Contents

Chapter 1

Vector Algebra

1.1 Vectors

A **vector** is simply a list of numbers, usually written in a column surrounded by square brackets. For example, consider the information in the following table:

Age	Male	Female
under 15	1.918	1.831
15-44	3.645	3.528
45-64	1.470	1.457
65 and over	0.642	0.889

Figure 1.1: Australian population in 1983 (in millions)

From this data we form two vectors, called **M** and **F**, containing the male and female populations in the various age groups:

$$\mathbf{M} = \begin{bmatrix} 1.918 \\ 3.645 \\ 1.470 \\ 0.642 \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} 1.831 \\ 3.528 \\ 1.457 \\ 0.889 \end{bmatrix}.$$

Note that the names **M** and **F** refer to the lists (vectors) as a whole.

The individual numbers appearing in a vector are called the **elements** or **components** of the vector. The number of components of a vector is called the **dimension** of the vector. Thus the vectors **M** and **F** above each have dimension 4. The individual components of a vector are distinguished by subscripts running from 1 to N, where N is the dimension of the vector. The vector **M** above has components:

$$\begin{aligned} M_1 &= 1.918, \\ M_2 &= 3.645, \\ M_3 &= 1.470, \\ M_4 &= 0.642. \end{aligned}$$

Addition and subtraction of vectors

Addition and subtraction of vectors are done component by component. The sum and difference of the vectors $\mathbf{M}$ and $\mathbf{F}$ defined above are:

$$\mathbf{M}+\mathbf{F} = \begin{bmatrix} 1.918+1.831 \\ 3.645+3.528 \\ 1.470+1.457 \\ 0.642+0.889 \end{bmatrix} = \begin{bmatrix} 3.749 \\ 7.173 \\ 2.927 \\ 1.531 \end{bmatrix}$$

and

$$\mathbf{M}-\mathbf{F} = \begin{bmatrix} 1.918-1.831 \\ 3.645-3.528 \\ 1.470-1.457 \\ 0.642-0.889 \end{bmatrix} = \begin{bmatrix} 0.087 \\ 0.117 \\ 0.013 \\ -0.247 \end{bmatrix}$$

and they have obvious interpretations as the total population in each age group and the excess of males over females in each age group.

Multiplication of a vector by a number

Multiplication of a vector by a number is done by multiplying each individual component of the vector by the number. Let $\mathbf{P}=\mathbf{M}+\mathbf{F}$ be the vector giving the total population (male and female) in each age group

$$\mathbf{P} = \begin{bmatrix} 3.749 \\ 7.173 \\ 2.927 \\ 1.531 \end{bmatrix}$$

and suppose we predict the population in each age group will grow by 2% in the next year. To obtain the predicted number in each age group we must multiply $\mathbf{P}$ by 1.02

$$1.02\times\mathbf{P} = 1.02\times\begin{bmatrix} 3.749 \\ 7.173 \\ 2.927 \\ 1.531 \end{bmatrix} = \begin{bmatrix} 1.02\times 3.749 \\ 1.02\times 7.173 \\ 1.02\times 2.927 \\ 1.02\times 1.531 \end{bmatrix} = \begin{bmatrix} 3.824 \\ 7.316 \\ 2.986 \\ 1.562 \end{bmatrix}$$

and our prediction is that there will be 3.824 million people under the age of 15, 7.316 between the ages of 15 and 44, and so on.

