)\) and \((x, y) = (-1, 2)\). You should substitute both values in the equation \( y = g(x) \) as a check on the correctness of the solutions.

## 2.5 Power Functions

If \( m \) is a positive whole number, the **power function**

\[
y = x^m
\]

makes \( y \) the result of multiplying \( m \) \( x \)'s together. The number \( m \) is called the **exponent** of \( x \). Here are some examples:

\[
2^3 = 2 \times 2 \times 2 = 8, \quad 3^2 = 3 \times 3 = 9, \quad (1 \frac{1}{2})^3 = 1 \frac{1}{2} \times 1 \frac{1}{2} \times 1 \frac{1}{2} = 3 \frac{3}{8},
\]

\[
(-2 \frac{1}{3})^2 = (-2 \frac{1}{3}) \times (-2 \frac{1}{3}) = 5 \frac{4}{9}, \quad 99^1 = 99.
\]

These power functions are well-defined for any real number \( x \), hence the natural domain is \( \mathbb{R} \). From the graphs in Figure 2.11 we see that they all pass through the point with coordinates \((0, 0)\) (because \(0^n = 0\)) and the point with coordinates \((1, 1)\) (because \(1^n = 1\)).

The behaviour is slightly different for odd exponents and even exponents:

If \( m \) is an odd number the power function \( y = x^n \) is increasing everywhere and the range is the set of all real numbers \( \mathbb{R} \). This is similar the function \( y = x^1 = x \). These functions have an inverse which is defined for all real numbers and is called the \( n \)-th root function. The \( n \)-th root function is denoted by

\[
f^{-1}: x \mapsto \sqrt[n]{x}
\]

\[
f^{-1}: \mathbb{R} \rightarrow \mathbb{R}.
\]

Notice that the graphs for odd power functions are rotationally symmetric about the origin.

If \( m \) is an even number the power function \( y = x^n \) is increasing for non-negative inputs and decreasing for non-positive inputs. The range is the set of the non-negative real numbers \( \mathbb{R}^+ \). This is similar the quadratic function \( y = x^2 \). These functions only have an inverse after we restrict them to either non-negative or non-positive inputs. The inverse
for non-negative inputs is defined for all non-negative real numbers and is called the n-th root function. The n-th root function is denoted by

\[ f^{-1} : x \mapsto \sqrt[n]{x} \]
\[ f^{-1} : \mathbb{R}^+ \to \mathbb{R}^+. \]

The inverse of the restriction to non-positive inputs is

\[ f^{-1} : x \mapsto -\sqrt[n]{x} \]
\[ f^{-1} : \mathbb{R}^+ \to \mathbb{R}^- . \]

**Rules for combining powers**

If we multiply \( m \) \( x \)'s then multiply \( n \) \( x \)'s then multiply the two answers, what we have done is multiply \( m + n \) \( x \)'s together. So

\[ x^m x^n = x^{m+n} . \]

If we multiply \( m \) \( x \)'s then multiply \( n \) copies of the answer, what we have done is multiply \( mn \) \( x \)'s together. So

\[ (x^m)^n = x^{mn} . \]
Exponents that are not positive whole numbers

We established the above two rules when \( m \) and \( n \) were positive whole numbers. We now use these rules to give meaning to power functions \( x^p \) when \( p \) is not a positive whole number. The rules are still true for these new power functions even when the exponents are not positive whole numbers.

**The power function** \( x^0 \): \( 1 \times x = x = x^1 = x^{0+1} = x^0 \times x^1 = x^0 \times x \). Hence

\[
\boxed{x^0 = 1.}
\]

**The power function** \( x^{-1} \): \( x^{-1}x = x^{-1}x^1 = x^{-1+1} = x^0 = 1 \). Hence

\[
\boxed{x^{-1} = \frac{1}{x}.}
\]

**Other negative exponents:** \( x^{-p} = x^{p \times (-1)} = (x^p)^{-1} = \frac{1}{x^p} \). Notice that the power functions with negative exponent cannot be defined for \( x = 0 \).

**Special fractional exponents:** \( \left(x^{\frac{1}{p}}\right)^p = x^{\frac{1}{p}p} = x^1 = x \). Hence \( x \mapsto x^{1/p} \) is inverse to \( x \mapsto x^p \) and therefore

\[
\boxed{x^{1/p} = \sqrt[p]{x}.}
\]

**General fractional exponents:** \( x^{q/p} = x^{\frac{1}{p}q} = (x^q)^{\frac{1}{p}} = \sqrt[q]{x^q} \). Since powers with fractional exponents involve roots it only makes sense to define them for non-negative inputs (positive inputs if the exponent is negative).

**Example**

Find the value of \( y = x^{-3/2} \) when \( x = 4 \).

\[
y = x^{-3/2} = \frac{1}{x^{3/2}} = \frac{1}{\sqrt[2]{x^3}}.
\]

Now putting \( x = 4 \) gives

\[
y = \frac{1}{\sqrt[2]{4^3}} = \frac{1}{\sqrt[2]{64}} = \frac{1}{8}.
\]
Graphs of power functions with positive exponents

Here is a table of values for some power functions. Check the numbers on your calculator.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$1/4$</th>
<th>$1/2$</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^0$</td>
<td>1</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$x^{1/2}$</td>
<td>0</td>
<td>0.5000</td>
<td>0.7071</td>
<td>1.0000</td>
<td>1.4142</td>
<td>2.0000</td>
</tr>
<tr>
<td>$x^1$</td>
<td>0</td>
<td>0.2500</td>
<td>0.5000</td>
<td>1.0000</td>
<td>2.0000</td>
<td>4.0000</td>
</tr>
<tr>
<td>$x^{3/2}$</td>
<td>0</td>
<td>0.1250</td>
<td>0.3536</td>
<td>1.0000</td>
<td>2.8284</td>
<td>8.0000</td>
</tr>
<tr>
<td>$x^2$</td>
<td>0</td>
<td>0.0625</td>
<td>0.2500</td>
<td>1.0000</td>
<td>4.0000</td>
<td>16.0000</td>
</tr>
<tr>
<td>$x^4$</td>
<td>0</td>
<td>0.0039</td>
<td>0.0625</td>
<td>1.0000</td>
<td>16.0000</td>
<td>256.0000</td>
</tr>
</tbody>
</table>

Figure 2.12 shows the graphs of these functions.

![Figure 2.12: Graphs of power function](image)

Some points to notice:

1. All of these functions (except $x^0$) increase with $x$.
2. All of the graphs pass through the point $(1, 1)$.
3. For $p > 1$ the steepness of the graph increases as $x$ increases, while for $p < 1$ (but $> 0$) the steepness decreases as $x$ increases.
4. For $x > 1$ the graphs for larger $p$ lie above those with smaller $p$. For $0 < x < 1$ the graphs for larger $p$ lie below those with smaller $p$. 
2.5. POWER FUNCTIONS

Graphs of power functions with negative exponent

Here is a table of values for the power functions whose exponents are the negatives of the previous set. Since $x^{-p} = 1/x^p$ this table is easily computed from the previous one. Check the numbers.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1/4</th>
<th>1/2</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^{-1/3}$</td>
<td>1.5874</td>
<td>1.2600</td>
<td>1.0000</td>
<td>0.7937</td>
<td>0.6300</td>
</tr>
<tr>
<td>$x^{-1/2}$</td>
<td>2.0000</td>
<td>1.4142</td>
<td>1.0000</td>
<td>0.7071</td>
<td>0.5000</td>
</tr>
<tr>
<td>$x^{-1}$</td>
<td>4.0000</td>
<td>2.0000</td>
<td>1.0000</td>
<td>0.5000</td>
<td>0.2500</td>
</tr>
<tr>
<td>$x^{-2}$</td>
<td>16.0000</td>
<td>4.0000</td>
<td>1.0000</td>
<td>0.2500</td>
<td>0.0625</td>
</tr>
<tr>
<td>$x^{-4}$</td>
<td>256.0000</td>
<td>16.0000</td>
<td>1.0000</td>
<td>0.0625</td>
<td>0.0039</td>
</tr>
</tbody>
</table>

Figure 2.13 shows the graphs of these functions.

![Graphs of power function with negative exponents](image)

Figure 2.13: Graphs of power function with negative exponents.

Note that

1. These functions are not defined at $x = 0$.
2. All of these functions decrease with $x$.
3. All of the graphs again go through point $(1, 1)$.
4. As $p$ increases the value of $x^{-p}$ decreases for $x > 1$ and increases for $x < 1$.
5. All the graphs approach the x-axis as $x$ becomes large.
2.6 Exponential Functions

In the section 2.5 we looked at the power functions

\[ y = x^p \]

with fixed exponent and variable base. Now let’s look at the exponential functions

\[ y = a^x \]

with fixed base and variable exponent. We must take \( a > 0 \) here or there will be trouble with fractional powers of negative numbers. These functions are defined for all values of \( x \), however.

Here is a table of values of some exponential functions. You can check the numbers on your calculator.

<table>
<thead>
<tr>
<th></th>
<th>(-10)</th>
<th>(-1)</th>
<th>(-0.1)</th>
<th>(0)</th>
<th>(0.1)</th>
<th>(1)</th>
<th>(10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\left(\frac{1}{2}\right)^x)</td>
<td>1024</td>
<td>2</td>
<td>1.071</td>
<td>1</td>
<td>0.933</td>
<td>0.500</td>
<td>9.766 \times 10^{-4}</td>
</tr>
<tr>
<td>(\left(\frac{3}{4}\right)^x)</td>
<td>17.768</td>
<td>1.333</td>
<td>1.029</td>
<td>1</td>
<td>0.972</td>
<td>3/4</td>
<td>0.0563</td>
</tr>
<tr>
<td>(1^x)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(2^x)</td>
<td>9.766 \times 10^{-4}</td>
<td>1/2</td>
<td>0.933</td>
<td>1</td>
<td>1.071</td>
<td>2</td>
<td>1024</td>
</tr>
<tr>
<td>(5^x)</td>
<td>1.024 \times 10^{-7}</td>
<td>1/5</td>
<td>0.851</td>
<td>1</td>
<td>1.1746</td>
<td>5</td>
<td>9.766 \times 10^{6}</td>
</tr>
</tbody>
</table>

Graphs of these functions are shown in Figure 2.14.

Note that

1. The functions with base \( a > 1 \) are strictly increasing — the larger the base the faster they increase. It follows that the exponential functions with base \( a > 1 \) are injective.

2. The functions with base \( 0 < a < 1 \) are strictly decreasing — the smaller the base the faster they decrease. It follows that the exponential functions with base \( 0 < a < 1 \) are also injective.

3. All the graphs go through the point \((0, 1)\).

4. The range of the exponential functions with \( a \neq 1 \) is the set of positive real numbers. Therefore such exponential functions have an inverse. The inverse functions to exponential functions are called logarithmic functions. For \( y = f(x) = a^x \) the inverse function is denoted by

\[ y = f^{-1}(x) = \log_a x. \]

This is pronounced: “logarithm of \( x \) to the base \( a \)” or “ \( \log \) \( x \) to the base \( a \)”.
By thinking about what happens to the graphs as the base is gradually changed we see that there is some base \( a > 0 \) for which the graph has slope exactly 1 at the point \((0, 1)\); that is, the graph of \( a^x \) just touches the line \( y = x + 1 \) (which also has slope 1 and goes through \((0, 1)\)). This special base is the irrational number

\[
e = 2.71828182845\ldots
\]

The values of \( e^x \) for several values of \( x \) are tabulated below. (They can be found using the “exp” key of your calculator.)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( e^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-10)</td>
<td>(4.540 \times 10^{-5})</td>
</tr>
<tr>
<td>(-1)</td>
<td>0.368</td>
</tr>
<tr>
<td>0.1</td>
<td>0.905</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>1.105</td>
</tr>
<tr>
<td>1</td>
<td>2.718</td>
</tr>
<tr>
<td>10</td>
<td>2.202 \times 10^4</td>
</tr>
</tbody>
</table>

Sometimes the notation \( y = \exp x \) is used (as on calculators) instead of \( y = e^x \). Here “exp” is short for “exponential”. The importance of the number \( e \) will become apparent when we study calculus, beginning in Chapter 3.

## 2.7 Logarithmic Functions

We have defined logarithmic functions \( y = \log_a x \) in the previous section as the inverse of the exponential functions

\[
y = a^x.
\]
Finding the logarithm of $y$ to the base $a$ is the same as finding the power to which $a$ must be raised to give $y$. For example

$$64 = 4^3 \quad \text{so} \quad 3 = \log_4 64,$$

$$10000 = 10^4 \quad \text{so} \quad 4 = \log_{10} 10000$$

and

$$1/16 = 2^{-4} \quad \text{so} \quad -4 = \log_2 1/16.$$

The special notation “ln” is used for logarithms to the base $e$ (it is derived from Latin “logarithmus naturalis”, which means “natural logarithm”). So

$$y = e^x \quad \text{is equivalent to} \quad x = \ln y.$$  

On calculators a plain “log $x$” means “log $10 x$” (but “log $x$” is sometimes used to mean “ln $x$” as well). In computer science and information theory logarithms to the base 2 are often used. Logarithms of negative numbers are NOT defined for any base.

Here is a table of values of some logarithmic functions.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$1/10$</th>
<th>$1/2$</th>
<th>1</th>
<th>2</th>
<th>10</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\log_2 x$</td>
<td>-3.3219</td>
<td>-1.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>3.3219</td>
<td>6.6439</td>
</tr>
<tr>
<td>$\log_e x = \ln x$</td>
<td>-2.3026</td>
<td>-0.6931</td>
<td>0.0000</td>
<td>0.6931</td>
<td>2.3026</td>
<td>4.6052</td>
</tr>
<tr>
<td>$\log_{10} x = \log x$</td>
<td>-1.0000</td>
<td>-0.3010</td>
<td>0.0000</td>
<td>0.3010</td>
<td>1.0000</td>
<td>2.0000</td>
</tr>
</tbody>
</table>

Their graphs are shown in Figure 2.15.

Note that the graphs of logarithmic functions all go through the point $(1, 0)$.

Rules for logarithms of products and powers

Suppose $y_1 = a^{x_1}$ and $y_2 = a^{x_2}$. Then

$$y_1 y_2 = a^{x_1} a^{x_2} = a^{x_1 + x_2}$$

so

$$\log_a y_1 y_2 = x_1 + x_2 = \log_a y_1 + \log_a y_2.$$  

Also if $y = a^x$ then $y^k = (a^x)^k = a^{kx}$ so

$$\log_a y^k = kx = k \log_a y.$$  

$$\log_a y^k = k \log_a y$$
Taking \( k = -1 \) gives the useful rule

\[
\log_a \left( \frac{1}{y} \right) = -\log_a y.
\]

Hence it follows that

\[
\log_a \left( \frac{y_1}{y_2} \right) = \log_a \left( y_1 \times \frac{1}{y_2} \right) = \log_a (y_1) - \log_a (y_2).
\]

Here are some examples (in base \( e \)):

\[
\ln 20 = \ln(2 \times 10) = \ln 2 + \ln 10 = 0.6931 + 2.3026 = 2.9957,
\]

\[
\ln(10^5) = 5 \times \ln 10 = 5 \times 2.3026 = 11.513.
\]

**Equations with exponential and logarithmic functions**

Later in this unit we will need to solve equation that involve exponential or logarithmic functions. The function \( y = f(t) = 20 + 60e^{-0.05t} \) models restricted exponential growth. In order to determine for which argument \( t \) the value \( y = 40 \) is attained we need to solve the equation

\[
40 = 20 + 60e^{-0.05t}.
\]
Here we isolate the exponential term first by standard manipulations and then apply the \( \ln \) function, which inverts the exponential function.

\[
40 = 20 + 60e^{-0.05t}
\]

\[
20 = 60e^{-0.05t}
\]

\[
\frac{1}{3} = e^{-0.05t}
\]

\[
e^{-0.05t} = \frac{1}{3}
\]

\[
-0.05t = -\ln 3 \approx -1.1
\]

\[
t \approx 1.1/0.05 = 22.
\]

2.8 Trigonometric Functions

This section reviews some basic facts about trigonometric functions. We begin with a quick summary of the foundations of trigonometry itself.

Angle measure

Commonly angles are measured in degrees. E.g. a right angle corresponds to 90° and a full circle to 360°. Mathematicians and other scientists often measure angles in \textbf{radians} rather than degrees. The number of radians in an angle is the length of arc the angle cuts out of a circle of radius 1. Since the length of the complete circumference of the circle is

\[
2\pi \times \text{radius} = 2\pi,
\]

\[
360 \text{ degrees} = 2\pi \text{ (radians)}
\]

\[
= 6.283\ldots \text{ (radians)}.
\]

The radian measure \( x \) of any angle is in the same ratio to \( 2\pi \) as the degree measure \( \theta \) [greek letter theta] of the same angle is to 360 degrees, i.e.,

\[
\frac{\theta^\circ}{360} = \frac{x}{2\pi}.
\]

This relationship allows us to convert easily between the two units of angular measurement, in particular

\[
x = \frac{\pi}{180} \times \theta^\circ,
\]

so that, for example,

\[
180^\circ = \pi \text{ (rad)}, \quad 90^\circ = \frac{\pi}{2} \text{ (rad)}, \quad 60^\circ = \frac{\pi}{3} \text{ (rad)}, \quad 45^\circ = \frac{\pi}{4} \text{ (rad)}, \quad 30^\circ = \frac{\pi}{6} \text{ (rad)}.
\]

From now on all angles will be measured in radians and the unit rad will be suppressed, unless we explicitly state otherwise by using the \( ^\circ \) notation for degrees.
What are the trigonometric functions?

Suppose one angle of a right-angled triangle is \( x \) (radians). Let the length of the adjacent side, joining this angle to the right angle, be \( b \). Let the length of the hypotenuse be \( c \) and the length of the opposite side be \( a \). Then the functions \( \sin \), \( \cos \) and \( \tan \) are given by

\[
\sin x = \frac{a}{c} \quad \cos x = \frac{b}{c} \quad \tan x = \frac{a}{b}
\]

(see Figure 2.16).

![Figure 2.16: \( \sin x = \frac{a}{c} \), \( \cos x = \frac{b}{c} \), \( \tan x = \frac{a}{b} \).](image)

**Note:** When computing these functions on your calculator you must remember to put it into “rad” or “deg” mode depending on whether the angles are measured in radians or degrees. Think of the variable \( x \) as an angle, not a number. In the examples below, we show that some trigonometric ratios can be worked out precisely from geometric reasoning. These cases are rather special, since for most angles your calculator must use more advanced methods to find an *approximate* value of the sine, cosine or tangent (i.e., approximate to eight or more decimal places!). Note that the values are defined by the ratios of the sides of a right-angled triangle, rather than the actual lengths of those sides for any particular triangle. This means that we can fix the length of the hypotenuse to be “1”, as long as we keep the other two sides in the same proportion for a given angle. From now on, we’ll always assume that the length \( c \) of the hypotenuse is equal to 1.

Let’s give special attention to the following:

1. \( \sin\left(\frac{\pi}{4}\right) = 1/\sqrt{2} \).

If the angle \( x \) is \( \frac{\pi}{4} \) (or 45°), then the other two angles must be \( \frac{\pi}{4} \) and \( \frac{\pi}{2} \) (remember the sum of angles in any triangle must be \( \pi \) radians!). Hence the triangle is isosceles, i.e., \( a = b \). According to Pythagoras’ theorem (see below), \( a^2 + b^2 = 2a^2 = 1 \), and therefore \( a = b = 1/\sqrt{2} \). Now for the hypotenuse \( c = 1 \) this implies

1. \( \sin\left(\frac{\pi}{4}\right) = \frac{a}{1} = 1/\sqrt{2} \).

The same reasoning tells us

2. \( \cos\left(\frac{\pi}{4}\right) = \frac{b}{1} = 1/\sqrt{2} \),
3. \( \tan\left(\frac{\pi}{4}\right) = \frac{a}{b} = 1 \).

A similar geometrical argument involving equilateral triangles allows us to see that

4. \( \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \),

and

5. \( \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \), \quad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \).

The values of sine and cosine change places when \( x \) equals \( \frac{\pi}{3} \), since this angle is complementary to \( \frac{\pi}{6} \) within the right - angled triangle. Hence

6. \( \sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \),

and similarly

7. \( \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} ; \quad \tan\left(\frac{\pi}{3}\right) = \sqrt{3} \).

When the angle \( x = 0 \), we note that \( b = c = 1 \), while \( a = 0 \). This means

8. \( \cos 0 = 1 \), \quad \sin 0 = \tan 0 = 0. \)

On the other hand, when \( x = \frac{\pi}{2} \), we note that \( b = 0 \), and \( a = c = 1 \), so

9. \( \sin\left(\frac{\pi}{2}\right) = 1 \), \quad \cos\left(\frac{\pi}{2}\right) = 0. \)

Notice also that \( \tan\left(\frac{\pi}{2}\right) = \frac{a}{b} \) is not defined in this case.

### Relations among trigonometric functions

By Pythagoras’s theorem (discovered about 600 BC), the sides of the triangle in Figure 2.16 satisfy

\[
a^2 + b^2 = c^2.
\]
2.8. TRIGONOMETRIC FUNCTIONS

Divide both sides of the equation by \( c^2 \)

\[
\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1,
\]

rearrange

\[
\left( \frac{a}{c} \right)^2 + \left( \frac{b}{c} \right)^2 = 1.
\]

Thus

\[
\sin^2 x + \cos^2 x = 1.
\]

(Notice that \( \sin^2 x \) means \( \sin x \times \sin x \), and similarly for \( \cos^2 x \).)

Also

\[
\frac{a}{b} = \frac{a/c}{b/c}
\]

so

\[
\tan x = \frac{\sin x}{\cos x}.
\]

Application

Standing at 20 meters from the halls of residence boiler house chimney we measure that the chimney top subtends an angle 51°32' above the horizontal. How high is the chimney?

Let the height be \( h \) meters. Then

\[
\tan 51°32' = h/20
\]

so

\[
h = 20 \tan 51°32' = 20 \tan 51.53° = 20 \times 1.259 = 25.17 \text{ m}.
\]

(Notice that since the angle is measured in degrees we need to use the “deg” mode of the calculator to work this out.)

Angles greater than a right angle

In a right-angled triangle the other two angles are both less than a right angle, so our definition of \( \sin x \), \( \cos x \) and \( \tan x \) only works for \( x \leq \frac{\pi}{2} \) (= one right angle).

We can also define these functions as follows. Let \( \mathbf{v} \) be the line segment (of length 1) obtained by rotating the segment that joins the point \( (0,0) \) of a plane coordinate system to the point \( (1,0) \) of that system. Let \( (b, a) \) be the coordinates of the end point of \( \mathbf{v} \) (as in Figure 2.16). We will assume the rotation is anticlockwise through an angle \( x \) about \( (0,0) \).
For the sine function this is illustrated at [http://www.youtube.com/watch?v=Ohp6Okk_tww](http://www.youtube.com/watch?v=Ohp6Okk_tww).

Then

\[
\begin{align*}
\cos x &= b\text{-coordinate of } v, \\
\sin x &= a\text{-coordinate of } v, \\
\tan x &= \frac{\sin x}{\cos x} = \frac{a\text{-coordinate of } v}{b\text{-coordinate of } v}.
\end{align*}
\]

This is the same as the first definition when \(0 \leq x \leq \pi/2\) (taking \(c = 1\)) but works for other values of \(x\) too. For example, when \(v\) undergoes further rotation through a right angle, the coordinates \((b, a)\) of the point at the tip of \(v\) change to \((-a, b)\). The reason for this isn’t obvious, but it follows from a simple piece of deduction about congruent triangles. This gives us the very useful formula

\[
\sin \left( x + \frac{\pi}{2} \right) = \cos x \quad \text{and} \quad \cos \left( x + \frac{\pi}{2} \right) = -\sin x.
\]

It follows

\[
\sin(x + \pi) = \cos \left( x + \frac{\pi}{2} \right) = -\sin x \quad \text{and} \quad \cos (x + \pi) = -\cos x.
\]

The first of these equations tells us that the graph of \(\cos x\) is the same as the graph of \(\sin x\) translated left by \(\pi/2\). The formula allows us to compute more examples of precise trig ratios:

1. \(\sin \frac{3\pi}{4} = \cos \left( \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}}, \quad \cos \frac{3\pi}{4} = -\frac{1}{\sqrt{2}}, \quad \tan \frac{3\pi}{4} = -1,\)

2. \(\sin \frac{5\pi}{4} = \sin \left( \frac{3\pi}{4} + \frac{\pi}{2} \right) = \cos \left( \frac{3\pi}{4} \right) = -\frac{1}{\sqrt{2}}, \quad \cos \frac{5\pi}{4} = -\sin \left( \frac{3\pi}{4} \right) = -\frac{1}{\sqrt{2}}, \quad \tan \frac{5\pi}{4} = 1.\)

**Periodicity of trigonometric functions**

Adding \(2\pi\) radians to \(x\) rotates the line segment \(v\) through a complete revolution about the point \((0, 0)\), leaving its \(b\) and \(a\) coordinates as they were. So

\[
\cos(x + 2\pi) = \cos x \quad \text{and} \quad \sin(x + 2\pi) = \sin x.
\]

Also

\[
\tan(x + \pi) = \frac{\sin(x + \pi)}{\cos(x + \pi)} = \frac{-\sin x}{-\cos x} = \tan x.
\]

This means that adding \(2\pi\) to \(x\) does not change the value of \(\sin x\), \(\cos x\) or \(\tan x\). We say that \(\sin x\), \(\cos x\) and \(\tan x\) are **periodic functions** with **period** \(2\pi\). We infer from this
that the $x$-intercepts (i.e., $y = 0$) of the graph of the function $y = \sin(x)$ occur at $x = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \ldots$ while those of the function $y = \cos(x)$ occur at $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots$.

Using set notation we may describe the zeroes of sine by

$$\{2k\pi : k \in \mathbb{Z}\}$$

and the set of zeroes of cosine by

$$\{(2k + \frac{1}{2})\pi : k \in \mathbb{Z}\}.$$  

The periodicity of sin and cos makes them very useful for describing repetitive phenomena, which occur a great deal in biology as well as in all other sciences.

The graphs of $\sin x$, $\cos x$ and $\tan x$ are shown in Figure 2.17.

![Figure 2.17: Trigonometric functions](image)

**Amplitude, Frequency and Phase**

A **periodic function** is a function $f(x)$ with

$$f(x + l) = f(x)$$

for some $l$. Such a function repeats itself at intervals of length $l$ along the $x$-axis (because when it is shifted left by $l$ it has the same values). The length $l$ is called the **period** of the
function. We have already seen that \( \sin x \) and \( \cos x \) each have period \( 2\pi \) while \( \tan x \) has period \( \pi \).

By introducing 3 parameters \( A, \omega, \phi \) we can modify the sine function to define the following periodic functions

\[
y = A \sin(\omega x + \phi).
\]

The absolute value \( |A| > 0 \) is the amplitude of the function, \( \omega > 0 \) is the angular frequency and \( \frac{\phi}{\omega} \) is the phase. We now look at the significance of these parameters and the effect they have on how to sketch the graph of the function.

**Angular frequency** The angular frequency \( \omega \) is related to the period of the function. As \( x \) runs from 0 to \( 2\pi/\omega \), \( \omega x \) runs from 0 to \( 2\pi \) — a full cycle of the sin function — so the function \( y = A \sin(\omega x + \phi) \) has period \( 2\pi/\omega \).

