

MATH110 — Assignment 11

Solutions

Question 1.

(a)

$$\begin{aligned}\frac{\partial z}{\partial x} &= 4e^y \cos(4x - 1) \\ \frac{\partial z}{\partial y} &= e^y \sin(4x - 1)\end{aligned}$$

(b) Let

$$z = u^{1/2} \quad u = x^2 - x + y^2 + 1.$$

Then

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} = \frac{1}{2}u^{-1/2}(2x - 1) = (1/2)(2x - 1)(x^2 - x + y^2 + 1)^{-1/2} \\ \frac{\partial z}{\partial y} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} = \frac{1}{2}u^{-1/2}2y = y(x^2 + y^2)^{-1/2}\end{aligned}$$

Question 2.

(a)

$$\begin{aligned}\frac{\partial z}{\partial x} &= y \cos(x) \\ \frac{\partial z}{\partial y} &= \sin(x)\end{aligned}$$

If $\partial z/\partial x = y \cos(x) = 0$ then $y = 0$ (or $x = (k + \frac{1}{2})\pi$ for some integer k). When $y = 0$, we must have $x = k\pi$ in order to make $\sin(x) = 0$, and the two partial derivatives cannot simultaneously vanish otherwise. The Hessian matrix is

$$H = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -y \sin(x) & \cos x \\ \cos x & 0 \end{bmatrix} = \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$$

This matrix has determinant -1, and the stationary points are all saddle points.

(b)

$$\begin{aligned}\frac{\partial z}{\partial x} &= -y - 3 \\ \frac{\partial z}{\partial y} &= 4y - x\end{aligned}$$

Solving the pair of equations $\partial z/\partial x = 0$ and $\partial z/\partial y = 0$ gives the stationary point $y = -3$, $x = -12$. The Hessian matrix is

$$H = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 4 \end{bmatrix}$$

which has determinant -1 and so the stationary point is a saddle point.

(c)

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2x - y + y^2 \\ \frac{\partial z}{\partial y} &= -x + 2xy = x(2y - 1)\end{aligned}$$

Now $\partial z/\partial y = 0$ gives $y = 1/2$ or $x = 0$. When $y = 1/2$, $\partial z/\partial x = 0$ gives $x = 1/8$.
When $x = 0$

$$\frac{\partial z}{\partial x} = y^2 - y = y(y - 1)$$

and $\partial z/\partial x = 0$ gives $y = 0$ or $y = 1$. Thus there are three stationary points (i) $(1/8, 1/2)$, (ii) $(0, 0)$ and (iii) $(0, 1)$. The Hessian matrix is

$$H = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 2y - 1 \\ 2y - 1 & 2x \end{bmatrix}.$$

For the three stationary points, the Hessian matrices are:

$$(i) \quad \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \quad (iii) \quad \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \quad (ii) \quad \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

and the stationary points are (i) local minimum, (ii) and (iii) saddle points.

Question 3.

$$\nabla f = (2x, 6y), \quad \nabla g = (-1, -1), \quad \text{where } g(x, y) = 8 - x - y.$$

Then we solve $2x - t = 0$, ie., $x = t/2$, followed by $6y - t = 0$, ie., $y = t/6$. Finally $g(t/2, t/6) = 0$ implies $8 = 2t/3$, ie., $t = 12$, and hence $x = 6$ and $y = 2$. The graph of f is a concave up paraboloid sheet, so the intersection with the plane described by $g=0$ is also concave up, and so the constrained stationary point is a minimum. $f(6, 2) = 58$.