1.2 Sigma Notation

Look at the vectors $\mathbf{M}$ and $\mathbf{F}$, defined in the last section, giving the numbers of males and females (in millions) in various age groups for the 1983 Australian population:

$$\mathbf{M} = \begin{bmatrix} 1.918 \\ 3.645 \\ 1.470 \\ 0.642 \end{bmatrix} \quad \text{and} \quad \mathbf{F} = \begin{bmatrix} 1.831 \\ 3.528 \\ 1.457 \\ 0.889 \end{bmatrix}.$$

an obviously useful piece of information is the total male and female populations in that year. These are just the sums of the components of the vectors **M** and **F**. Let m and f be these sums:

$$\begin{aligned} m &= M_1 + M_2 + M_3 + M_4 && (1.1)\\ &= 1.918 + 3.645 + 1.470 + 0.642\\ &= 7.675,\\ f &= F_1 + F_2 + F_3 + F_4 && (1.2)\\ &= 1.831 + 3.528 + 1.457 + 0.889\\ &= 7.705. \end{aligned}$$

Expressions like (1) and (2) can take a lot of writing out. Suppose, for example, that instead of 4 age groups the population figures had been given to us in 5-year age brackets. Then the vectors **M** and **F** would have had more than 20 components each! A useful shorthand way of writing such expressions is the **sigma notation**:

$$\begin{aligned} m &= \sum M_i,\\ f &= \sum F_i. \end{aligned}$$

Here the index i ranges over all of the compenents of the vector. So, in general, if the vector **X** has N components

$$\sum X_i \qquad \text{stands for} \qquad X_1 + X_2 + X_3 + \cdots + X_N.$$

Alternative notations

$$\sum X_i = \sum_i X_i = \sum_{i=1}^{N} X_i$$

are used when we want to be explicit about the index over which the summation is performed or the dimension of the vector.

Properties of sigma notation

Let p be the total Australian population in 1983. This can be worked out from the population table in two ways:

1. Find the total male and female populations and add them together, or
2. Find the total populations (male and female) in each age group and add these age group populations together.

Using sigma notation the first can be written

$$p = m + f = \sum M_i + \sum F_i$$

and the second written

$$p = \sum \left(M_i + F_i\right).$$

Of course the two sigma expressions give the same result. In general, for any vectors **X** and **Y** of the same dimension,

Property 1: $$\sum (X_i + Y_i) = \sum X_i + \sum Y_i.$$

Similarly, there are two ways of working out the predicted 1984 population q from the 1983 population vector **P**.

1. Find the predicted population in each age group for 1984 (by multiplying each 1983 population by 1.02) and add, or
2. Find the total population for 1983 (by adding the age group populations) and multiply by 1.02.

Using sigma notation these are

$$q = \sum (1.02 \times P_i)$$

and

$$q = 1.02 \times \sum P_i,$$

where $P_i = M_i + F_i$ is the ith component of **P**=**M**+**F**. In general we have

Property 2: $$\sum (a \times X_i) = a \times \sum X_i.$$

As another example, if **X** is the vector

$$\mathbf{X} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

then

$$\sum X_i = 1 + 2 + 3 + 4 + 5 = 15$$

and

$$\begin{aligned} \sum (3 \times X_i) &= 3 \times 1 + 3 \times 2 + 3 \times 3 + 3 \times 4 + 3 \times 5 \\ &= 3 + 6 + 9 + 12 + 15 \\ &= 45 \\ &= 3 \times 15 \\ &= 3 \times \sum X_i. \end{aligned}$$

Finally, if we have a vector with all components the same, say

$$\mathbf{X} = \begin{bmatrix} k \\ k \\ \vdots \\ k \end{bmatrix}$$

and $\mathbf{X}$ has n components then

$$\begin{aligned} \sum X_i &= k + k + k + \cdots + k && n \text{ times} \\ &= n \times k. \end{aligned}$$

Since each $X_i = k$ we have

Property 3: $$\sum_{i=1}^{n} k = n \times k.$$

As an example,

$$\sum_{i=1}^{5} 3 = 3 + 3 + 3 + 3 + 3 = 15 = 5 \times 3.$$

1.3 Inner Products

Suppose you are organizing a large barbeque and part of the shopping list is as follows:

5 kg	beef sausages	–	\$ 2.99	per kg
5 kg	pork sausages	–	\$ 3.99	per kg
8 kg	steak	–	\$ 7.99	per kg
4 kg	lamb chops	–	\$ 5.99	per kg
7	lettuces	–	\$ 1.10	each
10 kg	tomatoes	–	\$ 2.39	per kg
9	cucumbers	–	50c	each
6 kg	onions	–	\$ 1.09	per kg

From this information we construct two vectors:

$$\mathbf{Q} = \begin{bmatrix} 5 \\ 5 \\ 8 \\ 4 \\ 7 \\ 10 \\ 9 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 2.99 \\ 3.99 \\ 7.99 \\ 5.99 \\ 1.10 \\ 2.39 \\ 0.50 \\ 1.09 \end{bmatrix}$$

representing the quantities and prices of the various requirements. To obtain the total cost we multiply corresponding elements in the two vectors and add them all together:

$$\begin{aligned} \text{total cost} &= 5 \times 2.99 + 5 \times 3.99 + \cdots + 6 \times 1.09 \\ &= 165.42 \end{aligned}$$

thus the total cost is \$165.42.

This process of multiplying corresponding components of vectors and adding the results together occurs so often it is given a name — the **inner product** of the two vectors. The inner product of two vectors $\mathbf{X}$ and $\mathbf{Y}$ is written $\langle \mathbf{X}, \mathbf{Y} \rangle$. Writing out the inner product term by term we get

$$\langle \mathbf{X}, \mathbf{Y} \rangle = X_1 Y_1 + X_2 Y_2 + \cdots + X_n Y_n$$

or using sigma notation

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \sum X_i Y_i.$$

For the inner product to be defined the two vectors must have the *same dimension*.

The value of the inner product (just like the value of the product of two numbers) doesn't depend on the order of the vectors:

$$\langle \mathbf{X}, \mathbf{Y} \rangle = \langle \mathbf{Y}, \mathbf{X} \rangle.$$

Warning: some formulas you might hope to be true turn out not to be so; for example

$$\sum \mathbf{X}_i \times \sum \mathbf{Y}_i \neq \sum (\mathbf{X}_i \times \mathbf{Y}_i)$$

in general. If

$$\mathbf{X} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} \qquad \text{and} \qquad \mathbf{Y} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$$

then

$$\begin{aligned} \sum \mathbf{X}_i \times \sum \mathbf{Y}_i &= (X_1 + X_2 + X_3) \times (Y_1 + Y_2 + Y_3) \\ &= (1 + 3 + 7) \times (3 - 4 + 2) \\ &= 11 \times 1 \end{aligned}$$

while

$$\begin{aligned} \sum (\mathbf{X}_i \times \mathbf{Y}_i) &= (X_1 \times Y_1) + (X_2 \times Y_2) + (X_3 \times Y_3) \\ &= (1 \times 3) + (3 \times -4) + (7 \times 2) \\ &= 3 - 12 + 14 \\ &= 5 \end{aligned}$$

Both in statistics and geometry it is often necessary to take the sum of squares of the components of a vector $\mathbf{X}$. This is the same as taking the inner product of $\mathbf{X}$ with itself:

$$\langle \mathbf{X}, \mathbf{X} \rangle = \sum X_i^2 = X_1^2 + X_2^2 + \cdots + X_n^2.$$

For the vector $\mathbf{X}$ above

$$\begin{aligned} \langle \mathbf{X}, \mathbf{X} \rangle &= 1^2 + 3^2 + 7^2 \\ &= 1 + 9 + 49 \\ &= 59. \end{aligned}$$

As before

$$\sum X_i^2 \neq \left(\sum X_i \right)^2$$

in general. For the vector **X** above

$$\begin{aligned}\left(\sum X_i\right)^2 &= (X_1 + X_2 + X_3)^2 \\ &= (1+3+7)^2 \\ &= 11^2 \\ &= 121 \\ &\neq \sum X_i^2 = 59.\end{aligned}$$

There are different notations for the inner product. It is sometimes written **X**.**Y** and called the **scalar product** or **dot product**.