Let’s start with the special case \( A = 1, \phi = 0 \). Then the function is

\[
y = \sin(\omega x).
\]

A good way to understand the effect of the angular frequency on the graph of this function is through the relative positioning of the \( x \)-intercepts. Recall that \( \sin(x) = 0 \) when \( x = 0, \pm\pi, \pm2\pi, \pm3\pi, \ldots \), hence \( \sin(\omega x) = 0 \) when \( \omega x = 0, \pm\pi, \pm2\pi, \pm3\pi, \ldots \), i.e., when \( x = 0, \pm\frac{\pi}{\omega}, \pm\frac{2\pi}{\omega}, \pm\frac{3\pi}{\omega}, \ldots \). If \( \omega > 1 \) then the intercepts of the graph of \( y = \sin(\omega x) \) are closer together than those of \( y = \sin(x) \), hence so are the “oscillations” of the graph. If \( 0 < \omega < 1 \), then the intercepts are further apart, hence the oscillations seem stretched out compared with those of \( y = \sin(x) \). See Figure 2.18.

**Phase** Keep \( A = 1 \) and look at the effect of the phase \( \frac{\phi}{\omega} \). The function

\[
y = \sin(\omega x + \phi) = \sin(\omega(x + \frac{\phi}{\omega}))
\]

has the same graph as \( \sin(\omega x) \) except that it is shifted left by an amount \( \phi/\omega \). This is because

\[
\omega(x + \frac{\phi}{\omega}) = \omega x + \phi.
\]

Note that \( \cos x \) is one of these functions with angular frequency \( \omega = 1 \) and phase \( \phi = \pi/2 \), since we have seen that

\[
\sin(x + \frac{\pi}{2}) = \cos x.
\]

See Figure 2.19.

**Amplitude** The amplitude \( |A| \) is the height of the crests of the periodic function. The function

\[
y = A \sin(\omega x + \phi)
\]
Graphing trigonometric functions

The following steps allow any trigonometric function of the form

\[ y = A \sin(\omega x + \phi) \]

or

\[ y = A \cos(\omega x + \phi) \]

to be graphed easily:

1. Start by drawing a sin or cos curve without scaling the \(x\)-axis. Note that all points on either graph must have \(y\)-values between 1 and −1.

2. Specifically plot the \(x\)-intercepts of the graph of \(y = \sin(\omega x)\) or \(y = \cos(\omega x)\). In the latter case we remark that these occur at \(x = \pm \frac{\pi}{2\omega}, \pm \frac{3\pi}{2\omega}, \ldots\). Make a second sketch of the graph in proportion to these intercepts.
Figure 2.19: The functions $\sin(2x)$, $\sin(2x - 2)$, $\sin(2x + 4)$.

3. Translate each of the intercepts to the left by $\phi/\omega$ (assuming $\phi > 0$). Make a third sketch of the graph (this time for $y = \sin(\omega x + \phi)$ or $y = \cos(\omega x + \phi)$) relative to these new intercepts.

4. Rescale the $y$-axis so that all points on either graph have $y$-values between $A$ and $-A$. The third sketch should now look like the graph of $y = A\sin(\omega x + \phi)$ or $y = A\cos(\omega x + \phi)$ if $A > 0$. Make a reflection of your graph in the $x$-axis if $A$ is negative.

Try this!

Inverse Trigonometric Functions and Equations involving trigonometric functions

We start with the simplest equation

$$\sin x = c.$$  

Since the range of $\sin$ is the set of numbers between $-1$ and $1$ this equation has no solution if $c > 1$ or if $c < -1$. $\sin$ is not a 1-to-1 function. Therefore we may expect many solutions if $-1 \leq c \leq 1$. In fact, there are infinitely many solutions. Imagine a the horizontal line $y = c$ intersecting the sine curve $y = \sin x$. Since $\sin$ is periodic with period $2\pi$ we may add
2.8. TRIGONOMETRIC FUNCTIONS

Figure 2.20: The functions \( \sin(2x - 2) \), \( 2 \sin(2x - 2) \), \( \frac{1}{2} \sin(2x - 2) \).

to a solution \( x_0 \) an arbitrary integer multiple of 2\( \pi \) to find another solution. This set of solutions can be described by \( \{ \ldots, x_0 - 4\pi, x_0 - 2\pi, x_0, x_0 + 2\pi, \ldots \} = \{ x_0 + 2n\pi \} \), where \( n \) is any integer. But there are more solutions, namely \( \pi - x_0 \) and all numbers occurring by adding integer multiples of 2\( \pi \), that is \( \{ -x_0 + (2n + 1)\pi \} \), where \( n \) is any integer. This comes from the relation \( \sin \theta = \sin \pi - \theta \) (\( \pi - \theta \) is the angle that results from reflecting \( \theta \) about the \( y - \text{axis} \)).

If we restrict the domain of \( \sin \) to one half of its period and the codomain to its range \( \sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1] \) the function becomes bijective and therefore has an inverse. This inverse is called \( \arcsin: [-1, 1] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}] \). All scientific calculators have a key for the \( \arcsin \) function, although it is also sometimes called the inverse \( \sin \) and written \( \sin^{-1} \). The \( \arcsin \) function can be used to find the initial solution \( x_0 \).

Similarly, there is an inverse function to the restricted \( \cos: [0, \pi] \rightarrow [-1, 1] \). It is called \( \arccos: [-1, 1] \rightarrow [0, \pi] \). It can be used to solve the equation

\[
\cos x = c.
\]

If \(-1 \leq c \leq 1\) we find one solution \( x_0 = \arccos c \). Then all other solutions can be found by adding integer multiples of 2\( \pi \) to \( x_0 \) or by adding integer multiples of 2\( \pi \) to \(-x_0 \).

Finally, \( \tan \) is surjective with codomain \( \mathbb{R} \) and becomes bijective if restricted to one period \( \tan: ]-\frac{\pi}{2}, \frac{\pi}{2}[ \rightarrow \mathbb{R} \). (Notice the inside-out brackets indicating that the ends are excluded.) To find a solution of

\[
\tan x = c
\]
we may apply the inverse tangent arctan = tan$^{-1}$ to find $x_0 = \arctan c$. Any other solution can be obtained by adding an integer multiple of $\pi$. Notice that the period of tan is $\pi$ and tan is injective (increasing) if restricted to one period.

Let us look at a more general example:

$$A \sin(\omega x + \phi) = c.$$  

This has no solutions if $c > A$ or $c < -A$ (since the sin function only takes values between $-1$ and 1). When $-A \leq c \leq A$ there are infinitely many solutions but they occur in a regular pattern, so are easily found once we have found one.

The equation can now be solved like this:

divide both sides by $A$

$$\sin(\omega x + \phi) = \frac{c}{A}$$

apply the arcsin function to both sides we find one solution

$$\omega x + \phi = \arcsin \left( \frac{c}{A} \right)$$

Now any other solution satisfies

$$\omega x + \phi = \arcsin \left( \frac{c}{A} \right) + 2n\pi \text{ or } \omega x + \phi = -\arcsin \left( \frac{c}{A} \right) + (2n + 1)\pi.$$  

Solve these linear equations for $x$

$$x = \frac{1}{\omega} \left( \arcsin \left( \frac{c}{A} \right) - \phi + 2n\pi \right) \text{ and } x = \frac{1}{\omega} \left( -\arcsin \left( \frac{c}{A} \right) - \phi + (2n + 1)\pi \right).$$

Other periodic functions

The functions

$$y = A \sin(\omega x + \phi)$$

are the simplest periodic functions. Much more complicated periodic functions can arise, for example in the readout from an electrocardiogram. An important branch of mathematics called Fourier analysis is concerned with how complicated periodic functions can be built up from these simple ones. The most important conclusion is that any function with period $\omega$ can be built up by adding together a collection of these simple periodic functions (perhaps infinitely many of them) whose angular frequencies are integer multiples of $\omega$.

2.9 Combining Functions

So far we have looked at a number of basic functions: linear, quadratic, powers, exponential, logarithmic and trigonometric. There are many ways they can be combined to form the more complicated functions that occur in practice.
2.9. COMBINING FUNCTIONS

Multiples of a function

Given a function \( f(x) \) and a number \( c \) we can form a new function \( cf(x) \) by multiplying \( f \) by \( c \). For example, if \( f(x) = x^{\frac{1}{2}} \) and \( c = \frac{1}{4} \), then

\[
 cf(x) = \frac{1}{4} x^{\frac{1}{2}}.
\]

Multiplying by \( c \) stretches out the graph of \( f(x) \) in the \( y \) direction if \( |c| > 1 \) and squashes it down if \( |c| < 1 \). When \( c \) is negative the graph gets reflected in the \( x \)-axis too. See Figure 2.21.

![Figure 2.21: Graphs of \( x^{\frac{1}{2}} \), \((1/4)x^{\frac{1}{2}}\) and \((-1/4)x^{\frac{1}{2}}\).](image)

Sums of functions

Given two functions \( f(x) \) and \( g(x) \) we can form a new function \( f(x) + g(x) \) by adding them. It is not hard to get the graph of \( f(x) + g(x) \) from the graphs of \( f(x) \) and \( g(x) \): for each value of \( x \) add the \( y \) values given by \( f \) and \( g \). See Figure 2.22.

Differences of functions

The difference \( f(x) - g(x) \) of the functions \( f(x) \) and \( g(x) \) is similarly defined by subtracting the functions. See Figure 2.22.
Products of functions

From two functions \( f(x) \) and \( g(x) \) we can form their product \( f(x)g(x) \) by multiplying them. When sketching the graph of a product it is useful to bear in mind the following facts about products of positive numbers:

- the product of two numbers greater than 1 is greater than either of them,
- the product of a number greater than 1 and a number less than 1 lies between them,
- the product of two numbers less than 1 is smaller than either of them.

We also need to take into account the signs of \( f(x) \) and \( g(x) \) and that zeroes of \( f(x)g(x) \) occur when either \( f \) or \( g \) has a zero. See Figure 2.23

Quotients

Similarly we can form the quotient \( f(x)/g(x) \) of two functions \( f(x) \) and \( g(x) \) by dividing them. In this case there is a new kind of behaviour: since we can’t divide by 0 we must pay special attention to the the points where \( g(x) = 0 \). At such points the function \( f(x)/g(x) \) is not defined. Near such points the function values may tend to plus or minus infinity.
A good example of such behaviour is provided by the function \( y = \tan(x) \), which can be understood as the quotient of \( f(x) = \sin(x) \) and \( g(x) = \cos(x) \).

**Composite functions**

We return to the concept of composite functions from Section 1.3. Thinking of a function as a device that produces numerical output from a given numerical input we can compose a function \( f: x \mapsto u = f(x) \) with domain \( D \) and codomain \( V \) and \( g: u \mapsto y = g(u) \) with domain \( E \) and codomain \( W \) by taking the output \( u \) of the first function \( f \) as the input of the second function \( g \). The composite function is a new function with input \( x \) and output \( y \). It can be denoted by \( y = g(f(x)) \).

\[
 x \mapsto u = f(x) \mapsto y = g(u) = g(f(x)).
\]

Of course, this is only possible if the codomain of \( f \), or at least the range of \( f \) belongs to the domain of \( g \). Usually we call the “first” function the in-side function and the “second” function the out-side function. Notice that we used different letters for the independent variables of the two functions. Using the same letter would cause confusion.

For example, the composite of the functions

\[
y = g(u) = \sqrt{u} \quad \text{and} \quad u = f(x) = e^x
\]
is
\[ g(f(x)) = \sqrt{e^x}. \]

Notice that \( f(g(u)) \) can only be defined for non-negative inputs \( u \geq 0 \) and it equals
\[ f(g(u)) = e^{\sqrt{u}} \]
which is different from \( g(f(x)) \)!

As another example, the composite of
\[ y = u^2 \quad \text{and} \quad u = \sin x \]
is
\[ y = (\sin x)^2. \]
The expression \( (\sin(x))^2 \) is often written alternatively as \( \sin^2(x) \). We obtain the graph of this function by first drawing the graph of \( \sin x \) and then for each \( x \) squaring the corresponding value of \( y \). See Figure 2.24.

![Figure 2.24: Graphs of \( \sin(x) \) and \( \sin^2(x) \).](image)

**Important:**

In the same way one can combine more than two functions, say \( h(g(f(x))) \).
3.1 Definition of the Derivative

Differential calculus is the study of the changes that occur in one quantity when other quantities on which it depends change. For instance, the change in the growth of a culture of bacteria with each additional hour, or the change in crop yield which occurs with each additional kg of added fertilizer.

More particularly, we shall be interested in an instantaneous rate of change. For example, suppose a car is traveling from a point $A$ to a point $B$. Then the average rate of change of distance is:

\[
\text{the distance between } A \text{ and } B \quad \frac{\text{time taken to get from } A \text{ to } B}{}. 
\]

This quantity is more generally called the average speed of the car. However, if the car hits a wall on the way, then the average speed of the car tells us nothing about the force of the impact. What we need to know is the speed of the car at the instant the car hits the wall, i.e. we need to know the instantaneous rate of change of distance.

Suppose $y = f(x)$ is a function of $x$, and suppose we want to find the instantaneous rate of change of $y$ with respect to $x$ at a particular point $x_0$. We take a point $x_0 + \Delta x$ that is nearby to $x$. (Here the symbol $\Delta$ is the Greek letter delta, and is used to indicate a small change. Hence $\Delta x$ means a small change in the quantity $x$.) Then the average rate of change of $y$ with respect to $x$ over the interval from $x_0$ to $x_0 + \Delta x$ is given by

\[
\frac{\text{change in } y}{\text{change in } x} = \text{slope of the straight line from } (x_0, f(x_0)) \text{ to } (x_0 + \Delta x, f(x_0 + \Delta x)).
\]

Therefore the average rate of change of $y$ with respect to $x$ is

\[
\frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\Delta f}{\Delta x}.
\]
CHAPTER 3. DIFFERENTIATION

Figure 3.1: Definition of the derivative

For example, suppose \( y = f(x) = x^2 \), and that the particular point is \( x_0 = 2 \). Then the average rate of change of \( y \) with respect to \( x \) over the interval from 2 to \( 2 + \Delta x \) is given by

\[
\frac{(2 + \Delta x)^2 - 4}{(2 + \Delta x) - 2} = \frac{4 + 4\Delta x + (\Delta x)^2 - 4}{\Delta x} = \frac{4\Delta x + (\Delta x)^2}{\Delta x} = 4 + \Delta x.
\]

Thus if \( \Delta x = 1 \), then the average rate of change of \( y \) over the interval from \( x = 2 \) to \( x = 3 \) is \( x + \Delta x = 4 + 1 = 5 \). For the interval from \( x = 2 \) to \( x = 2.1 \) (\( \Delta x = 0.1 \)), the average rate of change is \( 4 + 0.1 = 4.1 \); from 2 to 2.01 it’s 4.01; from 2 to 2.001 it’s 4.001; and so on. We can see that as \( \Delta x \) approaches 0 the values of the average rate of change are approaching 4. Thus it seems reasonable to say that the instantaneous rate of change of \( f(x) = x^2 \) at \( x = 2 \) is 4. This value of 4 is called the “derivative” of \( f(x) = x^2 \) with respect to \( x \) at \( x = 2 \).

In general, if \( y = f(x) \) is any function, then the derivative of \( y \) with respect to \( x \), written as \( \frac{dy}{dx} \) or \( f'(x) \), is defined as

\[
\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.
\]

The symbol \( \lim_{\Delta x \to 0} \) stands for ‘the limit as \( \Delta x \) approaches 0’, so the derivative is the limit of the average rates of change as the interval widths get smaller and smaller. \( \frac{dy}{dx} \) or \( f'(x) \)
3.1. DEFINITION OF THE DERIVATIVE

Figure 3.2: Derivative of $x^2$

is the instantaneous rate of change of $y$ with respect to $x$ at the point $x$, and its value is also the slope of the tangent line to the graph of $y = f(x)$ at the point $(x, f(x))$.

Examples

1. Find $\frac{dy}{dx}$ for $y = f(x) = x^2$, and use the result to find $f'(1)$ and $f'(3)$. What is the equation of the line tangent to the graph of $y = x^2$ at $x = 4$?

By definition,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 2x + \Delta x$$

$$= 2x$$

Thus if $y = x^2$, then $\frac{dy}{dx} = 2x$; or, in alternate notation, if $f(x) = x^2$, then $f'(x) = 2x$. In particular, $f'(-1) = -2$ and $f'(3) = 6$. We next find the tangent line at $x = 4$. This line
has slope \( f'(4) = 8 \), and passes through \((4, 16)\) (since \( x = 4 \) implies \( y = 4^2 = 16 \)). Thus the equation of the tangent line is \( y - 16 = 8(x - 4) \), or \( y = 8x - 16 \). See Figure 3.3.

2. Suppose \( y = f(x) = c \), where \( c \) is a constant. Then \( y \) never changes, and so we would expect the instantaneous rate of change to be 0. This can be proved using the definition of the derivative, i.e.

\[
\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0.
\]

Hence the derivative of a constant function is zero.

3. Suppose \( y = f(x) = mx + b \). The graph of \( y = f(x) \) is a straight line with slope \( m \). Thus for any triangle we have that the rate of change of \( y \) over the interval from \( x_1 \) to \( x_2 \) is \( \frac{y_2 - y_1}{x_2 - x_1} = m \). Thus we would expect that \( f'(x) = m \).

Again this can be proved using the definition of the derivative. We have

\[
f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
= \lim_{\Delta x \to 0} \frac{[m(x + \Delta x) + b] - (mx + b)}{\Delta x} \\
= \lim_{\Delta x \to 0} \frac{mx + m\Delta x + b - mx - b}{\Delta x} \\
= \lim_{\Delta x \to 0} \frac{m\Delta x}{\Delta x} \\
= \lim_{\Delta x \to 0} m \\
= m.
\]

4. Find the derivative of \( y = f(x) = x^3 \).
3.2. RULES FOR DIFFERENTIATION

We have

\[
\frac{d(x^3)}{dx} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\
= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2 \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\
= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x \Delta x + (\Delta x)^2)}{\Delta x} \\
= \lim_{\Delta x \to 0} (3x^2 + 3x \Delta x + (\Delta x)^2) \\
= 3x^2.
\]

Actually, it can be shown that for any value of \( n \) (not only for positive integers)

\[
\frac{d(x^n)}{dx} = nx^{n-1}.
\]

For example,

\[
\frac{d(x^7)}{dx} = 7x^6,
\]

\[
\frac{d(\sqrt{x})}{dx} = \frac{d(x^{1/2})}{dx} = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}
\]

and

\[
\frac{d(1/x^4)}{dx} = \frac{d(x^{-4})}{dx} = -4x^{-4-1} = -4x^{-5} = \frac{-4}{x^5}.
\]

[Note: The material in this section on differentiation using deltas and limits is included to explain the definition of the derivative and show how it can be calculated from first principles. You will not be required to perform these calculations in the examination.]

3.2 Rules for Differentiation

At this point we have discussed the definition of a derivative and found the derivatives of some particular functions. Now we introduce some general rules for computing derivatives. The first two rules were derived in the previous section.

1. If \( c \) is a constant then

\[
\frac{d(c)}{dx} = 0,
\]

i.e. the derivative of a constant is zero.
2. For any power of $x$ we have the “Power Rule”

$$\frac{d(x^n)}{dx} = nx^{n-1}.$$  

Recall from the previous section that the linear function $y = mx + b$ satisfies $\frac{dy}{dx} = m$. A special case of this expression corresponds to $y = x$ (i.e., $m = 1$ and $b = 0$), so that $\frac{dy}{dx} = 1$. Another way of thinking about this case is to write $y = x^1$ and apply the Power Rule:

$$\frac{d(x^1)}{dx} = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1.$$  

Bringing back rule 1 about the derivative of a constant, we now notice that

$$\frac{d(mx + b)}{dx} = m \cdot \frac{dx}{dx} + \frac{d(b)}{dx} = m \cdot 1 + 0 = m.$$  

This is actually a special case of two general rules of differentiation. One is about the effect on $\frac{dy}{dx}$ of multiplying a function $f(x)$ by a constant, the other describes how to go about differentiating a function that is a sum or difference of two simpler functions. We present these as rules 3 and 4 below.

3. If $c$ is a constant and $y = f(x)$ is a function then

$$\frac{d(cy)}{dx} = c \frac{dy}{dx},$$

i.e. the derivative of a constant times a function is the constant times the derivative of the function. For example,

$$\frac{d(3x^2)}{dx} = 3 \frac{d(x^2)}{dx} = (3)(2x) = 6x,$$

and

$$\frac{d(-\frac{9}{3}x^{-\frac{7}{2}})}{dx} = -\frac{9}{4} \frac{d(x^{-\frac{7}{2}})}{dx} = -\frac{9}{4}(-\frac{7}{2}x^{-\frac{9}{2}}) = \frac{63}{8}x^{-\frac{9}{2}}.$$  

4. If $u$ and $v$ are two functions of $x$, then

$$\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx},$$

i.e. the derivative of the sum or difference of two functions is the sum or difference of their derivatives. For example,

$$\frac{d(x^2 - 3x)}{dx} = \frac{d(x^2)}{dx} - \frac{d(3x)}{dx} = \frac{d(x^2)}{dx} - 3 \frac{d(x)}{dx} = 2x - 3.$$
This rule also extends to more than two functions. For example,

\[
\frac{d}{dx}(4 + 2x - x^2 + x^{-2}) = \frac{d(4)}{dx} + \frac{d(2x)}{dx} - \frac{d(x^2)}{dx} + \frac{d(x^{-2})}{dx}
\]

\[
= 0 + 2 - 2x - 2x^{-3}
\]

\[
= 2 - 2x - \frac{2}{x^3}.
\]

Taken together, rules 3 and 4 describe the “linearity” property of the derivative. If we combine them in the following form

\[
\frac{d}{dx}(mu + bv) = m\frac{du}{dx} + b\frac{dv}{dx},
\]

we can see how it is related to the special case of differentiating the straight line function \(y = mx + b\) with \(u(x) = x\) and \(v(x) = 1\).

5. Again let \(u\) and \(v\) be two functions of \(x\). To differentiate their product \(u \cdot v\) we need the quantity \(\Delta(uv)\). We have

\[
\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u \cdot \Delta v + v \cdot \Delta u + \Delta u \cdot \Delta v.
\]

Hence the ratio \(\Delta(uv)\) by \(\Delta x\) becomes

\[
\frac{\Delta(uv)}{\Delta x} = u \cdot \frac{\Delta v}{\Delta x} + v \cdot \frac{\Delta u}{\Delta x} + \frac{\Delta u \cdot \Delta v}{\Delta x}.
\]

Since \(\Delta u\) and \(\Delta v\) are both small, the last summand \(\frac{\Delta u \cdot \Delta v}{\Delta x}\) is also small and we may neglect it. Now, as \(\Delta x\) tends to zero we find the **product rule**

\[
\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.
\]

Here are a couple of examples: (1) to differentiate

\[
f(x) = (x + 1)(x^2 - 3)
\]

we let

\[
u(x) = x + 1 \quad \text{and} \quad v(x) = x^2 - 3.
\]

Thus \(\frac{du}{dx} = 1\) and \(\frac{dv}{dx} = 2x\), and so

\[
\frac{d}{dx}((x + 1)(x^2 - 3)) = (x + 1)(2x) + (x^2 - 3)(1) = 3x^2 + 2x - 3.
\]

(2) To differentiate

\[
f(x) = (x^2 + 3x - 2)(4x - 7x^4)
\]
we let
\[ u(x) = x^2 + 3x - 2 \quad \text{and} \quad v(x) = 4x - 7x^4. \]
Thus \( \frac{du}{dx} = 2x + 3 \) and \( \frac{dv}{dx} = 4 - 28x^3 \), and so

\[
\frac{d}{dx}((x^2 + 3x - 2)(4x - 7x^4)) = (x^2 + 3x - 2)(4 - 28x^3) + (4x - 7x^4)(2x + 3)
\]
\[ = -42x^5 - 105x^4 + 56x^3 + 12x^2 + 24x. \]

**Note.** Often students have problems to recognise a product and do not correctly apply product rule. Remember that products are usually written by ‘juxtaposition’ that is without any symbol between the factors: \( x \sin x \) means \( x \cdot \sin x = x \times \sin x \), \( x^2(5x - 2) \) means \( x^2 \cdot (5x - 2) = x^2 \times (5x - 2) \), \( x \ln x \) means \( x \cdot \ln x = x \times \ln x \) etc.

6. To find a rule for differentiating the quotient \( f(x) = \frac{u(x)}{v(x)} \) of two functions \( u \) and \( v \), we apply product rule to \( u(x) = f(x)v(x) \) resulting in

\[
\frac{du}{dx} = \frac{df}{dx}v + f \frac{dv}{dx} = \frac{df}{dx}v + \frac{u \ dv}{v \ dx}.
\]
Solving this with respect to \( \frac{df}{dx} \) yields the **quotient rule**

\[
\frac{df}{dx} = \frac{d}{dx} \left( \frac{u}{v} \right) = \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2}
\]

For example to differentiate
\[ f(x) = \frac{2x + 3}{x - 4} \]
we let
\[ u(x) = 2x + 3 \quad \text{and} \quad v(x) = x - 4. \]
Then
\[
\frac{d}{dx} \left( \frac{2x + 3}{x - 4} \right) = \frac{(x - 4)(2) - (2x + 3)(1)}{(x - 4)^2} = \frac{-11}{(x - 4)^2}.
\]
Thus far all our derivatives have been with respect to \( x \). However, this doesn’t have to be the case. For instance, if \( u = 3t^2 - 4t \), then it makes sense to talk about \( \frac{du}{dt} \). In fact, \( \frac{du}{dt} = 6t - 4 \). Similarly, if \( y = 4u^2 + 8u^{-3} \), then \( \frac{dy}{du} = 8u - 24u^{-4} \), and so on.
7. Let us next consider the function \( y = (3x^2 + 11x - 7)^{14} \). We could, of course, already find its derivative, but to multiply \( 3x^2 + 11x - 7 \) times itself fourteen times in order to find the derivative using the rules we already would have a great deal of work. To handle this sort of example, it turns out to be much better to think in terms of composition of functions (cf. Section 2.9). In other words, if we took \( f(v) = v^{14} \), where \( v = g(x) = 3x^2 + 11x - 7 \), we could write the composition

\[
y = f(g(x)) = (3x^2 + 11x - 7)^{14}
\]

and think about how to find the derivative of \( f(g(x)) \) in terms of the derivatives of \( f(v) \) and \( g(x) \). For this we write

\[
\frac{\Delta y}{\Delta x} = \frac{\Delta f(v)}{\Delta v} \frac{\Delta v}{\Delta x} = \frac{\Delta f(v)}{\Delta v} \frac{\Delta g}{\Delta x}.
\]

Here we used that \( g = v \) and therefore \( \Delta g = \Delta v \). Now, as \( \Delta x \) and \( \Delta v \) tend to 0 this becomes the chain rule for derivatives of composite functions

\[
\frac{dy}{dx} = \frac{df(v)}{dv} \frac{dg(x)}{dx} = f'(g(x))g'(x).
\]

Notice that \( g'(x) \) is the same thing as \( \frac{dv}{dx} \), but the expression we obtain for \( \frac{dv}{dx} \) needs to have the \( v \) replaced by the full expression for \( g(x) \). This is why it is more precise to write \( f'(g(x)) \) instead of \( \frac{dy}{dx} \).

Thus to find the derivative of \( y = (3x^2 + 11x - 7)^{14} \), we let \( v = 3x^2 + 11x - 7 \), so that \( y = v^{14} \) and \( \frac{dv}{dx} = 6x + 11 \). Then

\[
\frac{dy}{dx} = 14v^{13}(6x + 11) = 14(3x^2 + 11x - 7)^{13}(6x + 11).
\]

**Examples**

Quite complicated functions can be differentiated using the rules given above.

