MATH110 — Assignment 10
Solutions

Question 1.

The differential equation

$$\frac{dT}{dt} = 0.2(35 - T)$$

is an equation of restricted exponential growth with solution

$$T(t) = 35 - Ae^{-0.2t}.$$

Now

$$T(0) = 35 - A = 4$$

so $A = 31$ and

$$T(t) = 35 - 31e^{-0.2t}.$$

At $t = 5$

$$T = 35 - 31e^{-1} = 23.6°.$$

Let t_1 be the time to reach $30°$. Then

$$30 = 35 - 31e^{-0.2t_1}$$

and solving for t_1 we find

$$e^{-0.2t_1} = \frac{5}{31}$$

and taking logs gives

$$-0.1t_1 = \ln 5/31$$

so

$$t_1 = -\frac{\ln 5/31}{0.2} = 9.12 \text{ minutes}.$$

Question 2.

The solution of the differential equation is

$$y(t) = \frac{AL}{A + e^{-kLt}}$$

where $k = 2 \times 10^{-7}$ and $L = 10^6$. To find A, let y_0 be the initial population. Then

$$y_0 = \frac{AL}{A + 1}$$

and solving for A gives

$$A = \frac{y_0}{L - y_0}$$

Now y_0 is 20% of the steady-state population of 10^6, so $y_0 = 2 \times 10^5$ and we find $A = 0.25$. After 6 months

$$y(6) = \frac{AL}{A + e^{-kL\cdot 6}} = 4.54 \times 10^5.$$

Let T be the time to reach $F = 0.9$ of the steady-state population L. Then

$$y(T) = FL = \frac{AL}{A + e^{-kLT}}.$$

Now

$$F(A + e^{-kLt}) = A$$

and

$$e^{-kLT} = A(\frac{1}{F} - 1) = \frac{A}{9}.$$

Taking logs gives

$$-kLT = \ln(A/9)$$

and

$$T = -\frac{\ln(A/9)}{kL} = 17.92 \text{ months.}$$

Aside: In problems like this, where there are a lot of numerical constants involved, I find it convenient to work with symbols rather than numbers in the hope of avoiding arithmetic errors. There is nothing wrong with using numerical values throughout the calculation, it is a matter of personal taste.

Question 3.

We have the parameter values $W_0 = 0.25$, $\mu_0 = 0.068$ and $D = 0.022$.
(a) The solution is

$$W(t) = W_0 e^{\frac{\mu_0}{D}(1-e^{-Dt})} = 0.25e^{3.0909(1-e^{-0.022t})}.$$

(b) We need to find t so that $W(t) = 4$, that is

$$4 = W_0 e^{\frac{\mu_0}{D}(1-e^{-Dt})}.$$

To find t, we proceed as follows:

$$\begin{aligned}
\frac{4}{W_0} &= e^{\frac{\mu_0}{D}(1-e^{-Dt})} \\
\ln\frac{4}{W_0} &= \frac{\mu_0}{D}(1-e^{-Dt}) \\
\frac{D}{\mu_0}\ln\frac{4}{W_0} &= (1-e^{-Dt}) \\
e^{-Dt} &= 1 - \frac{D}{\mu_0}\ln\frac{4}{W_0} \\
-Dt &= \ln(1 - \frac{D}{\mu - 0}\ln\frac{4}{W_0}) \\
t &= -\frac{\ln(1 - \frac{D}{\mu_0}\ln\frac{4}{W_0})}{D}
\end{aligned}$$

Substituting for the parameter values gives

$$t = 103.3 \text{ days.}$$

(c) As $t \to \infty$

$$W \to W_0 e^{\frac{\mu_0}{D}} = 5.488 \text{ kg.}$$