1.4 Exercises

Let **X**, **Y** and **Z** be the vectors

$$\mathbf{X} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 7 \\ -9 \end{bmatrix} \qquad \mathbf{Z} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

1. What is the dimension of the vectors **X**, **Y** and **Z**.
2. Write down the components Z_1, Z_4 and Z_5.
3. Calculate $\mathbf{X} + \mathbf{Y}$.
4. Calculate $3 \times \mathbf{Z}$.
5. Calculate $\frac{1}{2} \times \mathbf{Y}$.
6. Calculate $\sum Z_i$.
7. Calculate $\langle \mathbf{Y}, \mathbf{Z} \rangle$.
8. Calculate $\sum Y_i^2$ and $(\sum Y_i)^2$.
9. Why is the inner product of a vector with itself always non-negative?

Chapter 2

Matrix Algebra

2.1 Matrices

A **matrix** is a rectangular array of numbers arranged in rows and columns. As with vectors, the array is enclosed in brackets. There is a natural way to arrange the population data from section 1.2 into a 4 by 2 rectangular array or matrix which we will call $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} 1.918 & 1.831 \\ 3.654 & 3.528 \\ 1.470 & 1.457 \\ 0.642 & 0.889 \end{bmatrix}.$$

We shall often need to refer to the rows or columns of a matrix.

$$\begin{array}{c} \begin{bmatrix} 1.918 & 1.831 \\ 3.654 & 3.528 \\ 1.470 & 1.457 \\ 0.642 & 0.889 \end{bmatrix} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{col. 1} & \text{col. 2} \end{array} \end{array} \begin{array}{l} \longleftarrow \text{ row 1} \\ \longleftarrow \text{ row 2} \\ \longleftarrow \text{ row 3} \\ \longleftarrow \text{ row 4} \\ \\ \\ \end{array}$$

We say that a matrix is an $m \times n$ matrix, or has size $m \times n$ if it has m rows and n columns. For example, the matrix $\mathbf{P}$ above is a 4×2 matrix, the matrix $\mathbf{A}$ below is a 3×4 matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -5 & 7 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & -9 & -2 \end{bmatrix}.$$

A specific element of a matrix is named by specifying its row and column. Thus the $(2,3)$ **element** or **component** of a matrix is the number in the 2nd row and 3rd column, in the matrix $\mathbf{A}$ above the $(2,3)$ element is 2. As for vectors, the components of a matrix are distinguished by subscripts — however because a matrix is a two dimensional array we need two subscripts to specify a component of a matrix. Thus we refer to the $(2,3)$ element of a matrix $\mathbf{A}$ as $A_{2\,3}$. In the matrix $\mathbf{A}$ above we have, for example:

$$A_{1\,2} = 3 \quad \text{and} \quad A_{3\,4} = -2.$$

Remember that the first index refers to the row, the second index to the column.

With this notation a general $m \times n$ matrix $\mathbf{A}$ has the form:

$$\mathbf{A} = \begin{bmatrix} A_{1\,1} & A_{1\,2} & \dots & A_{1\,n} \\ A_{2\,1} & A_{2\,2} & \dots & A_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m\,1} & A_{m\,2} & \dots & A_{m\,n} \end{bmatrix}.$$

Now that we know about matrices we can see that an n-dimensional vector is just an $n \times 1$ matrix: a matrix with n rows and one column.