1. If \( y = (2x^2 - 3)^4(2x + 9)^2 \), then by the product rule

\[
\frac{dy}{dx} = (2x^2 - 3)^4 \frac{d}{dx}(2x + 9)^2 + (2x + 9)^2 \frac{d}{dx}(2x^2 - 3)^4.
\]

Now to find the derivative of \( (2x + 9)^2 \), let \( u = 2x + 9 \). Then

\[
\frac{d}{dx}(2x + 9)^2 = \frac{d}{dx}(u^2) = \frac{d}{du}(u^2) \frac{du}{dx} = (2u)(2) = 4(2x + 9).
\]

To find the derivative of \( (2x^2 - 3)^4 \), let \( u = 2x^2 - 3 \). Then

\[
\frac{d}{dx}(2x^2 - 3)^4 = \frac{d}{dx}(u^4) = \frac{d}{du}(u^4) \frac{du}{dx} = (4u^3)(4x) = 16x(2x^2 - 3)^3.
\]
Thus we arrive at
\[
\frac{dy}{dx} = (2x^2 - 3)^4 4(2x + 9) + (2x + 9)^2 16x(2x^2 - 3)^3 \\
= 4(2x^2 - 3)^3 (2x + 9)[(2x^2 + 9) + 4x(2x + 9)] \\
= 4(2x^2 - 3)^3 (2x + 9)(10x^2 + 36x - 3).
\]

2. To differentiate the function
\[
f(x) = \frac{(x^2 - 3)^3}{(2x + 7)^2}
\]
we use the quotient rule
\[
\frac{d}{dx} \left( \frac{(x^2 - 3)^3}{(2x + 7)^2} \right) = \frac{(2x + 7)^2 \frac{d}{dx} ((x^2 - 3)^3) - (x^2 - 3)^3 \frac{d}{dx} ((2x + 7)^2)}{(2x + 7)^4} \\
= \frac{(2x + 7)^2 3(x^2 - 3)^2 2x - (x^2 - 3)^3 4(2x + 7)}{(2x + 7)^4} \\
= \frac{2(2x + 7)(x^2 - 3)^2 [3x(2x + 7) - 2(x^2 - 3)]}{(2x + 7)^4} \\
= \frac{2(x^2 - 3)^2 (4x^3 + 21x + 6)}{(2x + 7)^3}.
\]

3.3 Higher Derivatives

Just as we found the derivative, or first derivative of a function \(y = f(x)\), we can go on and find the derivative of the function \(y = f'(x)\). This is called the second derivative of \(y = f(x)\), and is written as \(f''(x)\) or \(\frac{d^2y}{dx^2}\).

Examples

1. For
\[
f(x) = 4x^5,
\]
then
\[
\frac{dy}{dx} = 20x^4,
\]
and so
\[
\frac{d^2y}{dx^2} = 80x^3.
\]

2. For
\[
f(x) = (2x^2 + 4x - 7)(2x - 3),
\]
3.4. DERIVATIVES OF EXP, LOG AND TRIG FUNCTIONS

then

\[ f'(x) = (2x^2 + 4x - 7)(2) + (2x - 3)(4x + 4) = 12x^2 + 4x - 26. \]

Hence

\[ f''(x) = 24x + 4. \]

If we can differentiate a function to obtain third, fourth, fifth, etc. derivatives, then these are written as \( f'''(x), \ f^{(4)}(x), \ f^{(5)}(x), \ldots, \) or as \( \frac{d^3y}{dx^3}, \ \frac{d^4y}{dx^4}, \ \frac{d^5y}{dx^5}, \ldots. \)

Note that one interpretation of the second derivative is as an acceleration. That is, if \( s(t) \) denotes the distance travelled from a reference point by a moving object at time \( t, \) then \( s'(t) \) is the velocity of the object (= instantaneous rate of change of distance), and \( s''(t) \) is the acceleration of the object (= instantaneous rate of change of velocity).

3.4 Derivatives of Exponential, Logarithmic and Trig Functions

Exponential and Logarithmic Functions

We introduced the exponential functions \( y = f(x) = a^x \) in Section 2.6, and particular mention was made of the base \( e. \)

To compute the derivative of the exponential functions we consider again the quotients

\[ \frac{\Delta f}{\Delta x} = \frac{a^{x+\Delta x} - a^x}{\Delta x}. \]

Using the rules for exponential functions we get

\[ \frac{\Delta f}{\Delta x} = \frac{a^x a^{\Delta x} - a^x}{\Delta x} = a^x \frac{a^{\Delta x} - 1}{\Delta x}. \]

While it is beyond this unit to compute the limits as \( \Delta x \) tends to 0 we notice that in the latter product the first factor depends only on \( x \) and not on the increment \( \Delta x \) whereas the second factor depends only on \( \Delta x \) but not on \( x. \) It follows that the first factor is not affected by the limit for \( \Delta x \) whereas the second factor tends to some number, which does not depend on \( x. \) We may conclude that the derivative of \( f(x) = a^x \) is \( a^x \) multiplied by some unknown number. This number is the slope of the exponential curve at \( x = 0. \) It turns out that this mysterious number is nothing but \( \ln a. \) In particular, if \( a = e \) then \( \ln e = 1 \) and the derivative of \( y = e^x \) is the exponential function \( y = e^x \) itself. This is the reason why \( e \) is special as the base of an exponential function. We summarise

\[ \frac{d}{dx}a^x = a^x \ln a \]
and
\[
\frac{d}{dx} e^x = e^x.
\]

Examples

1. To differentiate
\[ y = e^{x^2+2x} \]
we use the chain rule with \( u = x^2 + 2x \), so \( y = e^u \). Therefore
\[
\frac{dy}{dx} = \left( \frac{dy}{du} \right) \left( \frac{du}{dx} \right) = e^u (2x + 2) = (2x + 2)e^{x^2+2x}.
\]

2. For
\[ y = x^2 e^{2x+3} \]
the product rule gives
\[
\frac{dy}{dx} = \frac{d(x^2)}{dx} e^{2x+3} + x^2 \frac{d(e^{2x+3})}{dx} = 2xe^{2x+3} + x^2 e^{2x+3} (2) = 2xe^{2x+3}(x+1).
\]

3. To differentiate
\[ y = 10^{2x^2+3} \]
the chain rule (let \( u = 2x^2 + 3 \)) gives
\[
\frac{dy}{dx} = \left( \frac{dy}{du} \right) \left( \frac{du}{dx} \right) = (\ln(10)10^u) \left( \frac{d(2x^2 + 3)}{dx} \right) = (\ln(10)10^{2x^2+3})(4x).
\]

Now we can use the derivatives of the exponential function and the chain rule to derive the logarithmic functions. Let \( y = f(x) = \log_a x \) and \( x = g(y) = a^y \) its inverse. Then
\[
a^f(x) = a^{\log_a x} = x.
\]

Differentiating both sides yields
\[
a^f(x) \ln a f'(x) = 1.
\]
Since \( f(x) = \log_a x \) we get
\[
x \ln a f'(x) = 1
\]
and therefore
3.4. DERIVATIVES OF EXP, LOG AND TRIG FUNCTIONS

\[ f'(x) = \frac{d}{dx} \log_a x = \frac{1}{x \ln a}. \]

and, in particular.

\[ \frac{d}{dx} \ln x = \frac{1}{x}. \]

**Examples**

1. To differentiate

\[ y = \ln(3x^2 + 4x - 7) \]

we use the chain rule with \( u = 3x^2 + 4x - 7 \), so \( y = \ln u \). Therefore

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \left( \frac{1}{u} \right) (6x + 4) = \frac{6x + 4}{3x^2 + 4x - 7}.
\]

2. To differentiate

\[ y = \ln(3x^2 + 4x - 7)^4 \]

we use the chain rule with \( u = (3x^2 + 4x - 7)^4 \). Thus

\[
\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \left( \frac{1}{u} \right) 4(3x^2 + 4x - 7)^3(6x + 4) = \frac{4(6x + 4) (3x^2 + 4x - 7)}{3x^2 + 4x - 7}.
\]

An alternate approach is to first use the log rules before differentiating: first we write

\[ y = \ln(3x^3 + 4x - 7)^4 = 4 \ln(3x^2 + 4x - 7). \]

Then

\[
\frac{dy}{dx} = (4) \left( \frac{1}{3x^2 + 4x - 7} \right) (6x + 4) = \frac{4(6x + 4)}{3x^2 + 4 - 7}.
\]

3. For

\[ y = \log_{10}(3x^2 + 4x - 7) \]
we use the chain rule (let \( u = 3x^2 + 4x - 7 \));

\[
\frac{dy}{dx} = \left( \frac{dy}{du} \right) \left( \frac{du}{dx} \right) = \left( \frac{1}{\ln(10)} \right) \left( \frac{1}{u} \right) \frac{d}{dx} (3x^2 + 4x - 7) \\
= \left( \frac{1}{\ln(10)} \right) \left( \frac{1}{3x^2 + 4x - 7} \right) (6x + 4).
\]

3.5 Trigonometric Functions

At the outset we will mention two facts which are needed in our effort to understand how to differentiate \( y = \sin(x) \). One is the “addition formula” from basic trigonometry

\[
\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b).
\]

The other is a pair of special limiting ratios, the value of which is plausible enough to believe, but requires some careful geometric analysis to prove, so we will merely state this fact without proof,

\[
\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \quad \lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0.
\]

Consider the quotient

\[
\frac{\Delta \sin x}{\Delta x} = \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x} = \frac{\sin x \cos \Delta x + \cos x \sin \Delta x - \sin x}{\Delta x}.
\]

Similar to the exponential function we get products in which one factor depends only on \( x \) and the other only on \( \Delta x \):

\[
\frac{\Delta \sin x}{\Delta x} = \sin x \frac{\cos \Delta x - 1}{\Delta x} + \cos x \frac{\sin \Delta x}{\Delta x}.
\]

Even without knowing the limits for \( \Delta x \) tending to 0 we can conclude that the derivative of sine is a combination of sine and cosine. Taking into account the limits stated above we find

\[
\frac{d}{dx} (\sin x) = \cos x.
\]

The derivatives of cosine can be computed by

\[
\frac{d}{dx} \cos x = \frac{d}{dx} \sin(\frac{\pi}{2} - x) = (-1) \cos(\frac{\pi}{2} - x) = -\sin x.
\]

Here we have used chain rule.

Using these results, we can find the derivatives of other functions which involve the trigonometric functions.
3.5. TRIGONOMETRIC FUNCTIONS

Examples

1. For 
\[ y = \tan x = \frac{\sin x}{\cos x} \]
we use by the quotient rule:
\[
\frac{dy}{dx} = \cos^2 x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x) = \cos x \cos x - \sin x (-\sin x) = \cos^2 x + \sin^2 x = \frac{1}{\cos^2 x}.
\]

2. To differentiate 
\[ y = \sin(2x + 3) \]
we use the chain rule with \( y = \sin u, \ u = 2x + 3, \)
\[
\frac{dy}{dx} = \cos(2x + 3) \frac{d}{dx}(2x + 3) = 2 \cos(2x + 3).
\]

3. To differentiate 
\[ y = \sin^2 x \]
recall that \( y = \sin^2 x = (\sin x)^2 \). Thus by the chain rule,
\[
\frac{dy}{dx} = 2 \sin x \frac{d}{dx}(\sin x) = 2 \sin x \cos x.
\]

4. For 
\[ y = \sin(x^2) \]
the chain rule with \( y = \sin u, \ u = x^2 \) gives
\[
\frac{dy}{dx} = \cos(x^2) \frac{d}{dx}(x^2) = 2x \cos(x^2).
\]

5. Differentiating 
\[ y = \sin^2 x + \cos^2 x \]
we get
\[
\frac{d}{dx}(\sin^2 x + \cos^2 x) = \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\cos x)^2 = 2 \sin x \frac{d}{dx}(\sin x) + 2 \cos x \frac{d}{dx}(\cos x) = 2 \sin x \cos x - 2 \cos x \sin x = 0.
\]
(This result isn’t surprising, since we know that $\sin^2 x + \cos^2 x = 1$ for all values of $x$, and the derivative of the constant 1 is 0.)

6. To differentiate

$$y = e^{2x} \sin(2x^2 + 2)$$

we use the product rule combined with the chain rule

$$\frac{dy}{dx} = e^{2x} \frac{d}{dx}(\sin(2x^2 + 2)) + \sin(2x^2 + 2) \frac{d}{dx}(e^{2x})$$

$$= e^{2x} \cos(2x^2 + 2)4x + \sin(2x^2 + 2)2e^{2x}$$

$$= 2e^{2x}(2x \cos(2x^2 + 2) + \sin(2x^2 + 2)).$$

7. Differentiating

$$y = x^2 \cos(3x - 4)$$

is another application of the product rule,

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\cos(3x - 4)) + \cos(3x - 4) \frac{d}{dx}(x^2)$$

$$= -3x^2 \sin(3x - 4) + 2x \cos(3x - 4).$$
4.1 Stationary Points

An important application of differentiation is to find the maximum or minimum values of a function. For instance, we might want to find the amount of drug administered to a person that gives a maximum reaction, or the amount of fertilizer applied to a crop that gives a maximum yield, or the production level required to maximize profit.

Let us start by considering the graph of a function (Figure 4.1):

As can be seen from the graph, at a maximum or minimum point the tangent is horizontal, and so the slope of the tangent is zero. Thus for a maximum or minimum point we have \( f'(x) = 0 \). However, there is another type of point where \( f'(x) = 0 \). This is called a point of horizontal inflexion. A typical example of a point of horizontal inflexion is the point \( x = 0 \) for the function \( y = x^3 \).

To find the maximum or minimum points of a function \( y = f(x) \), we must find all values of \( x \) for which \( f'(x) = 0 \). Such points are called stationary points. Before we go on to discuss a more formal procedure for classifying stationary points as maxima, minima, or inflexions, however, let’s first look at a simple case that will serve as our intuitive model.

**Example:** Recall that the quadratic function \( y = ax^2 + bx + c \) describes a parabola, for which the vertex is a stationary point, since the tangent line at the vertex is always horizontal. To check this, we need only compute \( \frac{dy}{dx} = f'(x) = 2ax + b \) and observe that
\[
2ax + b = 0 \quad \text{when} \quad x = \frac{-b}{2a}.
\]

Since this is indeed the formula for the x-coordinate of the vertex (cf. Section 2.4), we see that the tangent really is horizontal there (and nowhere else). Now recall that the question of whether the vertex is a maximum or minimum is equivalent to asking whether the parabola is concave up or down - which in turn boils down to whether the leading coefficient
A in the equation is a negative or a positive number. In the first case the quadratic function is increasing before attaining the maximum and then becomes decreasing. In other words the slope of the tangents are positive before the vertex and turn negative after the vertex. In the case \( a > 0 \) the slopes of the tangents change from negative to positive when passing through the vertex indicating change from decreasing to increasing.

We can develop this last observation into a useful test for maxima and minima.

**First Derivative Test for Classifying Stationary Points**

Suppose \( c \) is a stationary point of \( y = f(x) \), so that \( f'(c) = 0 \).

(i) \( x = c \) is a **local maximum point** of \( y = f(x) \) if \( f'(x) \) changes sign from positive to negative as \( x \) changes from just below \( c \) to just above \( c \), that is \( f(x) \) turns at \( c \) from increasing to decreasing.

(ii) \( x = c \) is a **local minimum point** of \( y = f(x) \) if \( f'(x) \) changes sign from negative to positive as \( x \) changes from just below \( c \) to just above \( c \), that is \( f(x) \) turns at \( c \) from decreasing to increasing.

(iii) \( x = c \) is a horizontal **inflexion point** of \( y = f(x) \) if \( f'(x) \) doesn’t change sign
4.1. STATIONARY POINTS

as $x$ changes from just below $c$ to just above $c$, that is $f(x)$ remains increasing to decreasing when passing through $c$.

Examples.

(i) Find and classify all stationary points of $f(x) = x^4 - 4x^3 + 7$.

Solution. For a stationary point, $0 = f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$. This equation is satisfied if and only if either the factor $4x^2$ is zero or the factor $x - 3$ is zero, so either $x = 0$ or $x = 3$. Each of these values corresponds to a stationary point on the graph of $y = x^4 - 4x^3 + 7$, for which we also need the $y$-values. Thus $f(0) = 7$ and $f(3) = 3^4 - 4.3^3 + 7 = 81 - 108 + 7 = -20$, which means that the stationary points have coordinates $(0, 7)$ and $(3, -20)$.

Consider first the stationary point $(0, 7)$. If $x$ is just less than 0, then $x^2 > 0$ and $x - 3 < 0$. Therefore $f'(x) < 0$. On the other hand, if $x$ is just greater than 0, then again $x^2 > 0$ and $x - 3 < 0$. Thus $f'(x)$ doesn’t change sign, and so $(0, 7)$ is a point of inflexion.

Next consider the stationary point $(3, -20)$. If $x$ is just less than 3, then $x^2 > 0$ and $x - 3 < 0$. Therefore $f'(x) < 0$. On the other hand, if $x$ is just greater than 3, then $x^2 > 0$ and $x - 3 > 0$. Therefore $f'(x) > 0$. Thus $f'(x)$ changes sign from negative to positive, and so $(3, -20)$ is a local minimum point.

(ii) The size of a bacteria population that is introduced to a nutrient grows according to the formula

$$N(t) = 5,000 + \frac{30,000t}{100 + t^2},$$

where the time $t$ is measured in hours. Determine the maximum size of this population.

Solution. Using the quotient rule we see that

$$\frac{dN}{dt} = \frac{30,000(100 + t^2) - 30,000t(2t)}{(100 + t^2)^2} = \frac{30,000(100 - t^2)}{(100 + t^2)^2}.$$

Thus $\frac{dN}{dt} = 0$ if and only if the numerator is zero, i.e., when $t^2 = 100$, or $t = \pm 10$. Since the problem requires $t$ to be positive, we can ignore $t = -10$. Therefore $t = 10$ is the only stationary point of $N$.

Now if $t < 10$, then $100 - t^2 > 0$, and so $\frac{dN}{dt} > 0$. If $t > 10$, then $100 - t^2 < 0$, and so $\frac{dN}{dt} < 0$. Thus at $t = 10$ we see that $\frac{dN}{dt}$ changes from being positive to being negative. This means $t = 10$ gives a maximum value for $N$. Hence $N(10) = 5,000 + \frac{300,000}{100 + 100} = 6,500$ is the maximum size of the bacteria population, and this occurs after 10 hours.

(iii) After a drug has been administered, its reaction at time $t$ is given by

$$R(t) = t^2 e^{-t}.$$
At what time is the reaction a maximum?

**Solution.** Using the product rule,

\[
\frac{dR}{dt} = (2t) e^{-t} + t^2 e^{-t}(-1) = te^{-t}(2 - t).
\]

Now we can assume \( t > 0 \), and also the exponential function is never zero. Thus \( \frac{dR}{dt} = 0 \) only at \( t = 2 \). Suppose \( t \) is just less than 2. Then \( 2 - t > 0 \), and so \( \frac{dR}{dt} > 0 \). On the other hand, if \( t \) is just greater than 2, then \( 2 - t < 0 \), and \( \frac{dR}{dt} < 0 \). Therefore \( t = 2 \) is a maximum point for \( R \), and so this is when the reaction of the drug is a maximum.

**Remark:** At this point it is appropriate to point out that the terms “maximum” and “minimum” as we have used them are strictly local in nature. The first derivative test (like the second derivative test, which we will meet shortly) usually only tells us whether a stationary point is the highest or lowest point in some small “neighbourhood” of the stationary point in question. It does not tell us whether the graph of a given function has a maximum or minimum value overall.

To find global maxima or minima we need to compare all local minima and maxima as well as the end points of the domain and points where the function has no derivative.

### 4.2 Derivative of sine and cosine Revisited

Looking back at the graphs of \( \sin(x) \) and \( \cos(x) \) in Figure 2.17 you can see that, for example, \( y = \sin(x) \) has many (in fact infinitely many) stationary points. The maxima and minima occur at all the odd multiples

\[
x = (2k + 1)\frac{\pi}{2}, \quad k = 0, \pm 1, \pm 2, \ldots ,
\]

and hence \( \frac{dy}{dx} = 0 \) at each of these points. In fact, the places where the derivative of \( \sin(x) \) vanishes correspond exactly to the values of \( x \) where \( \cos(x) \) vanishes. This indicates that \( \cos(x) \) and the derivative of \( \sin(x) \) are really the same thing. Our next observation, from looking carefully at the graph of \( y = \sin(x) \), is that the tangent lines at various points on the graph have their steepest slopes at all \( x \) where \( \sin(x) = 0 \), i.e., all \( x \) corresponding to multiples

\[
k\pi, \quad k = 0, \pm 1, \pm 2, \ldots .
\]

These points are therefore the places where \( \frac{dy}{dx} \) has its maxima and minima. Now \( y = \cos(x) \) also takes its maximum and minimum \( y \)-values, namely \( y = \pm 1 \), at these same values of \( x \).
4.3 Second Derivative Test for Classifying Stationary Points

Returning briefly to our model of the general parabola $y = ax^2 + bx + c$, recall that the vertex is a maximum if the parabola is concave down ($a < 0$) and a minimum if the parabola is concave up ($a > 0$). We have already related this to the change of the slope of the parabola, corresponding to the first derivative $\frac{dy}{dx}$. Another point of view is to notice that the second derivative $\frac{d^2y}{dx^2} = 2a$, hence we may say that the vertex is a maximum if $\frac{d^2y}{dx^2} < 0$ at $x = -\frac{b}{2a}$, and a minimum if $\frac{d^2y}{dx^2} > 0$ at $x = -\frac{b}{2a}$. In the first case the first derivative is increasing hence it has to change from negative to positive as it passes through the stationary point. If the second derivative is negative then the first derivative is decreasing, so it has to change from positive to negative as it passes the stationary point.

The corresponding statement for a general function $y = f(x)$ is called the **second derivative test**: 

Suppose $c$ is a stationary point of $y = f(x)$, so that $f'(c) = 0$.

(i) If $f''(c) < 0$, then $c$ is a local maximum point of $y = f(x)$.

(ii) If $f''(c) > 0$, then $c$ is a local minimum point of $y = f(x)$.

(iii) If $f''(c) = 0$, then no conclusion can be drawn.

The only advantage of the second derivative test is that we only need to look at the stationary point itself, not at a neighbourhood. This can be readily done if the second derivative is easy to compute.

In most cases the first derivative test is preferable, since it is always conclusive and only requires the first derivative.

**Example.** Find and classify all stationary points of $f(x) = x^3 - 3x + 4$.

**Solution.**

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) \quad \text{and} \quad f''(x) = 6x.$$ 

The stationary points occur when $x^2 - 1 = 0$, or $x = \pm 1$ (hence $f(1) = 2$ and $f(-1) = 6$). Since $f''(1) = 6 > 0$, by the second derivative test $(1, 2)$ is a local minimum. Since $f''(-1) = -6 < 0$, by the second derivative test $(-1, 6)$ is a local maximum.
Chapter 5

Integration

5.1 Indefinite Integration

Suppose the velocity $v(t)$ of a moving object at time $t$ is known, and that we want to find $s(t)$, the distance travelled up to time $t$. In order to find $s(t)$ we have to answer the question: ‘What is the function whose derivative is $v(t)$?’ In other words, we have to reverse the differentiation. This reversal of differentiation is also needed if we are given the rate $p'(t)$ at which a population is growing, and wish to calculate $p(t)$, the population size at time $t$; or if we are given the rate at which a certain product is being produced in a chemical reaction, and wish to know the amount of product formed up to a certain point in time.

The reversal of differentiation is called integration. Suppose $f(x)$ is a given function, and $F(x)$ is a function whose derivative is $f(x)$, i.e.

$$\frac{dF}{dx} = f(x).$$

Then $F(x)$ is called a primitive or anti-derivative of $f(x)$. For example, $\frac{x^2}{2}$ is a primitive of $x$ since

$$\frac{d}{dx} \left( \frac{x^2}{2} \right) = \frac{1}{2}(2x) = x.$$

If $F(x)$ is a primitive of $f(x)$ then for any constant $C$ the function $F(x) + C$ is also a primitive because

$$\frac{d}{dx} (F(x) + C) = \frac{dF}{dx}(x) + \frac{d}{dx} C = f(x) + 0 = f(x).$$

The set of all primitives of $f(x)$ is called the indefinite integral and denoted by

$$\int f(x) \, dx = F(x) + C.$$
Here the symbol \( \int \) is called the ‘integral sign’, and \( f(x) \) is the **integrant**. The symbol \( dx \) tells us that the integration is with respect to \( x \).

\( C \) is called the **constant of integration** and omitting it is a mistake (and incomplete solution of the indefinite integration.)

For example,

\[
\int x^2 \, dx = \frac{x^3}{3} + C \quad \text{and} \quad \int t^2 \, dt = \frac{t^3}{3} + C
\]

(this second integral is with respect to \( t \)).

We may summarise with the remark that, in effect, “differentiation **undoes** what integration does to a function”, and vice versa. More precisely,

\[
\frac{d}{dx} \int f(x) \, dx = f(x) , \quad \text{and} \quad \int \frac{d}{dx} f(x) \, dx = f(x) + C .
\]

Notice that the second combination doesn’t quite return to its starting point \( f(x) \), because indefinite integration creates constants \( C \), while differentiation always destroys constants.

### Standard Integrals

The following standard integrals should be known:

1. \( \int x^n \, dx = \frac{x^{n+1}}{n+1} + c \) for \( n \neq -1 \).
2. \( \int \frac{1}{x} \, dx = \ln |x| + c \) \[1\]
3. \( \int e^x \, dx = e^x + c \)
4. \( \int e^{kx} \, dx = \frac{1}{k} e^{kx} + c \)
5. \( \int \cos x \, dx = \sin x + c \)
6. \( \int \sin x \, dx = -\cos x + c \)

\[1\] Note the absolute value sign. Without it the antiderivative would be valid only for positive \( x \).
5.1. INDEFINITE INTEGRATION

7. $\int \cos kx \, dx = \frac{1}{k} \sin kx + c$

8. $\int \sin kx \, dx = -\frac{1}{k} \cos kx + c$

All these standard integrals can be proved by showing the derivative of the function on the right-hand side is equal to the integrand on the left-hand side. For example, formula (7) is correct since

$$\frac{d}{dx} \left( -\frac{1}{k} \cos(kx) \right) = -\frac{1}{k} \frac{d}{dx} \cos(kx) = \left( -\frac{1}{k} \right) (-\sin(kx))(k) = \sin(kx).$$

Properties of integrals

In working out integrals, we shall also make use of the following rules: Let $F$ and $G$ be primitives of $f$ and $g$ respectively. Then

1. $\int kf(x) \, dx = kF(x) + C$ when $k$ is constant.

2. $\int f(kx + a) \, dx = \frac{1}{k} F(kx + a) + C$ when $k \neq 0$ and $a$ are constants.

3. $\int (f(x) + g(x)) \, dx = F(x) + G(x) + C$

4. $\int (f(x) - g(x)) \, dx = F(x) - G(x) + C$

These rules follow from the corresponding rules for differentiation.

Examples

1. $\int 5 \sin(7x) \, dx = 5 \int \sin(7x) \, dx = 5 \left( -\frac{1}{7} \cos(7x) \right) + c = -\frac{5}{7} \cos(7x) + c.$
2. 
\[
\int \left( \frac{t^2 + 3}{t} - 2e^{5t} \right) dt = \int \left( t + \frac{3}{t} - 2e^{5t} \right) dt \\
= \int t \, dt + 3 \int \frac{1}{t} \, dt - 2 \int e^{5t} \, dt \\
= \frac{t^2}{2} + 3 \ln |t| - \frac{2}{5} e^{5t} + c.
\]

3. 
\[
\int \frac{1}{L - t} \, dt = -\frac{1}{1} \ln |L - t| + c = -\ln |L - t| + c.
\]

4. We note that there are no rules which give simple ways of integrating either the product or quotient of functions. If there is a product to be integrated, one possible way of performing the integration is to multiply out the integrand. For instance,
\[
\int \left( x - \frac{4}{x^2} \right)^2 \, dx = \int \left( x^2 - \frac{8}{x} + \frac{16}{x^4} \right) \, dx \\
= \int x^2 \, dx - 8 \int \frac{1}{x} \, dx + 16 \int x^{-4} \, dx \\
= \frac{x^3}{3} - 8 \ln |x| - \frac{16}{3} x^{-3} + c.
\]

More Examples

1. According to a model, the rate of growth of a colony of fruit flies at time \( t \geq 1 \) is equal to \( \frac{10(t + 2)}{t} \). When \( t = 1 \) there are 20 flies in the colony. Calculate the number of flies at a general value of \( t > 1 \).

Solution. Let the population of flies at time \( t \) be given by \( P(t) \). Then 
\[
P'(t) = \frac{10(t + 2)}{t},
\]
so
\[
P(t) = \int \frac{10(t + 2)}{t} \, dt \\
= 10 \int \left( 1 + \frac{2}{t} \right) \, dt \\
= 10 \int 1 \, dt + 20 \int \frac{1}{t} \, dt \\
= 10t + 20 \ln(t) + C.
\]
In this case we may compute the specific value of \( C \) since we know \( P(1) = 20 \). Substituting \( t = 1 \) in the formula for \( P(t) \) gives

\[
20 = P(1) = 10(1) + 20\ln(1) + C = 10 + C,
\]
so \( C = 10 \). Thus the formula for \( P(t) \) is

\[
P(t) = 10t + 20\ln(t) + 10 = 10(1 + t + 2\ln(t)).
\]

2. According to a model, the velocity of a migrating goose is given by

\[
v(t) = 20 - \frac{t}{3} \text{ mph},
\]
where \( t \) is the time measured in hours and starts at dawn with \( t = 0 \). How many miles has the goose travelled up to time \( t \)? How far does the goose fly in a 12-hour day?

**Solution.** Let \( s(t) \) denote the distance in miles travelled by the goose up to time \( t \). Then

\[
v(t) = s'(t) = 20 - \frac{t}{3},
\]
so

\[
s(t) = \int (20 - \frac{t}{3}) \, dt = \int 20 \, dt - \frac{1}{3} \int t \, dt = 20t - \frac{t^2}{6} + c.
\]

Now \( s(0) = 0 \), so

\[
0 = 20(0) - \frac{(0)^2}{6} + c = c.
\]

Hence

\[
s(t) = 20t - \frac{t^2}{6}.
\]

In particular, the distance travelled in a 12-hour day is

\[
s(12) = (20)(12) - \frac{(12)^2}{6} = 216 \text{ miles}.
\]

5.2. **Definite Integration**

Suppose that \( f(x) \geq 0 \) for \( a \leq x \leq b \).

The area between the graph of \( y = f(x) \) and the \( x \)-axis for \( a \leq x \leq b \) will be denoted by

\[
\int_{a}^{b} f(x) \, dx,
\]
CHAPTER 5. INTEGRATION

Figure 5.1: The Area under a curve.

This quantity is called a definite integral, and the numbers \( a \) and \( b \) are called the limits of integration. The question remains how to find this area.

The problem of finding \( \int_{a}^{b} f(x) \, dx \) can be solved by using the Fundamental Theorem of Calculus. This theorem says that if \( F(x) \) is any primitive of \( f(x) \), then

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a).
\]

(We note that the symbol \([F(x)]_{a}^{b}\) or \( F(x)_{a}^{b}\) is often used to denote \( F(b) - F(a) \).)

Examples

1. Find the area between the graph of \( y = x \) and the \( x \)-axis for \( 0 \leq x \leq 5 \).

Solution.

The area is given by

\[
\int_{0}^{5} x \, dx = \left[ \frac{x^2}{2} \right]_{0}^{5} = \frac{5^2}{2} - \frac{0^2}{2} = \frac{25}{2}.
\]

Note that if any other primitive is used, the answer for the area remains the same.
5.2. **DEFINITE INTEGRATION**

For example, suppose we take

\[ F(x) = \frac{x^2}{2} + 17 = \]

then

\[ [F(x)]_0^5 = \left[ \frac{x^2}{2} + 17 \right]_0^5 = \left( \frac{25}{2} + 17 \right) - \left( \frac{0}{2} + 17 \right) = \frac{25}{2}. \]

Thus when evaluating definite integrals, the constant of integration doesn’t matter.

Also note that in this example the required area is that of a triangle with base 5 and height 5. Hence the area is equal to

\[ \frac{1}{2} \text{ (base) (height)} = \frac{1}{2} (5)(5) = \frac{25}{2}, \]

which agrees with the result obtained by integration.

2. Find the area between the graph of \( y = 2x^2 + x + e^{-2x} \) and the \( x \)-axis for \( 1 \leq x \leq 3 \).

**Solution.**

\[
\begin{align*}
\text{Area} &= \int_1^3 (2x^2 + x + e^{-2x}) \, dx \\
&= \left[ \frac{2}{3}x^3 + \frac{x^2}{2} - \frac{1}{2} e^{-2x} \right]_1^3 \\
&= \left( \frac{2}{3}(3)^3 + \frac{1}{2}(3)^2 - \frac{1}{2} e^{-6} \right) - \left( \frac{2}{3} + \frac{1}{2} - \frac{1}{2} e^{-2} \right) \\
&= \frac{64}{3} + \frac{1}{2} e^{-2} - \frac{1}{2} e^{-6}.
\end{align*}
\]
5.3 More on Areas

Definite integrals can be calculated even if the function $f$ is not always non-negative for $a \leq x \leq b$. For example,

$$
\int_{-1}^{3} (x^2 - 4) \, dx = \left[ \frac{x^3}{3} - 4x \right]_{-1}^{3} = \left( \frac{3^3}{3} - 12 \right) - \left( \frac{-1}{3} + 4 \right) = -\frac{20}{3}.
$$

See Figure 5.3.

![Figure 5.3: Areas for a function with negative values.](image)

However, in this case the value doesn’t represent an area. This is because $x^2 - 4$ is negative for some of the interval $-1 \leq x \leq 3$. The answer of $-\frac{20}{3}$ represents the shaded area, with area below the $x$-axis counted as being negative. If we want the actual shaded area, with all areas being counted as positive, we have to calculate $\int_{2}^{3} (x^2 - 4) \, dx$ and then subtract $\int_{-1}^{2} (x^2 - 4) \, dx$, i.e.

$$
\text{shaded area} = \left[ \frac{x^3}{3} - 4x \right]_{2}^{3} - \left[ \frac{x^3}{3} - 4x \right]_{-1}^{2} = \left\{ (9 - 12) - \left( \frac{8}{3} - 8 \right) \right\} - \left\{ \left( \frac{8}{3} - 8 \right) - \left( \frac{-1}{3} + 4 \right) \right\} = \frac{34}{3}.
$$

Definite integrals can also be used to find the area between two curves, (see Figure 5.4). Suppose $f(x) \geq g(x)$ for $a \leq x \leq b$. Then the area between the curves is given by

$$
\text{Area} = \int_{a}^{b} (f(x) - g(x)) \, dx.
$$
Example. Find the area between the graphs of $y = x^2$ and $y = 3x$ for $1 \leq x \leq 3$ (Figure 5.5).

Solution. We first see which function is the greater\footnote{If we made the wrong choice the area comes out with a negative sign. In this case the absolute value of this result is the correct area.}. The graphs of the functions intersect when $x^2 = 3x$. Thus $0 = x^2 - 3x = x(x - 3)$, so $x = 0$ or $3$. For $0 \leq x \leq 3$, we have
$3x \geq x^2$; and so the area between the graphs for $1 \leq x \leq 3$ is

$$
\int_1^3 (3x - x^2) \, dx = \left[ \frac{3}{2}x^2 - \frac{x^3}{3} \right]_1^3 = \left( \frac{27}{2} - 9 \right) - \left( \frac{3}{2} - \frac{1}{3} \right) = \frac{10}{3}.
$$

5.4 The natural logarithm revisited

By this stage of our tour through calculus, we have met the “Fundamental Theorem” that relates the concept of definite integration (involving area calculation as a special case) to the problem of “differentiating backwards”, or anti-differentiation. One of our most useful basic rules in the game of anti-differentiation is the reverse power rule (cf. Section 5.1), which says

$$
\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C.
$$

This rule applies, like the ordinary power rule for forwards differentiation, to a power function $x^n$ for any real number $n$, except $n = -1$, since in this case the fraction $\frac{1}{n+1}$ isn’t defined. So how do we find $\int \frac{1}{x} \, dx$?

The Fundamental Theorem of Calculus tells us that every ‘reasonable’ function has an indefinite integral. Define

$$
F(b) := \int_1^b \frac{1}{x} \, dx.
$$

It may come as a surprise that this can be used as a definition of $\ln(b)$, for any number $b > 0$. We write

$$
\ln(b) := \int_1^b \frac{1}{x} \, dx.
$$

The natural logarithm function was introduced back in Chapter 2, but its place among the logarithms has until now been something of a mystery, especially because its base, the irrational number $e = 2.718 \ldots$, seems to have come from nowhere. But if this is how $\ln(b)$ is really defined, what does it have to do with logarithms, and why is its base $e$?

We will answer this question by showing that the formula $\int_1^b \frac{1}{x} \, dx$ has properties just like a logarithm, though this isn’t obvious when you look at the definition.

(1) We begin by recalling that any logarithm must have the property

$$
\log_a(1) = 0.
$$

If we set $b = 1$ in our definition, we see that

$$
\ln(1) = \int_1^1 \frac{1}{x} \, dx = 0,
$$

simply because the definite integral has no width if we integrate from 1 to 1.
Another basic property of all logarithms is that \( \log_a(a) = 1 \).

In other words, the “base” of a logarithm is the particular number whose logarithm equals 1 (e.g., \( \log_{10}(10) = 1 \), \( \log_2(2) = 1 \)). This is actually how we define \( e \): it is the particular number for which

\[
\ln(e) = \int_1^e \frac{1}{x} \, dx = 1.
\]

We can work out the value of this number to any order of accuracy, but the fact is that its decimal expansion never terminates or repeats itself, which is why we call it an “irrational” number.

It is necessary that

\[
\log_a(bc) = \log_a(b) + \log_a(c),
\]

\[
\log_a\left(\frac{b}{c}\right) = \log_a(b) - \log_a(c),
\]

\[
\log_a(b^n) = n \log_a(b).
\]

Showing that our definition of \( \ln(b) \) really satisfies these properties is a bit more challenging, but we will carry it out for the first of these relations, and indicate that a similar argument will do the job for each of the other two. Recall from Section 5.1 that for any function \( f(x) \)

\[
\frac{d}{dx} \left( \int f(x) \, dx \right) = f(x),
\]

and in particular, our definition tells us that

\[
\frac{d}{dx} \left( \int \frac{1}{x} \, dx \right) = \frac{1}{x}.
\]

Now write \( v = bu \), so that

\[
\ln(v) = \ln(bu) = \int_1^{bu} \frac{1}{x} \, dx.
\]

Then we apply the chain rule, treating \( b \) as a constant, and \( u \) and \( v \) as variables:

\[
\frac{d}{du} \left( \ln(bu) \right) = \frac{d}{du} \left( \ln(v) \right) = \frac{d}{dv} \left( \ln(v) \right) \frac{dv}{du} = \frac{1}{v} \frac{dv}{du} = \frac{1}{bu} = \frac{1}{u}.
\]

This tells us that

\[
\frac{d}{du} \left( \ln(bu) \right) = \frac{d}{du} \left( \ln(u) \right),
\]

but what does it mean when two functions, \( f(u) = \ln(bu) \) and \( g(u) = \ln(u) \) have the same derivative? Clearly the function \( f(u) - g(u) \) has derivative equal to zero for all \( u \), so it must be equal to a constant, \( C \). Hence

\[
\ln(bu) = \ln(u) + C.
\]
This relation is true for any \( u \), so let’s see what it says for \( u = 1 \):

\[
\ln(b) = \ln(1) + C = C,
\]

since \( \ln(1) = 0 \). In conclusion

\[
\ln(bu) = \ln(u) + \ln(b),
\]

which verifies the basic property of logarithms via a shrewd use of the chain rule.

We can write our definition of \( \ln(x) \) as an indefinite integral also:

\[
\int \frac{1}{x} \, dx = \ln(x) + C,
\]

provided the variable \( x \) is always understood to be positive. Like all logarithms, \( \ln(x) \) only makes sense for \( x > 0 \), and this is reflected in the definition, since the function \( \frac{1}{x} \) has a “vertical asymptote” at \( x = 0 \).

## 5.5 The Average Value of a Function

The average \( \bar{a} \) of \( n \) numbers \( a_1, a_2, \ldots, a_n \) is their sum divided by \( n \), i.e.

\[
\bar{a} = \frac{a_1 + a_2 + \cdots + a_n}{n}.
\]

This is equivalent to

\[
n\bar{a} = a_1 + a_2 + \cdots + a_n,
\]

that is \( \bar{a} \) is the number whose \( n \)-fold equals the sum of the \( n \) numbers.

For a function \( f \) defined on some interval \( a \leq x \leq b \) the average is the value of the constant function \( \bar{f} \) whose definite integral over the interval \( a \leq x \leq b \) is the same as the definite integral of \( f \) over \( a \leq x \leq b \). This can be expressed by

\[
\int_a^b \bar{f} \, dx = \bar{f}(b-a) = \int_a^b f(x) \, dx.
\]

It follows

\[
\bar{f} = \frac{1}{b-a} \int_a^b f(x) \, dx.
\]

If \( f \) is positive for \( a \leq x \leq b \), then \( \bar{f} \) can be geometrically interpreted as the height of the rectangle whose base is the interval \( a \leq x \leq b \), and whose area is the same as the area under the graph of \( f \). See Figure 5.6.

**Example.** Find the average value of \( f(x) = \sin(x) \) for \( 0 \leq x \leq \pi \).
Solution. See Figure 5.7.

\[ \bar{f} = \frac{1}{\pi - 0} \int_{0}^{\pi} \sin(x) \, dx \]
\[ = \frac{1}{\pi} \left[ -\cos(x) \right]_{0}^{\pi} \]
\[ = \frac{1}{\pi} \left\{ -\cos(\pi) - (-\cos(0)) \right\} \]
\[ = \frac{1}{\pi} (-(-1) - (-1)) \]
\[ = \frac{2}{\pi} \]
Figure 5.7: The average value of $\sin(x)$, $0 \leq x \leq \pi$. 
Chapter 6

Functions of Several Variables

6.1 Partial Derivatives

A function of two (or more) variables is a rule \( f \) that assigns a single number \( z \) to a pair (or more) of numbers \((x, y)\). We write

\[
z = f(x, y).
\]

For most of this course we have referred to \( x \) as an independent variable, and \( y \) as a dependent one, meaning that \( x \) can be chosen freely from a domain of possible values, whereas \( y \) depends on the output obtained from the function. For functions of, say, two variables, the domain of free choices, or input, is made up of pairs \((x, y)\), and the dependent output is now denoted by \( z \). Examples of such functions could be

\[
\begin{align*}
z &= 3x - 2y + 10, \\
z &= 2x^2 + 3y^2, \\
z &= \ln(\sqrt{4x - 3y}).
\end{align*}
\]

Whereas the graph of a function \( y = f(x) \) is just a curve in the two-dimensional coordinate plane, the graphs of functions \( z = g(x, y) \) require a space of three dimensions, to display the variation of \( z \) depending on \( x, y \) in the form of a surface. Such graphs can be difficult to draw in practice, but two of the most basic examples have been represented for you in the next section. In the same way that points on the graph of \( y = f(x) \) are written as a pair \((x, f(x))\), we can also denote points on the graph of \( z = g(x, y) \) as triples \((x, y, g(x, y))\) in three-dimensional space.

The simplest functions of two variables are linear functions:

\[
z = mx + ny + b.
\]
We have now two slopes \( m \) and \( n \) corresponding to the two variables \( x, y \) and one \( z \)-intercept \( b \) which is the value of the function for \( x = y = 0 \).

The graphs of linear functions are planes in 3-dimensional space. Their position is determined by the slopes and the \( z \)-intercept, similar to the straight lines as graphs of linear functions of one variable. In particular, the plane is horizontal if both slopes are zero, that is \( z = b \) is a constant function.

Let \( z = f(x, y) \) be a function of two variables. Similar to the linear functions having two slopes, \( f(x, y) \) has now two different kinds of derivatives: If \( y \) is kept fixed and only \( x \) varied then \( z \) becomes a function of \( x \). The derivative of this function holding \( y \) constant is called the “\textbf{partial derivative} of \( z \) with respect to \( x \)” and is formally defined in the same way as the ordinary derivative, but with a new notation

\[
\frac{\partial z}{\partial x} := \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}.
\]

Similarly, we can define the partial derivative of \( f(x, y) \) in the “\( y \)-direction” (i.e., holding \( x \) fixed) by the formula

\[
\frac{\partial z}{\partial y} := \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.
\]

For the linear function \( z = f(x, y) = mx + ny + b \) we find

\[
\frac{\partial f}{\partial x} = m, \quad \frac{\partial f}{\partial y} = n.
\]

The partial derivatives of \( f(x, y) \) at a point are the slopes of the tangent plane to the surface \( z = f(x, y) \) at a given point.

Just as for ordinary derivatives, \( \frac{\partial z}{\partial x} \) defines a new function of two variables \( (x, y) \), and so also does \( \frac{\partial z}{\partial y} \). These formulae may appear complicated, but partial differentiation is no more difficult in practice than ordinary differentiation. The trick is to “close one eye”, in a sense, and always treat one variable (either the \( x \) or the \( y \), depending on the direction of differentiation) as if it is a constant.

**Computation of partial derivatives**

The rules for ordinary derivatives apply in the same way to partial derivatives. In particular,

1. \[
\frac{\partial}{\partial x} (f(x, y) \pm g(x, y)) = \frac{\partial f(x, y)}{\partial x} \pm \frac{\partial g(x, y)}{\partial x}
\]

2. \[
\frac{\partial}{\partial y} (f(x, y) \pm g(x, y)) = \frac{\partial f(x, y)}{\partial y} \pm \frac{\partial g(x, y)}{\partial y}
\]
6.2 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

3. \( \frac{\partial}{\partial x} kf(x, y) = k \frac{\partial f(x, y)}{\partial x} \), \( \frac{\partial}{\partial y} kf(x, y) = k \frac{\partial f(x, y)}{\partial y} \)

4. \( \frac{\partial}{\partial x} x^m y^n = mx^{m-1} y^n \), \( \frac{\partial}{\partial y} x^m y^n = n x^m y^{n-1} \).

Example. Let \( z = x^2 + y^3 + \sin(xy) \). In order to calculate \( \frac{\partial z}{\partial x} \), it may help the reader to replace \( y^3 \) by the letter \( C \) in this formula, and \( y \) by the letter \( D \), since they are both held constant, i.e.,

\[
\frac{\partial z}{\partial x} = 2x + D \cos(Dx).
\]

Notice that \( \frac{d(\sin(Dx))}{dx} = D \cos(Dx) \) if we apply the chain rule with \( v = Dx \), while \( \frac{d(x^2+C)}{dx} = 2x \). The important thing to understand is that once \( y \), and any powers of \( y \), have been re-assigned as constants, then partial differentiation just becomes ordinary differentiation with respect to \( x \), so we get

\[
\frac{\partial z}{\partial x} = 2x + D \cos(Dx) .
\]

But \( D \) was only introduced as a device to help us “close one eye” to the \( y \)-variable, so we should re-instate \( y \) in the final answer

\[
\frac{\partial z}{\partial x} = 2x + y \cos(xy) .
\]

A similar game can be played with \( x \) when we calculate \( \frac{\partial z}{\partial y} \), hence we may write

\[
z = C + y^3 + \sin(Dy) ,
\]

and find

\[
\frac{d(C + y^3 + \sin(Dy))}{dy} = 3y^2 + D \cos(Dy) .
\]

Here \( D \) “closes the eye” to the \( x \)-variable, so we conclude

\[
\frac{\partial z}{\partial y} = 3y^2 + x \cos(xy) .
\]

The usual rules for differentiation, such as the product and chain rules, also work for partial derivatives.

6.2 Maxima and Minima of Functions of Two Variables

Maxima and minima

Suppose a maximum of a function \( f(x, y) \) of two variables occurs at the point \((x_0, y_0)\). If we keep \( y \) fixed at \( y_0 \) then the function \( f(x, y_0) \) of the one variable \( x \) will have a maximum
CHAPTER 6. FUNCTIONS OF SEVERAL VARIABLES

at the point \( x_0 \). For this to be so the derivative must vanish at \( x_0 \), i.e

\[
\frac{\partial f}{\partial x}(x_0, y_0) = 0.
\]

Similarly, if we keep \( x \) fixed at \( x_0 \) the function \( f(x_0, y) \) will have a maximum at \( y_0 \) and

\[
\frac{\partial f}{\partial y}(x_0, y_0) = 0.
\]

Thus at a maximum (or minimum) both partial derivatives \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) are zero. Points where both partial derivatives are zero are called \textit{stationary points} of the function.

Recall from Chapter 4 that our intuition about maxima and minima of functions \( y = f(x) \) comes from thinking about the example of the parabola \( y = ax^2 + bx + c \), where \( a > 0 \) means that the vertex is a minimum, and \( a < 0 \) means that the vertex is a maximum. On the other hand, \( y = x^3 \) provides the basic example of a function with a stationary point at \( x = 0 \), which is neither a maximum nor a minimum, but a \textit{horizontal inflection}. The first and second examples below provide the corresponding models of behaviour at stationary points for functions \( z = f(x, y) \).

\textbf{Examples}

1. \( z = x^2 + y^2 \).

\[
\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y.
\]

If both partial derivatives are zero then \( x = 0 \) and \( y = 0 \) (and hence \( z = 0 \)). So there is only one stationary point on the graph at \((0, 0, 0)\). We have sketched the graph of this function below (upper half of Figure 6.1). Notice that \( z = x^2 + y^2 \) is greater than zero for all pairs \((x, y)\) different from \((0, 0)\), so the stationary point must be a minimum, as it is with the case of a concave-up parabola. If we were to take instead the example

\[
z = -x^2 - y^2,\]

we would find that the point \((0, 0, 0)\) is now a maximum, as in the case of a concave-down parabola (see lower half of Fig. 6.1).

2. \( z = x^2 - y^2 \).

\[
\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = -2y.
\]

If both partial derivatives are zero then \( x = 0 \) and \( y = 0 \). So again there is only one stationary point on the graph at \((0, 0, 0)\). Now \((x, y) = (0, 0)\) implies \( z = 0 \), but \( y = 0 \) implies \( z = x^2 > 0 \) when \( x \neq 0 \). So there are points close to \((0, 0)\) with larger
6.2. MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

$z$ values and $(0, 0, 0)$ can’t be a maximum. Also $x = 0$ implies $z = -y^2 < 0$ when $y \neq 0$. So there are points close to $(0, 0)$ with smaller $z$ values and $(0, 0, 0)$ can’t be a minimum. In fact $(0, 0, 0)$ is a kind of stationary point called a saddle point (see Fig. 6.2 below).

3. $z = x^2 + y^3 + x - 3y$.

$$\frac{\partial z}{\partial x} = 2x + 1, \quad \text{and} \quad \frac{\partial z}{\partial y} = 3y^2 - 3.$$  

In this example we see that the general problem of finding stationary points of a function $z = f(x, y)$ boils down to solving a system of two simultaneous equations. Here we have to solve

$$2x + 1 = 0$$

and $3y^2 - 3 = 0$.

The solution to the first equation is $x = -\frac{1}{2}$, while the second equation has $y^2 = 1$, or $y = \pm 1$. The fact that we have obtained one $x$-value and two $y$-values means that we have two stationary points: when $x = -\frac{1}{2}$, $y = +1$, then

$$z = (-\frac{1}{2})^2 + (+1)^3 + (-\frac{1}{2}) - 3(+1) = -2\frac{1}{4}.$$
On the other hand, \( x = -\frac{1}{2}, y = -1 \), then

\[ z = \left(-\frac{1}{2}\right)^2 + (-1)^3 + \left(-\frac{1}{2}\right) - 3(-1) = 1\frac{3}{4}. \]

So the stationary points are at \((-\frac{1}{2}, 1, -2\frac{1}{4})\) and \((-\frac{1}{2}, -1, 1\frac{3}{4})\).

There are three main kinds of stationary point a function of two variables can have:

1. local maximum
2. local minimum
3. saddle point

In the next section we will apply our intuition about stationary points to devise a “second derivative test” which will help us to identify which type of stationary point we may have for a given function of two variables.

**Second order partial derivatives**

If \( z = f(x, y) \) is a function of \( x \) and \( y \) then \( \frac{\partial z}{\partial x} \) is also a function of \( x \) and \( y \). We can differentiate it again with respect to either \( x \) or \( y \) to get

\[ \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial x^2} \quad \text{and} \quad \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial^2 z}{\partial y \partial x}. \]

We can also differentiate \( \frac{\partial z}{\partial y} \) to get

\[ \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial^2 z}{\partial y^2}. \]
Example

\[ z = x^4 + 2x^3y + 3x^2y^2 + 4xy^3 + 5y^4. \]

\[ \frac{\partial z}{\partial x} = 4x^3 + 6x^2y + 6xy^2 + 4y^3, \quad \frac{\partial z}{\partial y} = 2x^3 + 6x^2y + 12xy^2 + 20y^3. \]

\[ \frac{\partial^2 z}{\partial x^2} = 12x^2 + 12xy + 6y^2, \quad \frac{\partial^2 z}{\partial y \partial x} = 6x^2 + 12xy + 12y^2, \]

\[ \frac{\partial^2 z}{\partial x \partial y} = 6x^2 + 12xy + 12y^2, \quad \frac{\partial^2 z}{\partial y^2} = 6x^2 + 24xy + 60y^2. \]

We notice that \( \frac{\partial^2 z}{\partial y \partial x} \) is the same as \( \frac{\partial^2 z}{\partial x \partial y} \), though we obtained it in quite different ways.

\[ \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \]

for all (reasonable) functions \( z \).

The collection of second partial derivatives, written in the following way,

\[ H = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\ \frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} \]

is called the **Hessian Matrix** of the function \( z \). In the example above the Hessian matrix is

\[ H = \begin{bmatrix} 12x^2 + 12xy + 6y^2 & 6x^2 + 12xy + 12y^2 \\ 6x^2 + 12xy + 12y^2 & 6x^2 + 24xy + 60y^2 \end{bmatrix}. \]

A very convenient shorthand notation for partial derivatives of first and second order will be the following

\[ \frac{\partial f}{\partial x} := f_x; \quad \frac{\partial f}{\partial y} := f_y; \]

\[ \frac{\partial^2 f}{\partial x^2} := f_{xx}; \quad \frac{\partial^2 f}{\partial y \partial x} := f_{yx}; \]

\[ \frac{\partial^2 f}{\partial x \partial y} := f_{xy}; \quad \frac{\partial^2 f}{\partial y^2} := f_{yy}. \]

**Classifying stationary points**

As in the case of single variable functions the intuition about deciding whether a stationary point is a minimum, a maximum or neither of them comes from quadratic functions. In a sense functions behave like quadratic functions near their stationary points.
While for a quadratic function of one variable \( y = cx^2 \) the sign of the parameter \( c \) completely determines whether the vertex is a minimum or a maximum, a quadratic function of two variables involves 3 parameters:

\[
z = ax^2 + 2bxy + cy^2.
\]

If \( b = 0 \), that is \( z = ax^2 + cy^2 \), the signs of \( a \) and \( c \) tell us whether we have a maximum, a minimum or a saddle. In fact, a maximum corresponds to \( a \) and \( c \) both negative, a minimum corresponds to \( a \) and \( c \) both positive and a saddle to \( a \) and \( c \) of opposite sign.