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

Addition and subtraction of matrices

Addition and subraction of matrices is done by adding or subtracting corresponding components, as it is for vectors. It can only be done when the matrices are the *same size*. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 6 \\ 3 & 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & -3 & -1 \\ 1 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}.$$

Then

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1+0 & 0-3 & 3-1 \\ 2+1 & -1+2 & 6+3 \\ 3-1 & 1-1 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 3 & 1 & 9 \\ 2 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 1-0 & 0-(-3) & 3-(-1) \\ 2-1 & -1-2 & 6-3 \\ 3-(-1) & 1-(-1) & 0-0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 1 & -3 & 3 \\ 4 & 2 & 0 \end{bmatrix}.$$

Example

Suppose numbers of three species of kangaroo are counted in three different areas over a period. Initially the numbers are represented in a matrix $\mathbf{P}$:

$$\begin{array}{cc} & \text{species} \\ & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \mathbf{P} = & \begin{bmatrix} 460 & 54 & 210 \\ 830 & 42 & 365 \\ 670 & 63 & 288 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ 1 \\ 2 \quad \text{area} \\ 3 \end{array}$$

A matrix $\mathbf{B}$ (which gives the number of births during the period), and a matrix $\mathbf{D}$ (which gives the number of deaths during the period) are given by:

$$\mathbf{B} = \begin{bmatrix} 42 & 6 & 30 \\ 88 & 7 & 43 \\ 72 & 6 & 22 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 24 & 3 & 22 \\ 103 & 12 & 83 \\ 32 & 4 & 16 \end{bmatrix}.$$

To find the change in population for each species in each area, it is natural to subtract the entries in **D** from the corresponding entries in **B** (since population change = births − deaths). We obtain

$$\mathbf{B} - \mathbf{D} = \begin{bmatrix} 18 & 3 & 8 \\ -15 & -5 & -40 \\ 40 & 2 & 6 \end{bmatrix}.$$

(This tells us, for instance, that in area 1 the 3rd kangaroo species increased by 8. The fact that the numbers in the 2nd row are all negative means that in area 2 all species of kangaroo suffered a decline in number.) To find the total numbers of kangaroos of each species in each area at the end of the period, it is natural to add the entries of **P** to the corresponding entries of **B** − **D**. We obtain

$$\mathbf{P} + (\mathbf{B} - \mathbf{D}) = \begin{bmatrix} 478 & 57 & 218 \\ 815 & 37 & 325 \\ 710 & 65 & 294 \end{bmatrix}.$$

Thus, for example, there are 710 of species 1 in area 3 at the end of the experimental period.

Multiplication of a matrix by a number

Each component of the matrix is multiplied by the number. If **A** is the matrix above, then

$$2 \times \mathbf{A} = 2 \times \begin{bmatrix} 1 & 0 & 3 \\ 2 & -1 & 6 \\ 3 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2\times 1 & 2\times 0 & 2\times 3 \\ 2\times 2 & 2\times -1 & 2\times 6 \\ 2\times 3 & 2\times 1 & 2\times 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 6 \\ 4 & -2 & 12 \\ 6 & 2 & 0 \end{bmatrix}.$$

Multiplication of a matrix by a number is called **scalar multiplication**.

2.2 Matrix Multiplication

A steelworks uses the raw materials hematite, dolomite and coking coal to produce cast iron, iron and steel. The amounts of raw materials required for each product are given in following table.

	cast iron	iron	steel
hematite	3	3	3
dolomite	1	1	1.5
coking coal	10	14	19

All the quantities above are measured in tonnes; for example, to produce one tonne of iron requires 3 tonnes of hematite, 1 tonne of dolomite and 14 tonnes of coking coal. The data in the table can be arranged as a 3×3 matrix:

$$\mathbf{Q} = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 1.5 \\ 10 & 14 & 19 \end{bmatrix}$$

Now suppose the steelworks produces the following quantities of steel products.

	July	August
cast iron	900	1100
iron	1200	1500
steel	650	750

This data can be arranged as a 3×2 matrix

$$\mathbf{P} = \begin{bmatrix} 900 & 1100 \\ 1200 & 1500 \\ 650 & 750 \end{bmatrix}$$

We want to know what quantities of raw materials to order each month; that is, we want to fill in the following table.

	July	August
hematite	?	?
dolomite	?	?
coking coal	?	?