If \( b \neq 0 \) the situation is more complicated. Assume \( a \neq 0 \). Then completion of the square yields

\[
ax^2 + 2bxy + cy^2 = a(x^2 + \frac{2b}{a}xy + \frac{c}{a}y^2)
= a(x + \frac{b}{a}y)^2 + (c - \frac{b^2}{a})y^2
= a(x + \frac{b}{a}y)^2 + (\frac{ac - b^2}{a})y^2
\]

Now a maximum corresponds to a sum of two negative squares, that is \( a < 0 \) and \( ac - b^2 > 0 \), a minimum corresponds to two positive squares, that is \( a > 0 \) and \( ac - b^2 > 0 \) and a saddle corresponds to opposite signs, that is \( ac - b^2 < 0 \).

Notice that the numbers \( a, b, c \) are half of the second partial derivatives:

\[
f_{xx} = 2a, \quad f_{xy} = 2b, \quad f_{yy} = 2c.
\]

Up to a factor of 4, which does not affect the sign the expression \( ac - b^2 \) equals

\[
f_{xx} \cdot f_{yy} - f_{xy}^2
\]

which is called the determinant of the Hessian matrix

\[
H = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix},
\]

(More will be said about matrices and determinants in the next chapter). The complete classification is: let \((x_0, y_0)\) be a stationary point of \( f(x, y) \), i.e. \( f_x(x_0, y_0) = 0 \) and \( f_y(x_0, y_0) = 0 \) then

1. If

\[
\det(H(x_0, y_0)) < 0
\]

\((x_0, y_0)\) is a saddle point.
2. If  
\[
\det(H(x_0, y_0)) > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0
\]

\((x_0, y_0)\) is a minimum.

3. If  
\[
\det(H(x_0, y_0)) > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0
\]

\((x_0, y_0)\) is a maximum.

4. If  
\[
\det(H(x_0, y_0)) = 0
\]

no conclusion can be drawn.

Example

Find and classify the stationary points of \(x^3 + y^3 - 3xy\).

Solution

\[
\frac{\partial z}{\partial x} = 3x^2 - 3y, \quad \frac{\partial z}{\partial y} = 3y^2 - 3x.
\]

The stationary points are found by solving

\[
\begin{align*}
3x^2 - 3y &= 0 \\
3y^2 - 3x &= 0.
\end{align*}
\]

From the first equation we get \(y = x^2\), and substituting this value of \(y\) in the second equation gives

\[
\begin{align*}
3x^4 - 3x &= 0, \\
3x(x^3 - 1) &= 0, \\
3x(x - 1)(x^2 + x + 1) &= 0, \\
x &= 0 \text{ or } x - 1 = 0.
\end{align*}
\]

(Because \(x^2 + x + 1 = (x + 1/2)^2 + 3/4 > 0\) can never be 0.) So \(x = 0\) or 1. From the first equation we have \(y = x^2\), so \(y = 0\) when \(x = 0\) (hence \(z = 0\)) and \(y = 1\) when \(x = 1\) (hence \(z = 1 + 1 - 3 = -1\)). The stationary points are therefore \((0, 0, 0)\) and \((1, 1, -1)\). Now

\[
\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial x \partial y} = -3, \quad \frac{\partial^2 z}{\partial y^2} = 6y.
\]

So we can summarise as follows
### Functions of Several Variables

<table>
<thead>
<tr>
<th>Point</th>
<th>(0,0,0)</th>
<th>(1,1,−1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{xx}$</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>$f_{xy}$</td>
<td>-3</td>
<td>-3</td>
</tr>
<tr>
<td>$f_{yy}$</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>$\det(H)$</td>
<td>-9</td>
<td>27</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Type</th>
<th>Saddle</th>
<th>Minimum</th>
</tr>
</thead>
</table>
Chapter 7

Matrices and Linear Systems of Equations

7.1 Matrices

A firm produces four products, \( A, B, C \) and \( D \). The costs of producing each of these four products consist of the use of material \( X \), the use of material \( Y \) and labor. The table below shows a sample of such costs for each product.

<table>
<thead>
<tr>
<th>Product</th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Units of ( X )</td>
<td>250</td>
<td>300</td>
<td>170</td>
<td>200</td>
</tr>
<tr>
<td>Units of ( Y )</td>
<td>160</td>
<td>230</td>
<td>75</td>
<td>10</td>
</tr>
<tr>
<td>Units of labor</td>
<td>80</td>
<td>85</td>
<td>120</td>
<td>100</td>
</tr>
</tbody>
</table>

Observe that the data in this table naturally form a rectangular array. If the headings are removed, we obtain the following rectangular array of numbers.

\[
\begin{bmatrix}
250 & 300 & 170 & 200 \\
160 & 230 & 75 & 10 \\
80 & 85 & 120 & 100
\end{bmatrix}
\]

This array is an example of a matrix.

**Definition.** A **matrix** is a rectangular array of real numbers, which is enclosed in large brackets. Matrices are generally denoted by capital letters \( A, B \), etc.

Some examples of matrices are

\[
A = \begin{bmatrix} 2 & -3 & 7 \\ 1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 6 \\ 7 \\ 1 \\ 4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}.
\]
The real numbers in a matrix are called **entries** or **elements** of the matrix. The elements in any horizontal line form a **row** and those in any vertical line form a **column** of the matrix. For example, the matrix $B$ above has 3 rows and 2 columns, whereas $C$ has 4 rows and 1 column.

If a matrix has $m$ rows and $n$ columns, then it is said to be of **size** $m \times n$ (read $m$ by $n$). Of the matrices given above, $A$ is a $2 \times 3$ matrix, $B$ is a $3 \times 2$ matrix, $C$ is a $4 \times 1$ matrix, and $D$ is a $2 \times 2$ matrix.

To denote a general $m \times n$ matrix, we usually use

$$A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}$$

This is also denoted by $A = [a_{ij}]_{m \times n}$.

If a matrix has all its elements zero, it is called a **zero matrix**. Thus the following is the zero matrix of size $2 \times 3$

$$\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}$$

**Definition** Two matrices $A$ and $B$ are said to be **equal** if

1. they are of the same size, and
2. their corresponding elements are equal.

For example, let

$$A = \begin{bmatrix}
2 & x & 3 \\
y & -1 & 4
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
a & 1 & b \\
6 & -1 & 4
\end{bmatrix}.$$  

Clearly, $A$ and $B$ are of the same size, and $A = B$ if and only if $a = 2, x = 1, b = 3, y = 6$.

### 7.2 Operations of Matrices

1. **Scalar Multiplication:**

   If $c$ is a scalar (i.e., a number), and $A = [a_{ij}]_{m \times n}$ a matrix, then

   $$cA = [ca_{ij}]_{m \times n}$$

   i.e. $cA$ is obtained by multiplying each element of $A$ by $c$.

2. **Addition and Subtraction:**
If \( A \) and \( B \) are matrices of the same size, say they are both of size \( m \times n \), then \( A + B \) and \( A - B \) are defined. If \( A = [a_{ij}]_{m \times n} \) and \( B = [b_{ij}]_{m \times n} \), then

\[
A + B = [a_{ij} + b_{ij}]_{m \times n}, \quad A - B = [a_{ij} - b_{ij}]_{m \times n}.
\]

i.e., \( A + B \) is obtained by adding the corresponding elements of \( A \) and \( B \), whereas \( A - B \) is obtained by subtracting the corresponding elements of \( B \) from \( A \).

**Example 1** Let

\[
A = \begin{bmatrix}
2 & 1 & 0 \\
0 & 1 & 2 \\
1 & 2 & 3
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 1 & 2 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 1 \\
2 & 1 \\
1 & 3
\end{bmatrix}.
\]

Determine which of the following operations are defined, and when it is defined, perform the operation.

\( A + B, \quad 3A - B, \quad B + C, \quad 2C \)

**Solution** Since \( A \) and \( B \) are of the same size, \( A + B \) is defined, and

\[
A + B = \begin{bmatrix}
2 & 1 & 0 \\
0 & 1 & 2 \\
1 & 2 & 3
\end{bmatrix} + \begin{bmatrix}
1 & 1 & 2 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
3 & 2 & 2 \\
0 & 1 & 3 \\
1 & 3 & 3
\end{bmatrix}
\]

\( 3A = 3 \begin{bmatrix}
2 & 1 & 0 \\
0 & 1 & 2 \\
1 & 2 & 3
\end{bmatrix} = \begin{bmatrix}
6 & 3 & 0 \\
0 & 3 & 6 \\
3 & 6 & 9
\end{bmatrix} \)

\( 3A \) and \( B \) are of the same size, therefore \( 3A - B \) is defined, and

\[
3A - B = \begin{bmatrix}
6 & 3 & 0 \\
0 & 3 & 6 \\
3 & 6 & 9
\end{bmatrix} - \begin{bmatrix}
1 & 1 & 2 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
5 & 2 & -2 \\
0 & 3 & 5 \\
3 & 5 & 9
\end{bmatrix}
\]

\( B \) and \( C \) are not of the same size, therefore \( B + C \) is not defined.

\[
2C = 2 \begin{bmatrix}
1 & 1 \\
2 & 1 \\
1 & 3
\end{bmatrix} = \begin{bmatrix}
2 & 2 \\
4 & 2 \\
2 & 6
\end{bmatrix}.
\]

**Example 2.** Let

\[
A = \begin{bmatrix} x^2 & y - 1 \\ u & v^3 + 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}.
\]

Suppose \( A = B \). Find \( x, y, u \) and \( v \).

**Solution** Since \( A = B \), their corresponding elements are equal. Hence we have \( x^2 = 4, y - 1 = 6, u = 3, v^3 + 1 = 2 \). It follows \( x = \pm \sqrt{4} = \pm 2, y = 6 + 1 = 7, u = 3, v^3 = 2 - 1 = 1, v = \sqrt[3]{1} = 1 \).
CHAPTER 7. MATRICES AND LINEAR SYSTEMS OF EQUATIONS

7.3 Multiplication of Matrices

In the last lecture, we have learned the scalar multiplication, addition, and subtraction of matrices. These operations are very similar to the corresponding operations for real numbers. In this lecture, we discuss matrix multiplications. We will see that this is very different from multiplications of real numbers.

Let us look at a situation when matrix multiplication arises. Suppose a firm manufactures a product using three different amounts of three inputs, $P, Q$ and $R$ (they could be materials or labour, for example). Let the number of units of these inputs used for each unit of the product be given by the following row matrix.

$$A = \begin{bmatrix} P & Q & R \\ 3 & 2 & 4 \end{bmatrix}$$

Then let the cost per unit of each of the three inputs be given by the following column matrix

$$B = \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix}$$

Then the total cost of producing one unit of the product is

$$3 \times 10 + 2 \times 8 + 4 \times 6 = 70.$$ 

We refer to this number as the product of the row matrix $A$ and the column matrix $B$, written $AB$. Observe that in forming $AB$, the first elements of $A$ and $B$ are multiplied together, the second elements are multiplied together, the third elements are multiplied together, and then these three products are added.

This method of forming products applies to row and column matrices of any size.

**Definition** Let $C$ be a $1 \times n$ row matrix and $D$ be an $n \times 1$ column matrix. Then the product $CD$ is obtained by calculating the products of corresponding elements in $C$ and $D$ and then adding all $n$ of these products. Note: We do not use $DC$ to denote this product of $C$ and $D$.

**Example 1**

Given $K = \begin{bmatrix} 2 & 5 \end{bmatrix}$, $L = \begin{bmatrix} 1 & -2 & -3 & 2 \end{bmatrix}$, $M = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $N = \begin{bmatrix} 2 \\ 5 \\ -3 \\ 4 \end{bmatrix}$. Then

$$KM = (2)(-3) + (5)(2) = -6 + 10 = 4.$$ 

$$LN = (1)(2) + (-2)(5) + (-3)(-3) + (2)(4) = 2 - 10 + 9 + 8 = 9$$

$$LN = (1)(2) + (-2)(5) + (-3)(-3) + (2)(4) = 2 - 10 + 9 + 8 = 9$$
Note: The row and column matrices must have the same number of elements in order to make a product. In example 1, the products $LM$ and $KN$ are not defined.

In more general situations, one needs to find the product of two matrices $A$ and $B$, but $A$ may not be a row matrix, and $B$ may not be a column matrix. The general definition is given below.

**Definition** If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the product $AB$ is an $m \times p$ matrix $C = [c_{ij}]$, where the $ij$-th element $c_{ij}$ of $C$ is obtained by multiplying the $i$-th row of $A$ and the $j$-th column of $B$. Note that the number of columns in $A$ and the number of rows in $B$ must be the same in this definition.

**Example 2** Let

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}$$

Find $AB$ and $BA$ if they exist.

**Solution** Here $A$ is $2 \times 2$ and $B$ is $2 \times 3$. Since the number of columns in $A$ is equal to the number of rows in $B$, the product $AB$ is defined. It is of size $2 \times 3$. If $C = AB$, then we can write $C$ in the form

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

The element $c_{ij}$ is found by multiplying the $i$-th row of $A$ and the $j$-th column of $B$. For example,

$$c_{12} = [2 \ 3] \begin{bmatrix} 1 \\ -3 \end{bmatrix} = (2)(1) + (3)(-3) = 2 - 9 = -7$$

In full, we have

$$AB = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix} = \begin{bmatrix} (2)(3) + (3)(2) & (2)(1) + (3)(-3) & (2)(0) + (3)(4) \\ (4)(3) + (1)(2) & (4)(1) + (1)(-3) & (4)(0) + (1)(4) \end{bmatrix} = \begin{bmatrix} 12 & -7 & 12 \\ 14 & 1 & 4 \end{bmatrix}.$$  

The product $BA$ is not defined because the number of columns in the left matrix $B$ is not equal to the number of rows in the right matrix $A$.

**Example 3** Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$, find $AB$ and $BA$. 


**Solution** Here $A$ and $B$ are both of size $3 \times 3$. Thus $AB$ and $BA$ are both defined and both have size $3 \times 3$. We have the following

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$


$$= \begin{bmatrix} 7 & 14 & 10 \\ 13 & 32 & 25 \\ 3 & 16 & 13 \end{bmatrix}$$

$$BA = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 4 \end{bmatrix}$$


$$= \begin{bmatrix} 6 & 3 & 8 \\ 13 & 17 & 25 \\ 17 & 19 & 29 \end{bmatrix}$$

Let us note that $AB \neq BA$, even though both products are defined.

### 7.4 Multiplication of Matrices (continued)

Recall that if $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix, then $AB$ is defined and is of size $m \times p$.

If $A$ and $B$ are as above, and $C$ is a $p \times q$ matrix, then the products $AB$, $BC$, $(AB)C$ and $A(BC)$ are all defined. It can be proved that

$$(AB)C = A(BC).$$

Therefore we usually omit the brackets and write simply $ABC$. Note that $AB$ is of size $m \times q$, $BC$ is of size $n \times q$ and $ABC$ is of size $m \times q$.

Recall that a matrix with all its elements 0 is called a zero matrix. If $A$ is an $m \times n$ matrix and 0 is the zero matrix of size $m \times n$, then it follows easily from the definition of matrix addition that

$$A + 0 = 0 + A = A.$$
Thus in matrix addition the zero matrix plays the role the number 0 plays in addition of real numbers.

There is also a matrix which plays the role as the number 1 in multiplications of real numbers. This is the identity matrix defined below.

**Definition** A square matrix is called an **identity matrix** if all the elements on its diagonal are equal to 1 and all the other elements are equal to zero. Here by a square matrix we mean a matrix with the same number of rows and columns, and diagonal elements of a square matrix are the elements \( a_{ij} \) with \( i = j \).

For example,
\[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\quad \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
are the identity matrices of size \( 2 \times 2 \) and \( 3 \times 3 \), respectively.

The identity matrix is usually denoted by \( I \) when its size is understood without ambiguity. To emphasize that an identity matrix has size \( n \times n \), we usually write \( I_{n \times n} \).

**Example 1.** Let
\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]
Find \( AI \) and \( IA \), where I denote the \( 2 \times 2 \) identity matrix.

**Solution**
\[
AI = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
a(1) + b(0) & a(0) + b(1) \\
c(1) + d(0) & c(0) + d(1)
\end{bmatrix} = A
\]
\[
IA = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} = \begin{bmatrix}
(1)a + (0)c & (1)b + (0)d \\
(0)a + (1)c & (0)b + (1)d
\end{bmatrix} = A.
\]

Thus \( AI = IA = A \).

It can be easily checked that the identity
\[
AI = IA = A
\]
is true for any square matrix \( A \), where \( I \) is the identity matrix the same size as \( A \). Moreover, if \( I \) is \( n \times n \), \( A \) is \( m \times n \) and \( B \) is \( n \times p \), then \( AI = A, IB = B \).

**Example 2.** Let \( A = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} \) and \( B = \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix} \). Find \( AB \) and \( BA \).
Solution

\[
AB = \begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
= \begin{bmatrix}
(1)(0) + (0)(1) & (1)(0) + (0)(0) \\
(0)(0) + (0)(1) & (0)(0) + (0)(0)
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
BA = \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
(0)(1) + (0)(0) & (0)(0) + (0)(0) \\
(1)(1) + (0)(0) & (1)(0) + (0)(0)
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
= B.
\]

Let us note that in Example 2, neither \(A\) nor \(B\) is a zero matrix, but \(AB = 0\), the zero matrix. We know that \(BI = B\) where \(I\) is the \(2 \times 2\) matrix, and clearly \(A \neq I\), but we have \(BA = B\). These are properties which are very different from that of real number multiplications.

By using the idea of matrix multiplications, we can write systems of linear equations in the form of matrix equations. Consider, for example, the system

\[
\begin{cases}
2x - 3y &= 7 \\
4x + y &= 21
\end{cases}
\]

This is equivalent to

\[
\begin{bmatrix}
2x - 3y \\
4x + y
\end{bmatrix}
= \begin{bmatrix}
7 \\
21
\end{bmatrix}
\]

But it is easily checked that

\[
\begin{bmatrix}
2 & -3 \\
4 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
2x - 3y \\
4x + y
\end{bmatrix}
\]

Therefore the original system can be written as

\[
\begin{bmatrix}
2 & -3 \\
4 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
7 \\
21
\end{bmatrix}
\]

If we define matrices \(A\), \(B\) and \(X\) as

\[
A = \begin{bmatrix}
2 & -3 \\
4 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
7 \\
21
\end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix}
x \\
y
\end{bmatrix},
\]

then this matrix equation can be written as

\[
AX = B.
\]
The advantage of writing an equation system in matrix form is the following. If we can find a matrix \( C \) such that \( CA = I \), the identity matrix, then from \( AX = B \), we obtain

\[
C(AX) = CB, (CA)X = CB, IX = CB, X = CB.
\]

For example, in the case \( A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \), we can check that \( C = \begin{bmatrix} \frac{1}{13} & \frac{3}{14} \\ \frac{1}{7} & \frac{1}{7} \end{bmatrix} \) satisfies \( CA = I \)

Hence \( X = CB = \begin{bmatrix} \frac{1}{13} & \frac{3}{14} \\ \frac{1}{7} & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{13}\right)(7) + \left(\frac{3}{14}\right) (21) \\ \left(-\frac{1}{7}\right)(7) + \left(\frac{1}{7}\right) (21) \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \)

That is \( x = 5, y = 1 \) is a solution of the original system.

**Example 3.** Express the following system of equations in matrix form.

\[
\begin{align*}
2x + 3y + 4z &= 7 \\
4y - 5z &= 2 \\
-2x + 3z &= -6
\end{align*}
\]

**Solution** The system can be rewritten as

\[
\begin{align*}
2x + 3y + 4z &= 7 \\
0x + 4y - 5z &= 2 \\
-2x + 0y + 3z &= -6
\end{align*}
\]

If we define

\[
A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 4 & -5 \\ -2 & 0 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix},
\]

then the system is equivalent to

\[
AX = B.
\]

### 7.5 The Inverse of a Matrix and Determinant

In the last lecture, we mentioned that systems of linear equations can be written as matrix equations of the form

\[
AX = B
\]

where \( X \) is the column matrix containing all the unknowns. For example,

\[
\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 3 \end{bmatrix}
\]
is equivalent to a system of linear equations with three unknowns \( x, y \) and \( z \).

If we know a matrix \( C \) such that \( CA = I \), the identity matrix, then \( X = CAX = CB \). In other words, once such a matrix \( C \) can be found, then the solutions can be found through a simple calculation of the matrix multiplication \( CB \). However, finding such a matrix \( C \) is usually not easy. Nevertheless, this way of solving equation systems is particularly useful when one tries to use computers to find the solutions, and we will discuss this in detail later.

Finding a matrix \( C \) such that \( CA = I \) is closely related to finding the inverse of a matrix. Indeed, if \( A \) is a square matrix, such a matrix \( C \) is actually the inverse of \( A \). A formal definition is given below.

**Definition** Let \( A \) be an \( n \times n \) square matrix. If there is an \( n \times n \) matrix \( B \) such that \( AB = I \) and \( BA = I \), then we say \( A \) is invertible or nonsingular, and call \( B \) the inverse of \( A \), and usually denote it by \( B = A^{-1} \).

**Example 1.** Show that \( A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \) has no inverse.

**Proof** Let \( B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) be an arbitrary \( 2 \times 2 \) matrix. Then

\[
AB = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 2c & b + 2d \\ 0 & 0 \end{bmatrix} \neq I.
\]

Hence \( A \) has no inverse.

If a square matrix has no inverse, then it is called a singular matrix. Note that we do not talk about the inverses for non-square matrices.

**Example 2.** Find the inverse of \( A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \).

**Solution.** Let \( B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) be the inverse of \( A \). Then

\[
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 2a + b \\ c & 2c + d \end{bmatrix}
\]

Therefore, \( a = 1, 2a + b = 0, c = 0, 2c + d = 1 \).

It follows, \( b = -2a = -2, d = 1 - 2c = 1 \). Thus

\[
B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.
\]

Checking \( AB \), we have

\[
AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.
\]
Hence $B$ is indeed the inverse of $A$.

Note that in Example 2, we only used $BA = I$ to find $B$, and it turns out that this $B$ also satisfies $AB = I$. This is no coincident. It can be proved that if the $n \times n$ matrices $A$ and $B$ satisfy $BA = I$, then we must have $AB = I$.

Moreover, if $A, B, C$ are all $n \times n$ matrices, and satisfy

$$AB = I, AC = I$$

then we must have $B = C$. In other words, the inverse of a matrix $A$ is unique. This can be proved as follows.

From $AB = I$ and the property mentioned above, we know $BA = I$. Hence, using $AC = I$, we obtain

$$B(AC) = BI = B$$
$$B(AC) = (BA)C = B, IC = B, C = B$$

Though there are straightforward procedures to compute the inverse of an $n \times n$ matrix $A$, the number of arithmetical operations involved grows fast with growing $n$. It would be useful to know whether $A$ has an inverse before starting such arduous procedure. But even this is a problem which requires a large number of arithmetical computations for big $n$. Computer Algebra System are of great help here. In the Economics stream, we will learn to use Excel to determine whether $A$ has an inverse, and when the inverse exists, we can use Excel to find it. The details are contained in the Practical Class Instructions for the Economics stream. *The precise mathematical process for finding the inverse of a matrix will not be discussed further in this course. When it is needed, the inverse will simply be given for you to use.*

One effective method to determine whether a given $n \times n$ matrix $A$ has an inverse is to calculate its **determinant**, denoted by $\det(A)$. Below we give the definition of the determinant for $2 \times 2$ and $3 \times 3$ matrices. In general, the determinant of an $n \times n$ matrix is a sum of $1 \cdot 2 \cdot 3 \cdots n$ terms, each of which is a product of $n$ matrix coefficients. For $n = 2$ there are $1 \cdot 2 = 2$ terms, each of them a product of two coefficients, for $n = 3$ there are already $1 \cdot 2 \cdot 3 = 6$ terms, each of them a product of three coefficients. For $n = 4$ we would need to write down 24 products of 4 coefficients, for $n = 5$ it is 120 products of 5 coefficients etc.

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$\det(A) = a_{11}a_{22} - a_{21}a_{12}. \quad (7.1)$$
When \( A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \) then

\[
\det(A) = a_{11} \det\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}
\]

\[= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}.\]

For \( A \) of the size bigger than \( 3 \times 3 \), its determinant is defined in the fashion of the above definition for \( 3 \times 3 \) matrices, but we will not go into the details. The interested reader is referred to the reference book, where more material on the topic of inverses and determinants can be found.

Now let us see how the determinant can be used to determine whether a matrix has an inverse.

**Theorem** An \( n \times n \) matrix has an inverse if and only if its determinant is nonzero.

The proof of this theorem is beyond the scope of this unit and you are only required to know the conclusion.

**Example 3.** Determine which of the following matrices has an inverse.

\[
A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}
\]

**Solution** \( \det(A) = 1 \times 4 - 2 \times 2 = 4 - 4 = 0 \). Hence \( A \) has no inverse. \( \det(B) = 1 \times 0 - 0 \times 0 = 0 \). Hence \( B \) has no inverse. \( \det(C) = 2 \times 4 - 1 \times 3 = 8 - 3 = 5 \neq 0 \). Hence \( C \) has an inverse.

We conclude this section with a useful formula for the inverse of a \( 2 \times 2 \)-matrix. Let

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\]

Then we know that its inverse exists if \( \det A = ad - bc \neq 0 \). The inverse is given by the formula

\[
A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

7.6 Solutions of Linear Systems

A general system of \( m \) linear equations involving \( n \) variables, denoted by \( x_1, x_2, \ldots, x_n \), can be expressed in the following form
If we denote by $A = [a_{ij}]_{m \times n}$ the $m \times n$ matrix consisting of the coefficients of the $n$ variables $x_1, \ldots, x_n$, in the $m$ equations, and
\[
X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},
\]
then the system can be rewritten as a matrix equation
\[
AX = B
\]
It can be proved that a given system of linear equations can have no solution, or a unique solution, or infinitely many solutions. But it can never happen that such a system has exactly two, or three, etc. solutions.

**Example 1.** The system
\[
\begin{align*}
x + 2y &= 3 \\
3x + 6y &= 2
\end{align*}
\]
has no solution, since if $x, y$ satisfy both equations, then we multiply the first equation by 3 and obtain
\[
3(x + 2y) = 9, \quad 3x + 6y = 9
\]
which is in contradiction to the second equation in the system.

**Example 2.** The system
\[
\begin{align*}
x + 2y &= 3 \\
x + y &= 1
\end{align*}
\]
has a unique solution $x = -1, y = 2$. Indeed, subtracting the second equation from the first, we obtain
\[
(x + 2y) - (x + y) = 301, \quad y = 2.
\]
Substituting $y = 2$ into either the first or the second equation, we obtain $x = -1$. 
Example 3. The System

\[
\begin{align*}
x + 2y &= 3 \\
3x + 6y &= 6
\end{align*}
\]

has infinitely many solutions. Indeed, let \( t \) be any real number, and \( y = t, x = 3 - 2t \). Then

\[
\begin{align*}
x + 2y &= (3 - 2t) + 2t = 3 \\
3x + 6y &= 3(3 - 2t) + 6t = 9.
\end{align*}
\]

Hence, for any \( t, x = 3 - 2t, y = t \) is a solution to the system.

Let us now return to the general equation

\[ AX = B \]

where \( A = [a_{ij}]_{m \times n}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \).

If the number of equations, \( m \), and the number of unknowns, \( n \), are equal, then \( A \) becomes an \( n \times n \) matrix and hence we can talk about its inverse \( A^{-1} \). We know from the last lecture that \( A^{-1} \) exists if and only if its determinant, \( \det(A) \), is not zero. In such a case,

\[
\begin{align*}
A^{-1}(Ax) &= A^{-1}B \\
(A^{-1}A)X &= A^{-1}B \\
IX &= A^{-1}B \\
X &= A^{-1}B
\end{align*}
\]

That is, the system has a unique solution which is given by

\[ X = A^{-1}B \]

Example 4 Find the solutions to the following system

\[
\begin{align*}
2x - 3y + 4z &= 13 \\
x + y + 2z &= 4 \\
3x + 5y - z &= -4
\end{align*}
\]
Solution  Let $A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 1 & 2 \\ 3 & 5 & -1 \end{bmatrix}$.

$\det(A) = -35$ (By the Formula $[7.2]$) and if a computer program is used to calculate the inverse, it is given approximately as

$$A^{-1} \approx \begin{bmatrix} 0.314285714 & -0.484714286 & 0.285714286 \\ -0.2 & 0.4 & 0 \\ -0.057142857 & 0.542857143 & -0.142857143 \end{bmatrix}$$

so then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 13 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Example 5. Two products $A$ and $B$ are competitive. The demands $x_A$ and $x_B$ for these products are related to their prices $P_A$ and $P_B$ according to the demand equations

$$x_A = 17 - 2P_A + 0.5P_B, \quad x_B = 20 - 3P_B + 0.5P_A$$

The supply equations are

$$P_A = 2 + x_A + \frac{1}{3}x_B, \quad P_B = 2 + \frac{1}{2}x_B + \frac{1}{4}x_A$$

For market equilibrium, all four equations must be satisfied (i.e. supply equals demand). Find the equilibrium values of $x_A, x_B, P_A$ and $P_B$.

Solution  The four equations can be rewritten into the standard form:

$$x_A + 0x_B + 2P_A - 0.5P_B = 17$$
$$0x_A + x_B - 0.5P_A + 3P_B = 20$$
$$x_A + \frac{1}{3}x_B - P_A + 0P_B = -2$$
$$\frac{1}{4}x_A + \frac{1}{2}x_B + 0P_A - P_B = -2$$

Thus the coefficient matrix is

$$C = \begin{bmatrix} 1 & 0 & 2 & -0.5 \\ 0 & 1 & -0.5 & 3 \\ 1 & \frac{1}{3} & -1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & -1 \end{bmatrix}$$

$\det(C) = 6.6 \neq 0$ so then the inverse can be calculated approximately by a computer as

$$C^{-1} = \begin{bmatrix} 0.353531230 & -0.063091483 & 0.738170347 & -0.3659306 \\ -0.03785489 & 0.43533123 & -0.293375394 & 1.324921136 \\ 0.340694006 & 0.082018927 & -0.359621451 & 0.075709779 \\ 0.069400631 & 0.201892744 & 0.03785489 & -0.42902208 \end{bmatrix}$$
and

\[
\begin{bmatrix}
  x_A \\
  x_B \\
  P_A \\
  P_B 
\end{bmatrix}
= C^{-1}
\begin{bmatrix}
  17 \\
  20 \\
  -2 \\
  -2 
\end{bmatrix}
\approx
\begin{bmatrix}
  4 \\
  6 \\
  8 \\
  6 
\end{bmatrix}.
\]

i.e., \( x_A = 4, x_B = 6, P_A = 8, P_B = 6 \).
Chapter 8

Applications of Matrices in Biology
(Science stream only)

8.1 The Leslie Matrix Model of Population Growth

In this section we apply matrices to the biological problem of population growth.

Suppose a population is divided into a number of age groups of equal time span $T$ and let

\[
\begin{align*}
P_1 & = \text{number of individuals aged from 0 to } T \\
P_2 & = \text{number of individuals aged from } T \text{ to } 2T \\
P_3 & = \text{number of individuals aged from } 2T \text{ to } 3T \\
\vdots & \vdots \\
P_k & = \text{number of individuals aged from } (k-1)T \text{ to } kT.
\end{align*}
\]

Suppose we know what the numbers $P_1, P_2, P_3, \ldots, P_k$ are now and we want to work out what they will be at time $T$ from now. Before we can do that there are two sets of numbers we need to know:

1. $S_i = \text{proportion of individuals in the } (i-1)\text{th age group who survive to the } i\text{th age group.}$
2. $B_i = \text{birth rate per individual in the } i\text{th age group.}$

For example, if we are working with a human population and $T = 10$ yrs then $S_7$ is the number of people now in their 60’s who survive another 10 years (when they will be in their 70’s) and $B_3$ is the total number of children born in the next 10 years to people now in their 30’s divided by the total number of those people.

Let $P'_i$ be the number of individuals in the $i$th age group after a time $T$. These numbers
can be worked out as follows:

\[
\begin{align*}
P'_1 &= B_1P_1 + B_2P_2 + \cdots + B_kP_k \\
P'_2 &= S_2P_1 \\
P'_3 &= S_3P_2 \\
\vdots &= \vdots \\
P'_k &= S_kP_{k-1}.
\end{align*}
\]

The reason is this: Firstly, \( P'_1 \), the number of individuals aged from 0 to \( T \) after time \( T \), must all be new births, since those originally in this age group will have aged to the next age group or died. So \( P'_1 \) is the sum of the number of surviving births for each age group, which is got by multiplying the birth rate \( B_i \) for the age group by the number of individuals \( P_i \) in the age group. Secondly, the numbers \( P'_2 \) up to \( P'_k \) are exactly the numbers of individuals surviving from the preceding age group, which is the number in the preceding age group \( P_{i-1} \) multiplied by the proportion \( S_i \) of this age group which survives.

The equations for the numbers \( P'_i \) can be written in matrix form

\[
\begin{bmatrix}
P'_1 \\
P'_2 \\
P'_3 \\
\vdots \\
P'_k
\end{bmatrix}
= \begin{bmatrix}
B_1 & B_2 & B_3 & \cdots & B_k \\
S_2 & 0 & 0 & \cdots & 0 \\
0 & S_3 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & S_k
\end{bmatrix}
\begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
\vdots \\
P_k
\end{bmatrix}.
\]

Let

\[
P' = \begin{bmatrix}
P'_1 \\
P'_2 \\
P'_3 \\
\vdots \\
P'_k
\end{bmatrix}, \quad P = \begin{bmatrix}
P_1 \\
P_2 \\
P_3 \\
\vdots \\
P_k
\end{bmatrix}
\]

and

\[
L = \begin{bmatrix}
B_1 & B_2 & B_3 & \cdots & B_{k-1} & B_k \\
S_2 & 0 & 0 & \cdots & 0 & 0 \\
0 & S_3 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & S_k & 0
\end{bmatrix}.
\]

Then

\[
P' = L \times P.
\]

The \( k \times k \) matrix \( L \) is called the Leslie matrix of the population.
Example

Suppose a certain species of beetle lives for only three months. The proportion of beetles aged 0–1 months surviving to age 1–2 months is 50% and the proportion aged 1–2 months surviving to age 2–3 months is 40%. The beetles give birth only in their third month and produce on average 6 surviving young. If initially there are 1000 beetles in each of the age groups 0–1, 1–2 and 2–3, how many beetles will there be in each age group after 1 month?

Here we have 3 age groups and

\[ T = 1 \text{ month}. \]

The initial population vector is

\[
P = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}.
\]

The birth rates are

\[
B_1 = 0 \\
B_2 = 0 \\
B_3 = 6
\]

and the survival rates are

\[
S_2 = 0.5 \\
S_3 = 0.4,
\]

so the Leslie matrix for the beetle population is

\[
L = \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}
\]

and

\[
P' = L \times P \\
= \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} \\
= \begin{bmatrix} 6000 \\ 500 \\ 400 \end{bmatrix},
\]

which tells us that after one month there will be 6000 beetles aged 0–1 months, 500 beetles aged 1–2 months and 400 beetles aged 2–3 months.
Applications of inverse matrices to the Leslie Model

There are many situations where we know matrices $A$ and $B$ and we need to find a matrix $X$ with

$$A \times X = B$$

(i.e., we would like to divide $B$ by $A$). How can $X$ be found? Multiplying both sides of the above equation by $A^{-1}$ gives

$$A^{-1} \times A \times X = A^{-1} \times B$$

$$I \times X = A^{-1} \times B$$

$$X = A^{-1} \times B.$$

So $X = A^{-1} \times B$. For this to work $A$ must be a square matrix with an inverse.

A special case that is often met is an equation $A \times X = Y$ with $A$ a square $n \times n$ matrix and $X$ and $Y$ $n$-dimensional vectors. If $A$ and $Y$ are known and $A$ has an inverse then

$$X = A^{-1} \times Y.$$

In the previous example we had a Leslie matrix

$$L = \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

and population vector

$$P = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}.$$

We found the population vector a month later by

$$P_1 = L \times P.$$

What if we had wanted to know the population vector a month before? This is easily found if we know the inverse of $L$. If the population vector a month before was $P_{-1}$, then

$$P = L \times P_{-1}$$

because multiplying by the Leslie matrix updates the population vector from one month to the next. Multiplying both sides of this equation by the matrix $L^{-1}$ gives

$$L^{-1} \times P = L^{-1} \times L \times P_{-1} = I \times P_{-1} = P_{-1},$$

so

$$P_{-1} = L^{-1} \times P.$$
8.1. THE LESLIE MATRIX MODEL OF POPULATION GROWTH

We found \( L^{-1} \) earlier, and with the vector \( P \) as above we get

\[
P_{-1} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 5/2 \\ 1/6 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}
\]

\[
= \begin{bmatrix} 2000 \\ 2500 \\ 166\frac{2}{3} \end{bmatrix}.
\]

By repeating this process we can obtain the population vectors for two months before, three months before, and so on.

It is important to realise that we could do the calculation above only because \( L \) had an inverse. If \( L \) did not have an inverse there would be no way to deduce what the past populations were from the present population.

We will find out more about Leslie matrices after we learn some more matrix algebra.
8.2 Powers of Matrices

If \( A \) is a square \( n \times n \) matrix then the product of \( A \) with itself is also a square \( n \times n \) matrix which is denoted by \( A^2 \):

\[
A^2 = A \times A.
\]

We get higher powers of \( A \) in the same way:

\[
A^3 = A \times A \times A = A \times A^2
\]
\[
A^4 = A \times A \times A \times A = A \times A^3
\]
\[
A^5 = A \times A^4, \quad \text{etc.}
\]

For example, if

\[
A = \begin{bmatrix}
1 & 2 & 0 \\
0 & 3 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

then

\[
A^2 = A \times A
\]

\[
= \begin{bmatrix}
1 & 2 & 0 \\
0 & 3 & 1 \\
1 & 0 & 1
\end{bmatrix} \times \begin{bmatrix}
1 & 2 & 0 \\
0 & 3 & 1 \\
1 & 0 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 \times 1 + 2 \times 0 + 0 \times 1 & 1 \times 2 + 2 \times 3 + 0 \times 0 & 1 \times 0 + 2 \times 1 + 0 \times 1 \\
0 \times 1 + 3 \times 0 + 1 \times 1 & 0 \times 2 + 3 \times 3 + 1 \times 0 & 0 \times 0 + 3 \times 1 + 1 \times 1 \\
1 \times 1 + 0 \times 0 + 1 \times 1 & 1 \times 2 + 0 \times 3 + 1 \times 0 & 1 \times 0 + 0 \times 1 + 1 \times 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 + 0 + 0 & 2 + 6 + 0 & 0 + 2 + 0 \\
0 + 0 + 1 & 0 + 9 + 0 & 0 + 3 + 1 \\
1 + 0 + 1 & 2 + 0 + 0 & 0 + 0 + 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 8 & 2 \\
1 & 9 & 4 \\
2 & 2 & 1
\end{bmatrix}
\]

and

\[
A^3 = A \times A^2
\]

\[
= \begin{bmatrix}
1 & 2 & 0 \\
0 & 3 & 1 \\
1 & 0 & 1
\end{bmatrix} \times \begin{bmatrix}
1 & 8 & 2 \\
1 & 9 & 4 \\
2 & 2 & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
3 & 26 & 10 \\
5 & 29 & 13 \\
3 & 10 & 3
\end{bmatrix}
\]
Higher powers of $A$ are calculated similarly.

Raising matrices to high powers is a tedious exercise, especially for matrices of large size. However in the special case of diagonal matrices multiplication and raising to powers is very easy:

$$
\begin{bmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{bmatrix}
\times
\begin{bmatrix}
b_1 & 0 & 0 \\
0 & b_2 & 0 \\
0 & 0 & b_3
\end{bmatrix}
= 
\begin{bmatrix}
a_1b_1 & 0 & 0 \\
0 & a_2b_2 & 0 \\
0 & 0 & a_3b_3
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
a_1 & 0 & 0 \\
0 & a_2 & 0 \\
0 & 0 & a_3
\end{bmatrix}^n
= 
\begin{bmatrix}
a_1^n & 0 & 0 \\
0 & a_2^n & 0 \\
0 & 0 & a_3^n
\end{bmatrix}.
$$

It turns out that most square matrices $A$ can be decomposed into a product of three matrices

$$
A = C^{-1}BC
$$

where $B$ is a diagonal matrix. Then

$$
A^n = C^{-1}BC \cdots C^{-1}BC = C^{-1}B^nC.
$$

If $B$ and $C$ are known then it is easy to compute $C^{-1}B^nC$.

**Application to Leslie matrices**

We can apply this to the Leslie matrix of the example in section 3

$$
L = 
\begin{bmatrix}
0 & 0 & 6 \\
0.5 & 0 & 0 \\
0 & 0.4 & 0
\end{bmatrix},
$$

with initial population vector

$$
P = 
\begin{bmatrix}
1000 \\
1000 \\
1000
\end{bmatrix}.
$$

Let $P_n$ be the population vector after $n$ months. Then

$$
P_1 = L \times P,$$

$$
P_2 = L \times P_1 = L \times L \times P = L^2 \times P,$$

$$
P_3 = L \times P_2 = L \times L^2 \times P = L^3 \times P,$$

etc.
In general

\[ P_n = L^n \times P. \]

Now

\[ L^2 = L \times L \]

\[ = \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 \times 0 + 0 \times 0.5 + 6 \times 0 & 0 \times 0 + 0 \times 0 + 6 \times 0.4 & 0 \times 6 + 0 \times 0 + 6 \times 0 \\ 0.5 \times 0 + 0 \times 0.5 + 0 \times 0 & 0.5 \times 0 + 0 \times 0 + 0 \times 0.4 & 0.5 \times 6 + 0 \times 0 + 0 \times 0 \\ 0 \times 0 + 0.4 \times 0.5 + 0 \times 0 & 0 \times 0 + 0.4 \times 0 + 0 \times 0.4 & 0 \times 6 + 0.4 \times 0 + 0 \times 0 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 & 2.4 & 0 \\ 0 & 0 & 3 \\ 0.2 & 0 & 0 \end{bmatrix} \]

and

\[ L^3 = L \times L^2 \]

\[ = \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 2.4 & 0 \\ 0 & 0 & 3 \\ 0.2 & 0 & 0 \end{bmatrix} \]

\[ = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}. \]

So we get

\[ P_2 = L^2 \times P \]

\[ = \begin{bmatrix} 0 & 2.4 & 0 \\ 0 & 0 & 3 \\ 0.2 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 \times 1000 + 2.4 \times 1000 + 0 \times 1000 \\ 0 \times 1000 + 0 \times 1000 + 3 \times 1000 \\ 0.2 \times 1000 + 0 \times 1000 + 0 \times 1000 \end{bmatrix} \]

\[ = \begin{bmatrix} 2400 \\ 3000 \\ 200 \end{bmatrix} \]
and

\[
P_3 = L^3 \times P
\]

\[
= \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1200 \\ 1200 \\ 1200 \end{bmatrix}.
\]

Another way of doing this last calculation is to note

\[
L^3 = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}
\]

\[
= 1.2 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
= 1.2 \times I,
\]

where \( I \) is the 3 \( \times \) 3 identity matrix, so

\[
P_3 = L^3 \times P = 1.2 \times I \times P = 1.2 \times P,
\]

using the fact that \( I \times P = P \). Hence

\[
P_3 = 1.2 \times P = 1.2 \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1200 \\ 1200 \\ 1200 \end{bmatrix}.
\]

Although matrix powers are useful in calculating future populations, it requires less work if we multiply the initial population repeatedly by \( L \). That is, it is easier to compute, for example,

\[
P_3 = L(\times L(\times L \times P))
\]

than it is to compute \( L^3 \) and then

\[
P_3 = L^3 \times P.
\]
Chapter 9

Differential Equations (Science stream only)

9.1 Introduction of Differential Equations

In many applications, one has to solve an equation that involves the derivative of an unknown function. Such an equation is called a differential equation. For example,

\[ y' = x^2, \quad z' = x^2 z, \quad s' = t^2 + s^2, \quad P'' = xP' + xP \]

are all differential equations, and \( y = y(x), \quad z = z(x), \quad s = s(t) \) and \( P = P(x) \) are used to stand for the unknown functions, respectively. Moreover, the first three are called first order differential equations because only first order derivatives are involved in the equations, while the last one is a second order differential equation for the obvious reason.

Solving a given differential equation is usually a difficult problem. For complicated differential equations, one usually relies on a combination of theoretical analysis and numerical approximation. We will concentrate on some simple differential equations, whose solution can be obtained through integration techniques.

The simplest case is an equation of the form \( y' = f(x) \), with \( f(x) \) a given function, such as \( x^2 \). Then the solution is given by \( y = \int f(x)dx \); in the case \( f(x) = x^2 \), we have

\[ y = \int x^2dx = x^3/3 + C, \]

where \( C \) is an arbitrary constant. To determine \( C \), one usually needs an initial condition, such as \( y(0) = 1 \), in which case, we should choose \( C = 1 \) and the solution to the initial value problem

\[ y' = x^2, \quad y(0) = 1 \]
is uniquely given by \( y = x^3/3 + 1 \).

More generally, for a differential equation of the form

\[ y' = f(x)g(y), \]

we can apply a so called \textit{separation of variables technique} to find its solutions through integration. To see how this technique works, let us suppose that \( y = y(x) \) is a solution to this equation, and try to find a formula for \( y(x) \) through integration. We firstly rewrite the equation in the following form:

\[ \frac{1}{g(y)}y' = f(x). \]

Notice that we may only divide by \( g(y) \) if it is different from 0. Assume that \( y_1 \) is a solution of the equation

\[ g(y) = 0. \]

Then the constant function \( y(x) = y_1 \) solves the differential equation. In fact, \( y'(x) = 0 \) being the derivative of a constant, and \( f(x)g(y) = f(x)g(y_1) = 0 \) being a product with a zero factor. Solution that occur this way are called \textit{steady state solutions} (see Section 9.3).

Assume now \( g(y) \neq 0 \). Then

\[ \frac{1}{g(y(x))}y'(x) = f(x), \]

since we have assumed that \( y(x) \) solves the differential equation.

Suppose that \( H(y) \) is an antiderivative of \( \frac{1}{g(y)} \), in other words,

\[ H'(y) = \frac{1}{g(y)}, \quad \text{or} \quad H(y) = \int \frac{1}{g(y)}dy. \]

Then by the chain rule we have

\[ \frac{d}{dx}H(y(x)) = H'(y(x))y'(x) = \frac{1}{g(y(x))}y'(x). \]

In view of the differential equation satisfied by \( y(x) \), we obtain from this identity that

\[ \frac{d}{dx}H(y(x)) = f(x). \]

Therefore

\[ H(y(x)) = \int f(x)dx. \]
The above rigorous argument can be better remembered by the following formal deduction:

Rewrite the original differential equation as
\[ \frac{dy}{dx} = f(x)g(y). \]

We think of \( dy/dx \) as a quotient and rewrite this equation so that each side contains only one variable (separation of variables):
\[ \frac{1}{g(y)} \, dy = f(x) \, dx. \]

Integrating both sides we obtain
\[ \int \frac{1}{g(y)} \, dy = \int f(x) \, dx, \]
that is
\[ H(y) = \int f(x) \, dx. \]
This is the same identity that we have obtained before by using rigorous argument.

**Example**

Find the solutions to the differential equation
\[ y' = x^2 y^2. \]

**Solution**

We apply the separation of variables technique. Rewrite the equation in the form
\[ \frac{dy}{dx} = x^2 y^2, \quad y^{-2} \, dy = x^2 \, dx. \]
Integrating we obtain
\[ \int y^{-2} \, dy = \int x^2 \, dx, \quad y^{-1}/(-1) = x^3/3 + C. \]
Therefore
\[ y = \frac{-1}{x^3/3 + C} = \frac{-3}{x^3 + 3C}, \]
where \( C \) is an arbitrary constant.
9.2 Exponential Growth and Decay

Example

Let \( y(t) \) be the population of a country at time \( t \). Let \( B \) be the annual birth rate per head of population and let \( D \) be the annual death rate per head. The statistics \( B \) and \( D \) are usually nearly constant and known quite accurately. How will the population \( y(t) \) change with time?

\[
\text{No. of births per year} = By \\
\text{No. of deaths per year} = Dy
\]

So rate of change of population per year is

\[
By - Dy = ky,
\]

where \( k = B - D \) is called the **specific growth rate** of the population and assumed to be constant. Thus the rate of change of the population \( y \) with respect to time \( t \) is

\[
\frac{dy}{dt} = ky.
\]

This is a **differential equation** since it is an equation involving the function \( y \) and its derivative \( \frac{dy}{dt} \).

We can solve this differential equation by “separation of variables”:

For \( y \neq 0 \) divide by \( y \):

\[
\frac{1}{y} \frac{dy}{dt} = k,
\]

multiply by \( dt \):

\[
\frac{1}{y} dy = k dt,
\]

integrate:

\[
\int \frac{1}{y} dy = \int k dt \\
\ln |y| = kt + c \\
e^{\ln|y|} = e^{kt+c} \\
|y| = e^{kt+c} = e^{kt}e^c
\]

Note that the constant of integration in the second step above is needed on only one side of the equation (since the constant on the other side of could be absorbed into this one).
The constant expression $e^c$ in the final step can be relabelled as a constant $A$ which is necessarily positive (since $e^c$ is always positive). By resolving the expression $|y|$ we find two options $y = Ae^{kt}$ or $y = -Ae^{kt}$. Therefore we may admit also negative values of $A$. Finally, if we add the steady state solution $y(x) = 0$ we see that $A$ can be any real number. Of course, in our population model only non-negative values of $y$ and $A$ make sense.

Thus the solution of the differential equation is

$$y = Ae^{kt}$$

where $A$ is any real constant. If the population at time zero is given, say $y(0) = y_0$, then from the solution we find

$$y(0) = Ae^{k\cdot0} = A$$

so that $A = y_0$ and

$$y = y_0e^{kt}.$$  

This is the equation of **simple exponential growth**.

**Example**

A culture of bacteria increases its population by one fifth daily. If the initial population is 10,000 how large is the population after 10 days? How long does it take the population to reach 1,000,000?

**Solution**

Here $k = \frac{1}{5}$ so the differential equation is

$$\frac{dy}{dt} = \frac{1}{5}y$$

with initial population 10,000 which has the solution

$$y = 10,000e^{\frac{1}{5}t}$$

When $t = 10$,

$$y = 10,000e^{(1/5)\cdot10} = 10,000e^2 \approx 73,891.$$  

So the population after 10 days is about 73,900.
When \( y = 1,000,000 \),
\[
10,000e^{\frac{1}{5}t} = 1,000,000 \\
e^{\frac{1}{5}t} = 100 \\
\frac{1}{5}t = \ln 100 \\
t = 5\ln 100 \approx 23.
\]
Thus the population reaches 1,000,000 after 23 days.

Exponential decay

Each kind of radio-active atom has a certain probability \( p \) of decaying in any given second. So in a sample of \( y \) atoms an expected number \( py \) decay per second.

Rate of change of \( y \) per sec. = \(-py\)

\[
\frac{dy}{dt} = -py \\
y = Ae^{-pt}
\]
(Like the population model with birth rate \( B = 0 \).)

This is the equation of simple exponential decay.

The half-life of a radio-active element is the time it takes for half the atoms in a sample to decay (see Figure 9.1). Let the half-life be \( T \).

When \( t = 0 \) no. of atoms = \( Ae^0 = A \).

When \( t = T \) no. of atoms = \( Ae^{-pT} = \frac{1}{2}A \).

So

\[
e^{-pT} = \frac{1}{2} \\
e^{pT} = 2 \\
pT = \ln 2 \\
T = \frac{1}{p}\ln 2.
\]

The half-life does not depend on the size of the sample.
9.3 Restricted Exponential Growth and Decay

The equations of exponential growth and decay came from the situation where the rate of change of a variable $y$ was proportional to $y$:

$$\frac{dy}{dx} = ky = k \times \text{(distance of } y \text{ from 0)}.$$  

This gave exponential growth when $k > 0$ and decay when $k < 0$. There are many situations where the rate of change of $y$ is proportional to the distance of $y$ from a steady value $L$:

$$\frac{dy}{dx} = k \times \text{(distance of } y \text{ from } L) = k(L - y).$$

where $k$ and $L$ are constants. Note that when $L = 0$ this reduces to equation for exponential decay

$$\frac{dy}{dx} = -ky.$$ 

The case when $k$ is positive is the most important in practice and we will assume this is the case throughout this section.

Again the equation can be solved by “separation of variables”:

$$dy = k(L - y)dt$$

$$\frac{dy}{L - y} = kdt$$

$$\int \frac{dy}{L - y} = \int kdt$$
We find the steady state solution \( y = y_1 \) by solving the equation \( L - y = 0 \). Hence \( y = L \) is a solution.

The integrand is of the form \( \frac{1}{z} \) with \( z = L - y \). This is Example 3 from Section 5.1.

\[
\int \frac{dy}{L - y} = -\ln |z| + c_1 = -\ln |L - y| + c_1.
\]

Hence

\[
-\ln |L - y| + c_1 = \int kdt = kt + c_2
\]

\[
\ln |L - y| = -kt + c \quad \text{(where } c = c_1 - c_2) \]

\[
|L - y| = e^{-kt}e^c
\]

\[
y - L = -Ae^{-kt} \quad \text{(where } A \text{ is any real constant)}.
\]

\[
y = L - Ae^{-kt}
\]

the equation of restricted exponential decay. The unknown constant \( A \) can be found from the starting value of \( y \). If \( y(0) = y_0 \) then

\[
y_0 = y(0) = L - Ae^{-0t} = L - A
\]

thus

\[
A = L - y_0
\]

Figure 9.2: Solutions of the differential equation for restricted exponential growth.

Note that for large values of \( t \), \( y(t) \) approaches the value \( L \).
Steady state solutions

It is much easier to find what values variables will have when a system reaches a steady state or equilibrium than to find exactly how they will change on the way to a steady state. In a steady state, when \( y \) is not changing, \( \frac{dy}{dt} = 0 \). For our equation this gives \( k(L - y) = 0 \) so \( y = L \). Hence \( y = L \) is the steady state solution of the equation. If \( y \) has the value \( L \) it stays at \( L \).

Example

A freshly poured cup of tea has temperature \( 80^\circ \text{C} \). The air temperature is \( 20^\circ \text{C} \). Let the temperature of the tea at time \( t \) be \( T(t) \). (So \( T(0) = 80 \).) Then \( T(t) \) satisfies

\[
\frac{dT}{dt} = k(20 - T) \text{ deg/min},
\]

where \( k \) is constant. Suppose \( k = 0.05 \). What is the temperature of the tea after 10 minutes? How long does it take to reach \( 40^\circ \?)