The unknown numbers form a 3×2 matrix $\mathbf{R}$ whose components can be worked out like this:

$$\begin{aligned} \text{hematite required in July} &= \text{hematite required to produce one tonne of cast iron} \\ &\quad\ \times \text{tonnes of cast iron produced in July} \\ &\quad + \text{hematite required to produce one tonne of iron} \\ &\quad\ \times \text{tonnes of iron produced in July} \\ &\quad + \text{hematite required to produce one tonne of steel} \\ &\quad\ \times \text{tonnes of steel produced in July} \\ &= 3 \times 900 + 3 \times 1200 + 3 \times 650 \\ &= 8250. \end{aligned}$$

Similar calculations for the other components of $\mathbf{R}$ give

$$\begin{aligned} \mathbf{R} &= \begin{bmatrix} 3 \times 900 + 3 \times 1200 + 3 \times 650 & 3 \times 1100 + 3 \times 1500 + 3 \times 750 \\ 1 \times 900 + 1 \times 1200 + 1.5 \times 650 & 1 \times 1100 + 1 \times 1500 + 1.5 \times 750 \\ 10 \times 900 + 14 \times 1200 + 19 \times 650 & 10 \times 1100 + 14 \times 1500 + 19 \times 750 \end{bmatrix} \\ &= \begin{bmatrix} 2700 + 3600 + 1950 & 3300 + 4500 + 2250 \\ 900 + 1200 + 975 & 1100 + 1500 + 1125 \\ 9000 + 16800 + 12350 & 11000 + 21000 + 14250 \end{bmatrix} \\ &= \begin{bmatrix} 8250 & 10050 \\ 3075 & 3725 \\ 38150 & 46250 \end{bmatrix}. \end{aligned}$$

So, for example, 3075 tonnes of dolomite are required in July and 46250 tonnes of coking coal in August.

You can see from this calculation that the components of $\mathbf{R}$ are obtained from the components of $\mathbf{Q}$ and $\mathbf{P}$ as follows:

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \\ R_{31} & R_{32} \end{bmatrix} = \begin{bmatrix} Q_{11}P_{11} + Q_{12}P_{21} + Q_{13}P_{31} & Q_{11}P_{12} + Q_{12}P_{22} + Q_{13}P_{32} \\ Q_{21}P_{11} + Q_{22}P_{21} + Q_{23}P_{31} & Q_{21}P_{12} + Q_{22}P_{22} + Q_{23}P_{32} \\ Q_{31}P_{11} + Q_{32}P_{21} + Q_{33}P_{31} & Q_{31}P_{12} + Q_{32}P_{22} + Q_{33}P_{32} \end{bmatrix}.$$

We write

$$\mathbf{R} = \mathbf{QP},$$

the **product** of the matrices $\mathbf{Q}$ and $\mathbf{P}$.

The value of R_{31} can be written in sigma notation like this:

$$R_{31} = \sum_k Q_{3k} P_{k1}.$$

In general

$$\begin{aligned} R_{ij} &= \sum_k Q_{ik} P_{kj} \\ &= Q_{i1}P_{1j} + Q_{i2}P_{2j} + Q_{i3}P_{3j}, \end{aligned}$$

where i can be 1, 2 or 3 and j can be 1 or 2.

The product of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined like this in the general case: Suppose $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times p$ matrix. Then $\mathbf{A}$ has m *rows* each of length n and $\mathbf{B}$ has p columns each of length n. Since the rows of $\mathbf{A}$ have the same dimension as the columns of $\mathbf{B}$ it is possible to form the inner product of any row of $\mathbf{A}$ with any column of $\mathbf{B}$. Now the **product** $\mathbf{AB}$ is the $m \times p$ matrix $\mathbf{C}$ whose (i, j) component is the inner product of ith row of $\mathbf{A}$ with jth column of $\mathbf{B}$, thus

$$C_{ij} = \sum_k A_{ik} B_{kj}.$$

Another way to look at the same thing is as follows: to find the element C_{ij} run across the ith row of $\mathbf{A}$ and down the jth column of B multiplying and adding as you go