Solution

The differential equation is

\[
\frac{dT}{dt} = 0.05(20 - T),
\]

so its solution is

\[
T = 20 - Ae^{-0.05t}. 
\]

Find \( A \):

\[
A = L - T_0 = 20 - 80 = -60. 
\]

Therefore

\[
T = 20 + 60e^{-0.05t}. 
\]

Find \( T \) when \( t = 10 \):

\[
T = 20 + 60e^{-0.05 \times 10} = 20 + 60e^{-0.5} = 56.4^\circ \text{C}. 
\]

Find \( t \) when \( T = 40 \):

\[
40 = 20 + 60e^{-0.05t} \\
20 = 60e^{-0.05t} \\
1 = 3e^{-0.05t} \\
e^{0.05t} = 3 \\
0.05t = \ln 3 \approx 1.1 \\
t \approx 1.1/0.05 = 22 \text{ mins.}
\]
So the time taken to reach 40°C is about 22 mins.

Applications of restricted exponential growth equation

1. **Newton’s Law of Cooling**: If an isolated object $O$ has temperature $T$ and its environment has temperature $L$ then

\[
\frac{dT}{dt} = k(L - T),
\]

where $k$ is a constant. This is illustrated in the example above.

2. **Restricted population growth**. Suppose a population $y$ has a fixed food supply $F$ and each individual consumes $f$ per year. When the population is close to maximum capacity we would expect:

\[
\text{Rate of increase of population } \propto \text{ excess food supply } = F - yf.
\]

So

\[
\frac{dy}{dt} = c(F - yf) = k(L - y),
\]

where $k = cf$ and $L = F/f$. For large values of $t$ the population approaches $L = F/f$ the situation where each individual has just enough food. This equation also applies to plant populations growing in a restricted area, where it gives quite a good approximation.
9.4 The Logistic Equation

Models of population growth

1. Simple exponential growth

\[ \frac{dy}{dt} = ky \]

This is a good approximation when \( y \) is small, but can’t continue to be satisfied when \( y \) is large since the solution \( y(t) \) grows rapidly forever which is physically impossible.

2. Restricted exponential growth

\[ \frac{dy}{dt} = k(L - y) \]

This is a good approximation when \( y \) is near \( L \), not good when \( y \) is small since when the population \( y \) is small the equation gives growth rate \( \approx kL \), but the growth rate should be small when the population is small.

3. Logistic equation Assume the birth rate \( B \) per head is constant but the death rate \( D \) per head increases with population size \( y \) (due to reduced food ration, not enough space, more exposure to infection, for example). Let’s assume \( D = ky \). Then rate of change of population is

\[ \frac{dy}{dt} = By - Dy = By - ky^2 = ky \left( \frac{B}{k} - y \right). \]

Put \( B/k = L \). Then

\[ \frac{dy}{dt} = k(y(L - y)) \]

the logistic equation.

When \( y \) is small \( dy/dt \approx kLy \propto y \): the logistic equation behaves like simple exponential growth when \( y \) is small.

When \( y \) is near \( L \), \( dy/dt \approx kL(L - y) \): the logistic equation behaves like restricted exponential growth when \( y \) is near \( L \).

Steady state solutions

The steady state solutions are the values of \( y \) for which \( dy/dt = 0 \). So \( k(y(L - y)) = 0 \) and \( y = 0 \) or \( L \). If the population starts as 0 or \( L \) it stays there.
CHAPTER 9. DIFFERENTIAL EQUATIONS (SCIENCE STREAM ONLY)

Solution of the logistic equation

Again we solve by separation of variables.

\[
\frac{dy}{dt} = ky(L - y)
\]

\[
\frac{dy}{y(L - y)} = kdt
\]

\[
\frac{L}{y(L - y)}dy = Lkdt
\]

\[
\left\{ \frac{1}{y} + \frac{1}{L - y} \right\} dy = Lkdt \quad (9.1)
\]

(In the last step we have used that \( \frac{L}{y(L - y)} = \frac{(L - y) + y}{y(L - y)} = \frac{L - y}{y(L - y)} + \frac{y}{y(L - y)} \). Notice that subtracting and then adding \( y \) in the numerator of the fraction does not change it.)

\[
\int \frac{dy}{y} = \ln |y| + c_1,
\]

\[
\int \frac{dy}{(L - y)} = -\ln |L - y| + c_2, \quad \text{(as we found in the previous section)}
\]

\[
\int Lkdt = Lkt + c_3.
\]

So (9.1) gives

\[
\ln |y| - \ln |L - y| = Lkt + c \quad \text{(where } c = c_3 - c_1 - c_2)\]

\[
\ln \left| \frac{y}{L - y} \right| = Lkt + c
\]

\[
\frac{y}{L - y} = \pm e^{Lkt}e^{c} = Ae^{Lkt} \quad \text{(where } A = \pm e^{c} \text{ or } 0)\]

\[
y = (L - y)Ae^{Lkt}
\]

\[
e^{-Lkt}y = AL - Ay
\]

\[
(A + e^{-Lkt})y = AL
\]

\[
y = \frac{LA}{A + e^{-Lkt}}
\]

The constant \( A \) can be found from the initial value of \( y \). If \( y(0) = y_0 \) then

\[
y_0 = y(0) = \frac{LA}{A + 1}.
\]
Solving this equation for $A$ gives

$$A = \frac{y_0}{L - y_0}.$$  

Note that this formula only works if $y_0 \neq L$. But we know that for $y_0 = L$ we have the steady state solution $y(t) = L$.

![Figure 9.3: Solution of the logistic equation.](image)

**Example**

Assume that the world population of blue whales has been modelled by the logistic equation with $k = 4 \times 10^{-7}$ (per year) and steady state size 200,000. The whale population when a moratorium was introduced in 1972 was 500. What would we estimate the population to have been in the year 2012, according to this model? In what year would the population be expected to have reached 100,000, according to this model?

**Solution**

Let $t$ be the time in years since 1972. The required logistic differential equation is

$$\frac{dy}{dt} = 4 \times 10^{-7} y (200,000 - y).$$

$Lk = 200,000 \times 4 \times 10^{-7} = 0.08$, so its solution is

$$y = \frac{200,000A}{A + e^{-0.08t}}.$$
CHAPTER 9. DIFFERENTIAL EQUATIONS (SCIENCE STREAM ONLY)

Find $A$:

$$A = \frac{y_0}{L - y_0} = \frac{500}{200,000 - 500} = \frac{1}{399} \approx 0.0025.$$  

So

$$y = \frac{200,000A}{A + e^{-0.08t}} = \frac{500}{0.0025 + e^{-0.08t}}.$$  

Find $y$ when $t = 40$:

$$y = \frac{500}{0.0025 + e^{-0.08\times40}} = 11,600.$$  

According to the model the population of blue whales is about 11,600.

Find $t$ when $y = 100,000$:

$$100,000 = \frac{500}{0.0025 + e^{-0.08t}}$$

$$0.0025 + e^{-0.08t} = \frac{1}{200}$$

$$e^{-0.08t} = \frac{1}{400}$$

$$0.08t = \ln 400$$

$$t \approx 75$$

According to this model the population of blue whales will have reached 100,000 by about the year 2047.
Chapter 10

Applications in Economics
(Economics stream only)

10.1 The Input-Output Model

The input-output model was first introduced in the late forties by Leontief, the recipient of a 1973 Nobel Prize, in a study of the U.S. economy. The main feature of this model is that it incorporates the interactions between different industries or sectors which make up the economy. The aim of the model is to allow economists to forecast the future production levels of each industry in order to meet future demands for the various products. Such forecasting is complicated as a change in the demand for one product can induce a change in production levels of many industries. For example, an increase in the demand for cars leads not only to an increase in the production level of automobile manufacturers, but also in the levels of many other industries in the economy, such as the steel industry, the rubber industry, and so on. In Leontief’s original model, he divided the U.S. economy into 500 interacting sectors of this type.

In order to discuss the model in the simplest possible terms, we consider a hypothetical economy with only two industries $P$ and $Q$. The interaction between $P$ and $Q$ are described by the following table:

<table>
<thead>
<tr>
<th>Industry</th>
<th>Industry P inputs</th>
<th>Industry Q inputs</th>
<th>Consumer demands</th>
<th>Total Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>60</td>
<td>64</td>
<td>76</td>
<td>200</td>
</tr>
<tr>
<td>Industry Q</td>
<td>100</td>
<td>48</td>
<td>12</td>
<td>160</td>
</tr>
<tr>
<td>Labor</td>
<td>40</td>
<td>48</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total inputs</td>
<td>200</td>
<td>160</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The first two columns in this table give the inputs of $P$ and $Q$, measured in suitable
units (e.g., millions of dollars). From the first column, we see that \( P \) uses 60 units of its own product, 100 units of \( Q \)'s product and 40 units of labor. Similarly, from the second column, \( Q \) uses 64 units of \( P \)'s product, 48 units of its own product and 48 units of labor. Totalling the columns, we see that the total inputs for \( P \) is 200 units, and that for \( Q \) is 160 units.

Now consider the first two rows in the table. The first row shows how the outputs of \( P \) are used: 60 units for \( P \) itself, 64 units for \( Q \) and 76 units for consumers, totalling 200, which is the same as \( P \)'s total inputs. Similarly, the outputs of \( Q \) are used as follows: 100 units for \( P \), 48 units for \( Q \) itself, and 12 units for consumers, totalling 160 units, the same as \( Q \)'s total inputs. The fact that the total inputs and total outputs of \( P \) and \( Q \), respectively, are the same shows that the economy is perfectly balanced. Therefore the structure of the economy should be kept when changes are needed in the future.

Suppose that market research predicts that in 5 years, the consumer demand for \( P \) will decrease from 76 to 70 units, whereas for \( Q \), it will increase considerably from 12 to 60 units. Then how much should each industry adjust its production level in order to meet these projected consumer demands?

Let us suppose that in order to meet these new demands, \( P \) must produce \( x_1 \) units and \( Q \) must produce \( x_2 \) units.

From the first column in the table we see that to produce 200 units, \( P \) uses 60 units of its own produce and 100 units of \( Q \)'s product. Thus to produce \( x_1 \) units, \( P \) must use

\[
\frac{60}{200}x_1 \quad \text{units of its own product and} \quad \frac{100}{200}x_1 \quad \text{units of \( Q \)'s product.}
\]

Similarly, using the second column in the table we see that to produce \( x_2 \) units, \( Q \) must use

\[
\frac{64}{160}x_2 \quad \text{units of \( P \)'s product and} \quad \frac{48}{160}x_2 \quad \text{units of its own product.}
\]

Thus we have a new table describing the interaction of \( P \) and \( Q \) under the new demands.

\begin{tabular}{|c|c|c|c|c|}
\hline
 & \textbf{\( P \) inputs} & \textbf{\( Q \) inputs} & \textbf{Consumer demands} & \textbf{Total outputs} \\
\hline
\textbf{\( P \) outputs} & \frac{60}{200}x_1 & \frac{64}{160}x_2 & 70 & \text{\( x_1 \)} \\
\hline
\textbf{\( Q \) outputs} & \frac{100}{200}x_1 & \frac{48}{160}x_2 & 60 & \text{\( x_2 \)} \\
\hline
\end{tabular}
Thus we have

\[
\begin{align*}
x_1 &= \frac{60}{200} x_1 + \frac{64}{160} x_2 + 70 \\
x_2 &= \frac{200}{100} x_1 + \frac{48}{160} x_2 + 60
\end{align*}
\]

These two equations can be written in matrix form as

\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 60 & 64 \\ 200 & 160 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 70 \\ 60 \end{bmatrix}
\]

Thus

\[
X = AX + D,
\]

where \( X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) is called the output matrix, \( D = \begin{bmatrix} 70 \\ 60 \end{bmatrix} \) is called the demand matrix, and \( A = \begin{bmatrix} 60 & 64 \\ 200 & 160 \end{bmatrix} \) is called the input-output matrix.

From \( X = AX + D \), we obtain

\[
X - AX = D, (I - A)X = D.
\]

If \((I - A)^{-1}\) exists, then

\[
X = (I - A)^{-1}D.
\]

In our case here,

\[
I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 60 & 64 \\ 200 & 160 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.4 \\ 0.5 & 0.3 \end{bmatrix} = \begin{bmatrix} 0.7 & -0.4 \\ -0.5 & 0.7 \end{bmatrix}
\]

\[
(I - A)^{-1} = \frac{1}{29} \begin{bmatrix} 70 & 40 \\ 50 & 70 \end{bmatrix},
\]

\[
X = (I - A)^{-1}D = \frac{1}{29} \begin{bmatrix} 70 & 40 \\ 50 & 70 \end{bmatrix} \begin{bmatrix} 70 \\ 60 \end{bmatrix} = \begin{bmatrix} 251.7 \\ 265.5 \end{bmatrix}
\]

Thus \( P \) must produce 251.7 units and \( Q \) should produce 265.5 units.

**Example 1.** Suppose that in a hypothetical economy with two industries \( A \) and \( B \), the interaction between \( A \) and \( B \) is as shown in the following table.
(a) Find the input-output matrix $A$.

(b) Determine the output matrix if the demands change to 312 units for industry $A$ and 299 units for industry $B$.

(c) What will then be the new primary inputs for $A$ and $B$?

Solution

(a) $A = \begin{bmatrix} 240 & 750 & 210 & 1200 \\ 720 & 450 & 330 & 1500 \\ 240 & 300 \\ 1200 & 1500 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.5 \\ 0.6 & 0.3 \end{bmatrix}$

(b) $D = \begin{bmatrix} 312 \\ 299 \end{bmatrix}$

$X = (I - A)^{-1}D = \frac{5}{13} \begin{bmatrix} 7 & 5 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} 312 \\ 299 \end{bmatrix} = \begin{bmatrix} 1415 \\ 1640 \end{bmatrix}$

(c) New primary inputs for $A = \frac{240}{1200} \times \text{(new outputs of } A) = 0.2 \times 1415 = 283$

New primary inputs for $B = \frac{300}{1500} \times \text{(new outputs of } B) = 0.2 \times 1640 = 328$

Note that the given table gives the proportions for the inputs for $A$ as:

$\frac{240}{1200}$ of $A$, $\frac{720}{1200}$ of $B$ and $\frac{240}{1200}$ of primary.

That for $B$ are:

$\frac{750}{1500}$ of $A$, $\frac{450}{1500}$ of $B$ and $\frac{300}{1500}$ of primary.
10.2 Linear Inequalities

We know that \( y = mx + b \) represents a straight line in the \( xy \)-plane. But what does the inequality \( y \geq mx + b \) represent? Let us look at a concrete case: \( y \geq 2x - 4 \). Here the two variables \( x \) and \( y \) are related by an inequality, and the right side of the inequality is a linear function of \( x \). Hence we call this a linear inequality.

The equation \( y = 2x - 4 \) has as its graph a straight line with slope 2 and \( y \)-intercept-4.

It turns out that the inequality \( y \geq 2x - 4 \) is satisfied by any point that lies on or above the straight line \( y = 2x - 4 \).

Therefore, all the points which satisfy \( y \geq 2x - 4 \) form a half plane that lies above the line \( y = 2x - 4 \). Similarly, the inequality \( y \leq 2x - 4 \) gives a half plane that lies below \( y = 2x - 4 \). If equality is not allowed, then \( y > 2x - 4 \) gives the half plane above \( y = 2x - 4 \) excluding the straight line \( y = 2x - 4 \). Similarly, \( y < 2x - 4 \) gives the half plane below \( y = 2x - 4 \) excluding this straight line itself.

**Example 1** Sketch the graph of the linear inequality \( 2x - 3y < 6 \).

**Solution.** We know \( 2x - 3y = 6 \) gives a straight line which can be rewritten as

\[
-3y = 6 - 2x, \quad y = \frac{1}{-3} (6 - 2x) = -2 + \frac{2}{3}x
\]
The inequality $2x - 3y < 6$ represents a half plane excluding the straight line $2x - 3y = 6$, but we need to determine whether the half plane is above or below the line $2x - 3y = 6$.

Adding $3y$ to both sides of the inequality we obtain

$$2x - 3y + 3y < 6 + 3y$$

or

$$2x < 6 + 3y$$

Subtracting 6 from both sides of the inequality we obtain

$$2x - 6 < 6 + 3y - 6$$

or

$$2x - 6 < 3y$$

or

$$3y > 2x - 6.$$ 

Dividing both sides by 3, we get

$$y > \frac{2}{3}x - 2$$

Hence the half plane is above the straight line $y = \frac{2}{3}x - 2$.

Note that if we subtract $2x$ from both sides of the original inequality, we deduce

$$-3y < 6 - 2x$$

and if we divide the inequality by $-3$, we should reverse the inequality sign and obtain

$$y > \frac{6 - 2x}{-3} = -2 + \frac{2}{3}x$$

Example 2. An investor plans to invest up to $30000 in two stocks, $A$ and $B$. Stock $A$ is currently priced at $165$ and stock $B$ at $90$ per share. If the investor buys $x$ shares of $A$ and $y$ shares of $B$, use an inequality to represent the relationship of $x$ and $y$.

Solution. The total cost of buying $x$ shares of $A$ and $y$ shares of $B$ is

$$165x + 90y \text{ dollars}$$
As the investor plans to invest up to $30000 in buying these two shares, apparently we should have
\[165x + 90y \leq 30000\]

Very often, inequalities of more than two variables are needed. Then it is usually difficult to find a graph for the inequality or inequalities. One then has to rely on analytical tools to analyse them.

**Example 3.** An electronics company makes television sets at two factories, \(F_1\) and \(F_2\). \(F_1\) can produce up to 100 sets per week and \(F_2\) can produce up to 200 sets per week. The company has three distribution centres, \(X\), \(Y\) and \(Z\). \(X\) requires 50 television sets per week, \(Y\) requires 75 sets per week, and \(Z\) requires 125 sets per week in order to meet the demands in their respective areas. If factory \(F_1\) supplies \(x\) sets per week to distribution centre \(X\), \(y\) sets to \(Y\), and \(z\) sets to \(Z\), write the inequalities satisfied by \(x\), \(y\), and \(z\).

**Solution.** It is better to use the diagram [10.3] to help us analyze the relationships between \(x\), \(y\), and \(z\).

Firstly we must have \(x \geq 0\), \(y \geq 0\), \(z \geq 0\).

Secondly, the total number of sets supplied by \(F_1\) should not be larger than 100. Hence
\[x + y + z \leq 100.\]

Thirdly, as \(X\) requires 50 sets per week, the supply by \(F_1\), should not be bigger than this
number, i.e.

\[ x \leq 50. \]

Similarly

\[ y \leq 75, \quad z \leq 125. \]

As \( F_1 \) supplies \( x \) sets to \( X \) and \( X \) requires 50 sets per week, \( F_2 \) must supply \( 50 - x \) sets to \( X \). Similarly, \( F_2 \) must supply \( 75 - y \) sets to \( Y \) and \( 125 - z \) sets to \( Z \). The total number of sets supplied by \( F_2 \) is then

\[
(50 - x) + (75 - y) + (125 - z) = 250 - (x + y + z)
\]

But this number should not be larger than 200 which \( F_2 \) can produce up to per week. Thus we have

\[
250 - (x + y + z) \leq 200,
\]

or

\[
x + y + z \geq 50.
\]

Thus we have eight inequalities:

\[
x \geq 0, \quad y \geq 0, \quad z \geq 0, \quad x \leq 50, \quad y \leq 75, \quad z \leq 125
\]

\[
x + y + z \leq 100, \quad x + y + z \geq 50
\]
10.3 Linear Optimization

A linear programming problem is one that involves finding the maximum or minimum value of some linear algebraic expression when the variables in this expression are subject to a number of linear inequalities. The following example is typical of such problems.

Example 1 (Maximum Profit). A company manufactures two products, $X$ and $Y$. Each of these products requires a certain time on the assembly line and a further amount of time in the finishing shop. Each item of $X$ needs 5 hours for assembly and 2 hours for finishing, and each item of type $Y$ needs 3 hours for assembly and 4 hours for finishing. In any week, the firm has available 105 hours on the assembly line and 70 hours in the finishing shop. The firm can sell all it can produce and makes a profit of $200 on each item of $X$ and $160 on each item of $Y$. Find the number of items of each type that should be manufactured per week to maximize the total profit.

Solution It is usually convenient to summarize the given information in the form of a table. The following table shows the information given in Example 1.

<table>
<thead>
<tr>
<th></th>
<th>Assembly</th>
<th>Finishing</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>5</td>
<td>2</td>
<td>200</td>
</tr>
<tr>
<td>$Y$</td>
<td>3</td>
<td>4</td>
<td>160</td>
</tr>
<tr>
<td>Available</td>
<td>105</td>
<td>70</td>
<td></td>
</tr>
</tbody>
</table>

Suppose that the firm produces $x$ items of $X$ per week, and $y$ items of $Y$ per week. Then the time needed on the assembly line will be $5x$ hours for $X$ and $3y$ hours for $Y$, or $(5x + 3y)$ hours in all. Since only 105 hours are available, we must have

$$5x + 3y \leq 105.$$ 

Similarly, considering the time needed in the finishing shop, we derive

$$2x + 4y \leq 70.$$ 

Each item of $X$ produces a profit of 200 dollars, $x$ items produce $200x$ dollars in profit. Similarly, $y$ items of $Y$ produce 160 $y$ dollars in profit. Hence the total weekly profit $P$ (in dollars) is given by

$$P = 200x + 160y.$$ 

Therefore the problem is to find values of $x$ and $y$ that maximize

$$P = 200x + 160y$$

when $x$ and $y$ are subject to the conditions

$$5x + 3y \leq 105, \quad 2x + 4y \leq 70, \quad x \geq 0, \quad y \geq 0.$$
Note that the inequalities \( x \geq 0, y \geq 0 \) are added for completeness. This follows from the definition of \( x \) and \( y \), and adding these two inequalities is necessary for the mathematical treatment of this problem.

To find \( x \) and \( y \) manually for this problem, one can use the methods in Chapter 12 of the textbook. In this unit, we are satisfied with using Excel to find the solutions, and the detailed steps for using Excel to solve this kind of problems are given in the booklet “Workbook for Practical Classes”.

The solution to this problem (by Excel) is

\[
\begin{align*}
  x &= 15, \\
  y &= 10
\end{align*}
\]

\( \Box \) In a general linear programming problem, the inequalities that must be satisfied by the variables are called the constraints, and the linear function to be maximized or minimized is called the objective function. In Example 1, the constraints are:

\[
\begin{align*}
  5x + 3y &\leq 105, & 2x + 4y &\leq 70, & x &\geq 0, & y &\geq 0,
\end{align*}
\]

and the objective function is

\[
P = 200x + 160y.\]

**Example 2.** A chemical firm makes two brands of fertilizer. The regular brand contains nitrates, phosphates, and potash in the ratio 3:6:1 (by weight) and the super brand contains these three ingredients in the ratio 4:3:3. Each month the firm can rely on a supply of 9 tons of nitrates, 13.5 tons of phosphates, and 6 tons of potash. The firm’s manufacturing plant can produce at most 25 tons of fertilizer per month. If the firm makes a profit of 300 dollars on each ton of regular fertilizer and 480 dollars on each ton of the super grade, what amounts of each grade should be produced in order to yield the maximum profit?

**Solution** The information given is summarized in the following table.

<table>
<thead>
<tr>
<th>Nitrates</th>
<th>Phosphates</th>
<th>Potash</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular</td>
<td>0.3</td>
<td>0.6</td>
<td>0.1</td>
</tr>
<tr>
<td>Super</td>
<td>0.4</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Available</td>
<td>9</td>
<td>13.5</td>
<td>6</td>
</tr>
</tbody>
</table>

Let the firm produce \( x \) tons of regular grade and \( y \) tons of super grade. Then the constraints are

\[
\begin{align*}
  x &\geq 0, & y &\geq 0, & 0.3x + 0.4y &\leq 9, & 0.6x + 0.3y &\leq 13.5 & 0.1x + 0.3y &\leq 6
\end{align*}
\]

and the objective function is

\[
P = 300x + 480y.\]

Using Excel, we obtain \( x = 6, y = 18 \).

\( \Box \)
Example 3. A chemical company is designing a plant for producing two types of polymer, \( P_1 \) and \( P_2 \). The plant must be capable of producing at least 100 units of \( P_1 \) and 420 units of \( P_2 \) per day. There are two possible designs for the basic reaction chambers which are to be included in the plant: each chamber of type A costs 600,000 dollars and is capable of producing 10 units of \( P_1 \) and 20 units of \( P_2 \) per day; type B is a cheaper design costing 300,000 dollars and capable of producing 4 units of \( P_1 \) and 30 units of \( P_2 \) per day. For some reason it is necessary to have at least 4 chambers of each type in the plant. How many chambers of each type should be included to minimize the cost while still meeting the required production schedule?

Solution The given information is summarized in the following table.

<table>
<thead>
<tr>
<th></th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>Cost (Thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chamber A</td>
<td>10</td>
<td>20</td>
<td>600</td>
</tr>
<tr>
<td>Chamber B</td>
<td>4</td>
<td>30</td>
<td>300</td>
</tr>
<tr>
<td>Required</td>
<td>100</td>
<td>420</td>
<td></td>
</tr>
</tbody>
</table>

Let the design include \( x \) chambers of type A and \( y \) chambers of type B. Then the constraints are

\[
x \geq 4, \quad y \geq 4, \quad 10x + 4y \geq 100, \quad 20x + 30y \geq 420.
\]

We want to minimize the objective function

\[
C = 600x + 300y.
\]

Using Excel, it gives \( x = 6 \) and \( y = 10 \). \( \square \)
10.4 Compound Interest

Consider a sum of money, say $100, that is invested at a fixed rate of interest, such as 6% per annum. After 1 year, the investment will have increased in value by 6% to $106. If the interest is compounded, then during the second year, this whole sum of $106 earns interest at 6%. Thus the value of the investment at the end of the second year will consist of the $106 existing at the beginning of that year plus 6% of $106 in interest, giving a total value of

\[
106 + (106)(0.06) = (106)(1 + 0.06) = (106)(1.06) = 100 \times (1.06)^2 = 112.36.
\]

During the third year, the value increases by an amount of interest equal to 6% of $112.36, giving a total value at the end of that year equal to

\[
112.36 + 112.36 \times 0.06 = 112.36(1 + 0.06) = 112.36(1.06) = 100 \times (1.06)^3.
\]

In general, the investment increases by a factor of 1.06 with each year that passes, so after \(n\) years, its value is $100 \times (1.06)^n$.

The above way of calculating the value of an investment growing at compound interest can be generalised to the general case, that is, if a sum \(P\) is invested at a rate of interest of \(R\) percent per annum, then the value of the investment after \(n\) years is given by the formula

\[
\text{Value after } n \text{ years} = P(1 + i)^n, \quad i = \frac{R}{100}.
\]

**Example 1 (Investment)** A sum of $200 is invested at 5% interest compounded annually. Find the value of the investment after 10 years.

**Solution** We have \(P = 200, R = 5, i = \frac{5}{100} = 0.05, \) and \(n = 10\). Hence,

\[
\text{Value after 10 years} = 200(1 + 0.05)^{10} = 200 \times (1.05)^{10} = 200 \times (1.628895) = 325.78 \text{ (dollars)}.
\]

In some cases, interest is compounded more than once per year, for example semiannually (2 times per year), quarterly (4 times per year) or monthly (12 times per year). In these cases, annual rate of interest \(R\) percent which is usually quoted is called the **nominal** rate. If compounding occurs \(k\) times per year with a nominal rate of interest \(R\) percent, then the interest rate at each compounding is equal to \(\frac{R}{k}\) percent, and if the number of compounding period is \(n\), the compound interest formula becomes
Value after $n$ periods $= P \left(1 + \frac{R}{100k}\right)^n$

Thus, after $N$ years, we have $kN$ compounding periods, and

Value after $N$ years $= P \left(1 + \frac{R}{100k}\right)^{kN}$

Example 2 (Monthly Compounding) A sum of $2000 is invested at a nominal rate of interest of 9% compounded monthly. Find the value of the investment after 3 years.

Solution We have $k = 12$, $R = 9$ and $P = 2000$. After 3 years, there are $n = 3 \times k = 3 \times 12 = 36$ compounding periods. Hence

$$\text{Value after 3 years} = P \left(1 + \frac{R}{100k}\right)^n = 2000 \left(1 + \frac{9}{100 \times 12}\right)^{36} = 2000(1.0075)^{36} = 2617.29 \text{ (dollars).}$$

The effective rate of interest of an investment is defined as the annual rate that would provide the same growth if compounded once per year. Consider an investment that is compounded $k$ times per year at nominal rate of interest of $R\%$. Then the value of investment after 1 year is

$$P \left(1 + \frac{R}{100k}\right)^k.$$  

i.e. the investment grows by a factor $\left(1 + \frac{R}{100k}\right)^k$ in one year. If we use $i_{\text{eff}}$ to denote the effective rate of interest, the investment grows by a factor $(1 + i_{\text{eff}})$ each year. Thus we must have

$$1 + i_{\text{eff}} = \left(1 + \frac{R}{100k}\right)^k, \quad i_{\text{eff}} = \left(1 + \frac{R}{100k}\right)^k - 1$$

Example 3 Which is better for the investor, 12% compounded monthly or 12.2% compounded quarterly?

Solution We compute the effective rate for each of the two investment plans, the one with a larger effective rate is better for the investor.

For the first plan, $R = 12, k = 12$,

$$i_{\text{eff}} = \left(1 + \frac{12}{100 \times 12}\right)^{12} - 1 = (1.01)^{12} - 1 = 0.126825$$
For the second plan, $R = 12.2$, $k = 4$, so

$$i_{\text{eff}} = \left(1 + \frac{12.2}{100 \times 4}\right)^4 - 1 = (1.0305)^4 - 1 = 0.127696.$$  

Therefore, the second plan, 12.2% compounded quarterly, is better for the investor. \qed
10.5 Geometric Progressions and Savings Plans

Suppose $1000 is deposited with a bank that calculates interest at the rate of 10% compounded annually. The value of this investment (in dollars) at the end of $n$ years is equal to

$$1000 \left(1 + \frac{10}{100}\right)^n = 1000(1.1)^n.$$  

Thus the values of the investment at the end of 0 years, 1 year, 2 years, 3 years, and so on, are

$$1000, 1000(1.1), 1000(1.1)^2, 1000(1.1)^3, \cdots$$

Note that the ratio of each term to its preceding term in the above sequence is the same, namely 1.1. Such sequences are called geometric progressions. A formal definition is given below.

**Definition** A sequence of terms is said to be a **geometric progression** (G.P.) if the ratio of each term to its preceding term is the same throughout. This ratio is called the **common ratio** of the G.P.

If $a$ is the first term in a G.P., and $r$ is the common ratio, then the G.P. has the form

$$a, ar, ar^2, ar^3, \cdots$$

Thus the $n$-th term is given by $T_n = ar^{n-1}$.

**Example 1.** Find the fifth and $n$-th term of the G.P. $2, 6, 18, 54, \cdots$.

**Solution** We have $a = 2, r = \frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$. Hence the fifth term is

$$T_5 = ar^{5-1} = ar^4 = 2(3)^4 = 162.$$  

The $n$-th term is

$$T_n = ar^{n-1} = 2(3)^{n-1}.$$  

**Example 2** (Depreciation) A machine is purchased for $10,000 and is depreciated annually at the rate of 20% of its declining value. Find an expression for the value after $n$ years. If the ultimate scrap value is $3000, what is the effective life of the machine (i.e., the number of years until its depreciated value is less than its scrap value)?

**Solution** Since the value of the machine depreciates each year by 20% of its value at the beginning of the year, the value of the machine at the end of any year is 80% or 0.8 of its value at the beginning of the year. Thus the value of the machine at the end of the $n$-th year is $10000(0.8)^n$. To find the effective life of the machine, we let

$$10000(0.8)^n = 3000$$  

□.
and find \( n \). We have
\[
(0.8)^n = \frac{3000}{10000} = 0.3
\]
\[
\ln(0.8)^n = \ln(0.3),
\]
\[
n \ln(0.8) = \ln(0.3)
\]
\[
n = \frac{\ln(0.8)}{\ln(0.8)} = \frac{-1.204}{-0.223} = 5.4
\]
Therefore the effective life of the machine is 5.4 years.

Let us use \( S_n \) to denote the sum of the first \( n \) terms of the G.P. given by
\[
a, ar, ar^2, ar^{n-2}, ar^{n-1}, \ldots,
\]
i.e.
\[
S_n = a + ar + \cdots + ar^{n-1}.
\]
Then
\[
-rS_n = -ar - ar^2 - \cdots - ar^{n-1} - ar^n.
\]
Hence
\[
S_n - rS_n = a + (ar - ar) + (ar^2 - ar^2) + \cdots + (ar^{n-1} - ar^{n-1}) - ar^n
\]
\[
= a - ar^n = a(1 - r^n).
\]
i.e
\[
(1 - r)S_n = a(1 - r^n),
\]
\[
S_n = a \frac{1 - r^n}{1 - r}
\]

**Example 3** (Savings Plan). Each year a person invests $1000 in a savings plan that pays interest at the fixed rate of 8% per annum. What is the value of this savings plan on the tenth anniversary of the first investment? (Including the current payment paid into the plan).

**Solution** The first $1000 has been invested for 10 years, so it has increased in value to
\[
1000 \left( 1 + \frac{R}{100} \right)^{10} = 1000(1.08)^{10}.
\]
The second $1000 has increased in value to
\[
1000(1.08)^9
\]
as it has been invested for 9 years.

The third $1000 has value \( 1000(1.08)^8, \ldots \), and the 11th $1000 has value 1000. Thus the value of the savings plan is equal to
\[
1000 + 1000(1.08) + \cdots + 1000(1.08)^{10}
\]
This is the sum of 11 terms of the G.P. with $a = 1000, r = 1.08$. Hence

$$S_{11} = \frac{a(1 - r^{11})}{1 - r} = \frac{1000(1 - 1.08^{11})}{1 - 1.08} = \frac{1000 - 1.3316}{-0.08} = 16645.$$  

Thus the value is $16645$.  

**Example 4** Every month Jane pays $100 into a savings plan that earns interest at 0.5% per month. Calculate the value of her savings immediately after making her $n$-th payment.

**Solution** The $n$-th payment is made after $n - 1$ months of the first payment. So the value of the savings plan is the sum of

$$100 \left(1 + \frac{0.5}{100}\right)^{n-1}, 100 \left(1 + \frac{0.5}{100}\right)^{n-2}, \ldots, 100 \left(1 + \frac{0.5}{100}\right), 100.$$  

This is the sum of $n$ terms of the G.P.

$$100, 100(1.005), 100(1.005)^2, \ldots$$

Hence

$$S_n = a\frac{1 - r^n}{1 - r} = 100\frac{1 - (1.005)^n}{1 - 1.005} = 100\frac{1 - (1.005)^n}{-0.005}$$

$$= 100\frac{(1.005)^n - 1}{0.005} = 20000(1.005^n - 1)$$  

\[\square\]
10.6 Annuities and Amortisation

In a savings plan, one deposits to the bank at regular intervals of time, and at the end, one gets a sum of money from the bank. This sum of money is often called the future value of the savings plan.

Another situation is described by an annuity, where a person makes a deposit to the bank (or insurance company, etc) at the very beginning, and the bank (or insurance company, etc) pays that person a certain amount of money at regular time intervals until the deposit is used up. Let us look at the following example.

Example 1 On his 65th birthday, Mr Hoskins wishes to purchase an annuity that will pay him $5000 per year for the next 10 years, the first payment to be made to him on his 66th birthday. His insurance company will give him an interest rate of 8% per annum on the investment. How much must he pay in order to purchase such an annuity?

Solution Mr Hoskins will get ten $5000 payments in the next 10 years and all his money comes from the deposit and the interest it earns.

To find the amount $A$ that he should pay initially, we imagine that the amount $A$ is divided into 10 parts, $A_1, A_2, \ldots, A_{10}$, in such a way that the value of $A_1$ after one year is the first $5000$ he gets paid at his 66th birthday, the value of $A_2$ after 2 years is the second $5000$ he gets paid, the value of $A_3$ after 3 years gives him the third $5000$, $\ldots$, the value of $A_{10}$ after 10 years is the last $5000$ he gets paid.

Then we should have

\[ A_1(1 + 0.08) = 5000, \quad A_1 = 5000(1.08)^{-1} \]
\[ A_2(1 + 0.08)^2 = 5000, \quad A_2 = 5000(1.08)^{-2} \]
\[ A_3(1 + 0.08)^3 = 5000, \quad A_3 = 5000(1.08)^{-3} \]
\[ \vdots \]
\[ A_{10}(1 + 0.08)^{10} = 5000, \quad A_{10} = 5000(1.08)^{-10} \]

Then

\[ A = A_1 + A_2 + A_3 + \cdots + A_{10}, \]

which is the sum of the first 10 terms of a G.P. with $a = 5000(1.08)^{-1}, r = 1.08^{-1}$.

Therefore,

\[ A = a \frac{1 - r^{10}}{1 - r} = 5000(1.8)^{-1} \frac{1 - (1.08^{-1})^{10}}{1 - 1.08^{-1}} = 5000 \frac{1 - 1.08^{-10}}{1.08 - 1} = 33,550 \text{ (dollars)}. \]

In general, suppose an annuity is purchased with a down payment $A$, and the payments made to the annuitant are equal to $P$ at regular intervals for $n$ periods starting one period
after the annuity is purchased, and assume the interest rate is $R$ percent per period. Then

$$A = P(1 + i)^{-1} + P(1 + i)^{-2} + \cdots + P(1 + i)^{-n} \quad (i = \frac{R}{100})$$

$$= a \frac{1 - r^n}{1 - r} \quad (\text{with } a = P(1 + i)^{-1}, r = (1 + i)^{-1})$$

$$= P(1 + i)^{-1} \frac{1 - (1 + i)^{-n}}{1 - (1 + i)^{-1}} = P \frac{1 - (1 + i)^{-n}}{(1 + i) - 1}$$

$$= \frac{P}{i} [1 - (1 + i)^{-n}]$$

i.e.

$$A = \frac{P}{i} [1 - (1 + i)^{-n}]$$

**Example 2.** Mrs Josephs retires at the age of 63 and uses her life savings of $120,000 to purchase an annuity. The life insurance company gives an interest rate of 6% per annum, and they estimate that her life expectancy is 15 years. How much annuity (i.e., how big an annual pension) will she receive?

**Solution** We use the formula

$$A = \frac{P}{i} [1 - (1 + i)^{-n}] ,$$

where we know $A = 120000, i = 6\% = 0.06$ and $n = 15$.

Thus

$$120000 = \frac{P}{0.06} [1 - (1 + 0.06)^{-15}]$$

$$= \frac{P}{0.06} [1 - 1.06^{-15}]$$

$$= P \times 9.712249$$

$$P = \frac{120000}{9.712249} = 12355.53 \quad \text{(dollars)}$$

Hence Mrs Josephs will receive an annual pension of $12,355.53. \Box$

When a debt is repaid by regular payments over a period of time, we say that the debt is amortized. Examples of this are car loans, mortgages, etc.

Mathematically speaking, the amortization of a debt presents exactly the same problem as paying an annuity. With an annuity, we can view the annuitant as lending a certain amount $A$ to the insurance company; the company then repays this loan by $n$ regular payments of amount $P$ each.
Example 3. A certain student borrowed $8000 from the bank to buy a car. The interest rate is 8% per annum and the student repays in single installments at the end of each year. How much must the student pay each year to repay the loan in 5 years?

Solution We use the formula

\[ A = \frac{P}{i} \left[ 1 - (1 + i)^{-n} \right] \]

where \( a = 8000, i = 8\% = 0.08, n = 5 \)

Hence

\[ 8000 = \frac{P}{0.08} \left[ 1 - (1 + 0.08)^{-5} \right] \]
\[ = \frac{P}{0.08} (1 - 1.08^{-5}) \]
\[ = P \times 3.99271 \]
\[ P = \frac{8000}{3.99271} = 2003.65 \text{ (dollars)}. \]

The student must repay $2003.65 each year.

Example 4 A married couple have a combined income of $45,000. Their mortgage company will allow them to borrow up to an amount at which the repayments are one-third of their income. If the interest rate is 1.2% per month, amortized over 25 years, how much can they borrow?

Solution Again we use the formula

\[ A = \frac{P}{i} \left[ 1 - (1 + i)^{-n} \right] \]

where \( P = \left( \frac{1}{3} \times 54000 \right) \div 12 = 1250 \)

\[ i = 1.2\% = 0.012 \]
\[ n = 12 \times 25 = 300 \]

Hence

\[ A = \frac{1250}{0.012} \left[ 1 - (1 + 0.012)^{-300} \right] \]
\[ = \frac{1250}{0.012} (1 - 1.012^{-300}) \]
\[ = 101,258.80 \text{ (dollars)}. \]

Therefore, they can borrow $101,258.80.
10.7 Other Applications

1. Linear Cost Model

In the production of any commodity by a firm, there are two types of costs involved; these are known as **fixed costs** and **variable costs**. **Fixed costs** are costs that have to be met no matter how much or how little of the commodity is produced; that is, they do not depend on the level of production. Examples of fixed costs are rents, interest on loans, and management salaries.

**Variable costs** are costs that depend on the level of production, that is, on the amount of commodity produced. Material costs and labour costs are examples of variable costs. The total cost is given by

\[
\text{Total Cost} = \text{Variable Costs} + \text{Fixed Costs}
\]

In a **linear cost model**, one assumes that the variable costs per unit of commodity is constant, and let us denote it by \( m \) (dollars). Then the total variable costs of producing \( x \) units of commodity is \( mx \) (dollars). If the fixed costs are \( b \) dollars, then the total cost \( y_c \) (in dollars) of producing \( x \) units is given by

\[
y_c = mx + b.
\]

**Example 1** The variable cost of processing 1 pound of coffee beans is 50 cents and the fixed costs per day are $300. Find the cost of processing 1000 pounds of coffee beans in one day.

**Solution** Total variable costs = 0.5 \( \times \) 1000. Fixed cost = 300. Therefore,

\[
\text{Total cost} = 0.5 \times 1000 + 300 = 800 \text{ (dollars)}.
\]

**Example 2.** The cost of manufacturing 10 typewriters per day is $350, while it costs $600 to produce 20 typewriters per day. Assuming a linear cost model, determine the relationship representing the total cost \( y_c \) of producing \( x \) typewriters per day.

**Solution** Since it is a linear cost model,

\[
y_c = mx + b
\]

where \( b \) is the fixed cost, \( m \) is the variable cost per commodity. By the conditions given, \( y_c = 350 \) when \( x = 10 \) and \( y_c = 600 \) when \( x = 20 \). We use these to determine \( m \) and \( b \).

\[
350 = m \times 10 + b \\
600 = m \times 20 + b
\]
Subtracting the first equation from the second we obtain

\[ 600 - 350 = (m \times 20 + b) - (m \times 10 + b) \]

\[ 250 = 10m, \quad m = \frac{250}{10} = 25. \]

Substituting this into the first equation we have

\[ 350 = 25 \times 10 + b = 250 + b \]

\[ b = 350 - 250 = 100. \]

Hence

\[ y_c = 25x + 100. \]

\[ \square \]

2. Break-even Analysis

If the total cost \( y_c \) of production exceeds the revenue \( y_R \) obtained from the sales, then a business is running at a loss. On the other hand, if the revenue exceeds the costs there is a profit. If the cost of production equals the revenue obtained from the sales, there is no profit or loss, so the business breaks even. The number of units produced and sold in this case is called the break-even point.

Example 3 For a watch maker, the cost of labour and materials per watch is $15 and the fixed costs are $2000 per day. If each watch sells for $20, how many watches should be produced and sold each day to guarantee that the business breaks even?

Solution The cost of producing \( x \) watches per day is

\[ y_c = 15x + 2000 \]

The revenue is

\[ y_R = 20x. \]

To break-even, we need \( y_c = y_R \), i.e.

\[ 15x + 2000 = 20x. \]

\[ 5x = 2000 \]

\[ x = \frac{2000}{5} = 400. \]

Hence 400 watches should be produced and sold each day to break even. \( \square \)

Example 4 Suppose the total daily cost (in dollars) of producing \( x \) chairs is given by

\[ y_c = 2.5x + 300, \]
and it is known that at least 150 chairs can be sold each day. What price should be charged to guarantee no loss?

**Solution** The total cost of producing 150 chairs per day is
\[ y_c = 2.5 \times 150 + 300 = 675. \]

If the price is \( p \) dollars per chair, then the revenue of selling 150 chairs is \( 150p \).

To break even, we should have
\[
150p = 675
\]
\[
p = \frac{675}{150} = 4.50.
\]

Thus, the price should be at least 4.50 dollars per chair to guarantee no loss. 

3. Supply and Demand.

The laws of demand and supply are two of the fundamental relationships in any economic analysis. The quantity \( x \) of any commodity that will be purchased by consumers depends on the price at which that commodity is made available; usually, if the price is decreased, then the demand will increase. A relationship that specifies the amount of a particular commodity that consumers are willing to buy at various price levels is called the law of demand. The simplest law is a linear relation of the type
\[ p = mx + b \]

where \( p \) is the price per unit of the commodity, \( m \) and \( b \) are constants. The graph of the demand law is called the demand curve. In this linear case, it is just a straight line. \( m \) usually has a negative value to represent that demand increases when price decreases.

A relationship specifying the amount of any commodity that manufacturers (or sellers) can make available in the market at various prices is called the supply law, its graph is called the supply curve. A linear supply law has the form
\[ p = m_1 x + b_1 \]

where \( p \) is the price per unit of the commodity, \( x \) is the number of units of the commodity and \( m_1, b_1 \) are constants, but this time, \( m_1 \) is usually positive (why?).

**Example 5.** A dealer can sell 20 electric shavers per day at $25 per shaver, but he can sell 30 shavers if he charges $20 per shaver. Determine the demand equation, assuming it is linear.

**Solution** The equation has the form
\[ p = mx + b \]
and we need to determine $m$ and $b$.

We know $x = 20$ when $p = 25$ and $x = 30$ when $p = 20$. i.e.

\[
25 = m(20) + b \\
20 = m(30) + b.
\]

Subtracting the first equation from the second, we obtain

\[
10m = -5 \quad m = \frac{-5}{10} = -0.5
\]

Substituting this into the first equation, we have

\[
25 = -0.5 \times 20 + b = -10 + b \\
b = 35.
\]

Thus the demand law is

\[
p = -0.5x + 35.
\]

\[\square\]

4. Market Equilibrium

If the price of a certain commodity is too high, consumers will not purchase it, whereas if the price is too low, suppliers will not sell it. In a competitive market, when the price per unit depends only on the quantity demanded and the supply available, there is always a tendency for the price to adjust itself so that the quantity demanded by the purchasers matches the quantity which suppliers are willing to supply. Market equilibrium is said to occur at the price when the quantity demanded is equal to the quantity supplied.

Example 6 Determine the equilibrium price and quantity for the following demand and supply laws.

\[
D: \quad p = 25 - 2x \\
S: \quad p = 3x + 5
\]

Solution. At equilibrium, the price $p$ and quantity $x$ in both equations are the same, i.e. they satisfy both equations. Hence $p$ and $x$ are solutions of the system of linear equations. Subtract the demand equation from the supply equation, it results

\[
3x + 5 - (25 - 2x) = p - p = 0 \\
5x - 20 = 0, 5x = 20, x = \frac{20}{5} = 4.
\]

Substitute this back to the demand equation,

\[
p = 25 - 2 \times 4 = 25 - 8 = 17.
\]
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Hence the equilibrium quantity and price are \( x = 4 \) and \( p = 17 \), respectively.

5. Marginal Analysis

Derivatives have a number of applications in business and economics in constructing what are called marginal rates. In this field, the word “marginal” is used to indicate a derivative, that is, a rate of change. For example, if \( C(x) \) is the cost function, where \( x \) represents the number of units, then \( C'(x) \) is called the marginal cost when \( x \) units are produced. Similarly, if \( R(x) \) is the revenue function, then \( R'(x) \) is called the marginal revenue, etc.

Example 7 (Marginal Cost) The cost function

\[
C(x) = 0.001x^3 - 0.3x^2 + 40x + 1000
\]
determines the cost as a function of \( x \). Evaluate the marginal cost when the production is given by \( x = 50 \), \( x = 100 \), and \( x = 150 \).

Solution The marginal cost is

\[
C'(x) = (0.001x^3 - 0.3x^2 + 40x + 1000)'
\]
\[
= 0.003x^2 - 0.6x + 40
\]

When \( x = 50 \)

\[
C'(50) = 0.003(50)^2 - 0.6(50) + 40
\]
\[
= 17.5
\]

When \( x = 100 \)

\[
C'(100) = 0.003(100)^2 - 0.6(100) + 40
\]
\[
= 10
\]

When \( x = 150 \)

\[
C'(150) = 0.003(150)^2 - 0.6(150) + 40
\]
\[
= 17.5
\]

Roughly speaking, we can say that the 51st item costs $17.50 to produce, the 101st item costs $10, and the 151st item costs $17.50. (Such statements as these are not quite accurate, since the derivative gives the rate for an infinitesimally small increment in production, not for a unit increment).

It is important not to confuse the marginal cost with the average cost. If \( C(x) \) is the cost function, then the average cost of producing \( x \) items is given by
Average Cost per item = \frac{C(x)}{x}

This is commonly denoted by \( \bar{C}(x) \), i.e.,

\[
\bar{C}(x) = \frac{C(x)}{x}
\]

**Example 8**  For the cost function \( C(x) = 1000 + 10x + 0.1x^2 \), the marginal cost is \( C'(x) = 10 + 0.2x \). The average cost of producing \( x \) items is

\[
\bar{C}(x) = \frac{C(x)}{x} = \frac{1000 + 10x + 0.1x^2}{x} = 1000x^{-1} + 10 + 0.1x
\]

**Example 9**  (Marginal Revenue) Determine the marginal revenue when \( x = 300 \) if the demand equation is

\[
x = 1000 - 100p
\]

**Solution**  We know that revenue \( R = xp \). From the demand equation, we can solve for \( p \) in terms of \( x \):

\[
x = 1000 - 100p, \\
100p = 1000 - x \\
p = \frac{1000 - x}{100} = 10 - 0.01x
\]

Thus

\[
R = xp = x(10 - 0.01x) = 10x - 0.01x^2.
\]

The marginal revenue is

\[
R'(x) = (10x - 0.01x^2)' = 10 - 0.02x
\]

When \( x = 300 \), the marginal revenue is

\[
R'(300) = 10 - 0.02(300) = 10 - 6 = 4
\]

**Example 10**  (Marginal Profit)  The demand equation for a certain item is

\[
p + 0.1x = 80
\]
and the cost function is
\[ C(x) = 5000 + 20x. \]

Compute the marginal profit when (a) 150 units are produced and sold, (b) when 400 units are produced and sold. Find \( x \) that maximizes the profit and find the marginal profit at this value of \( x \).

**Solution** The profit \( P(x) = R(x) - C(x) \), where \( R(x) \) is the revenue function.

We have \( R = xp \), and by the demand equation,

\[ p + 0.1x = 80, \quad p = 80 - 0.1x \]

Therefore,
\[ R(x) = xp = x(80 - 0.1x) = 80x - 0.1x^2. \]

It follows that
\[
\begin{align*}
P(x) &= R(x) - C(x) \\
&= (80x - 0.1x^2) - (5000 + 20x) \\
&= 80x - 0.1x^2 - 5000 - 20x \\
&= 60x - 0.1x^2 - 5000
\end{align*}
\]

\[ P'(x) = (60x - 0.1x^2 - 5000)' = 60 - 0.2x \]

When \( x = 150 \),
\[ P'(150) = 60 - 0.2(150) = 60 - 30 = 30. \]

Thus when 150 items are being produced and sold, the marginal profit, that is, roughly speaking the extra profit per additional item is $30.

When \( x = 400 \),
\[ P'(400) = 60 - 0.2(400) = 60 - 80 = -20 \]

Thus when 400 items are being produced and sold, the extra profit per additional item is -$20, or a loss of $20 per additional item.

Since the profit \( P(x) = 60x - 0.1x^2 - 5000 \) is a quadratic function of the form \( ax^2 + bx + c \) with \( a = -0.1, b = 60 \) and \( c = -5000 \), we know that it takes a maximum when
\[ x = \frac{-b}{2a} = \frac{-60}{2(-0.1)} = \frac{60}{0.2} = 300 \]
and

\[ P(300) = 60(300) - 0.1(300)^2 - 5000 = 18000 - 9000 - 5000 = 4000. \]

\[ P'(300) = 60 - 0.2(300) = 60 - 60 = 0 \]

It is interesting to see from the above example that at \( x = 300 \), where maximal profit is achieved, the marginal profit is 0.

6. Elasticity

A concept widely used in economics is that of elasticity. We shall introduce this idea via the so-called \textit{elasticity of demand}.

Let the demand law be given by

\[ x = f(p) \]

where \( x \) is the number of units that can be sold at price \( p \) per unit. The elasticity of demand is usually represented by the Greek letter \( \eta \) (read eta) and is defined as follows:

\[
\eta = \frac{p}{x} \frac{dx}{dp} = \frac{p}{x} f'(p) = \frac{pf'(p)}{f(p)}
\]

Recall that \( \frac{dx}{dp} \) is approximately the average rate of change \( \frac{\Delta x}{\Delta p} \), when \( \Delta p \) is small where \( \Delta x = f(p + \Delta p) - f(p) \). Therefore,

\[
\eta \approx \frac{p}{x} \frac{\Delta x}{\Delta p} = \frac{p}{x} \frac{\Delta x}{\Delta p} = \frac{\Delta x/x}{\Delta p/p}
\]

\[
\frac{\Delta x}{x} \approx \eta \frac{\Delta p}{p}
\]

\( \frac{\Delta x}{x} \) represents the percentage change in demand, for example, \( \frac{\Delta x}{x} = 0.1 \) would mean the change in demand is 10%. Similarly, \( \frac{\Delta p}{p} \) represents the percentage change in price. Therefore,

Percentage Change in Demand \( \approx \) (elasticity of demand) \( \times \) (percentage change in price)

Example 11. Calculate the elasticity of demand if \( x = 500(10 - p) \) for \( p = 4 \).
10.7. OTHER APPLICATIONS

Solution

\[
\frac{dx}{dp} = [500(10 - p)]' = 500(10 - p)' = 500(-1) = -500 \\
\eta = \frac{p \cdot dx}{x \cdot dp} = \frac{p}{500(10 - p)}(-500) = \frac{-500p}{500(10 - p)} = \frac{-p}{10 - p}.
\]

When \( p = 4 \).

\[
\eta = \frac{-4}{10 - 4} = \frac{-4}{6} = -\frac{2}{3}.
\]

The idea of elasticity can be used for any pair of variables that are related by a function. If \( y = f(x) \), then the **elasticity of \( y \) with respect to \( x \)** is defined as

\[
\eta = \frac{x \cdot dy}{y \cdot dx}.
\]

Note that

\[
\frac{d}{dx} (\ln y) = \frac{1}{y} \frac{dy}{dx}.
\]

Therefore, \( \eta = x \frac{d}{dx} (\ln y) \).