$$\begin{array}{ccccc} \mathbf{C} & = & \mathbf{A} & \times & \mathbf{B} \end{array}$$

$$\begin{array}{cc} & \begin{array}{c} \mathrm{j}\mathit{th} \\ \text{column} \\ \downarrow \end{array} \\ \mathrm{i}\mathit{th}\ \text{row} \ \rightarrow & \begin{bmatrix} & \vdots & \\ \cdots & \times & \cdots \\ & \vdots & \end{bmatrix} \end{array} = \mathrm{i}\mathit{th}\ \text{row} \begin{bmatrix} \rightarrow & \rightarrow & \rightarrow \end{bmatrix} \times \begin{array}{c} \mathrm{j}\mathit{th} \\ \text{column} \\ \begin{bmatrix} \quad \downarrow \quad \\ \downarrow \\ \downarrow \end{bmatrix} \end{array}$$

You perform this operation systematically, starting with the top lefthand element C_{11} which is computed from the first row of $\mathbf{A}$ and the first column of $\mathbf{B}$. Then compute C_{12} which is obtained from the first row of $\mathbf{A}$ and the second column of $\mathbf{B}$. Continue computing the first row of the product $\mathbf{C}$ by running across the first row of $\mathbf{A}$ and down successive columns of $\mathbf{B}$. Once the first row of the product is completed, start on the second row. The first element of this row C_{21} is computed from the second row of $\mathbf{A}$ and the first column of $\mathbf{B}$. The rest of the second row of the product is found by sticking with the second row of $\mathbf{A}$ and running down successive columns of $\mathbf{B}$. Continue in this way until the matrix product is complete. Although this procedure may seem complicated, it becomes fairly easy with enough practice.

When can matrices be multiplied?

To multiply **A** and **B** we form the inner product of the rows of **A** with columns of **B**. If **A** has n columns then its rows have dimension n:

$$\text{dimension of rows of } \mathbf{A} = \text{number of columns of } \mathbf{A}.$$

Similarly

$$\text{dimension of columns of } \mathbf{B} = \text{number of rows of } \mathbf{B}.$$

Since we can form the inner product of two vectors only if they have the same dimension

$$\boxed{\begin{array}{c}\text{the product } \mathbf{AB} \text{ can be formed only if} \\ \text{number of columns of } \mathbf{A} = \text{number of rows of } \mathbf{B}.\end{array}}$$

The fact that not all pairs of matrices can be multiplied fits in with similar facts we already know:

Two vectors can be added or subtracted only if they have the same dimenion.
The inner product of two vectors can be formed only if they have the same dimension.
Two matrices can be added or subtracted only if they are the same size.

The order of matrix multiplication can't be changed

Suppose that the number of columns of **A** is the same as the number of rows of **B**, so that the product **AB** can be formed. Usually $\mathbf{BA} \neq \mathbf{AB}$, so it is a mistake to change the order of multiplication in a matrix calculation. Here are some of the things that can happen.

1. **BA** doesn't exist.

   Example

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}.$$

$$\begin{aligned}
\mathbf{AB} &= \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1\times 0 + 0\times 2 & 1\times 1 + 0\times 3 & 1\times(-1) + 0\times 0 \\ 1\times 0 + (-1)\times 2 & 1\times 1 + (-1)\times 3 & 1\times(-1) + (-1)\times 0 \end{bmatrix} \\
&= \begin{bmatrix} 0+0 & 1+0 & (-1)+0 \\ 0+(-2) & 1+(-3) & (-1)+0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & -1 \end{bmatrix}.
\end{aligned}$$

   However the product **BA** is not defined since **B** has 3 columns but **A** has 2 rows, and these numbers are not the same.

2. BA exists but is not the same size as **AB**.

Example

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 0 & -1 \end{bmatrix}.$$

Then

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 2 \\ -2 & -3 \end{bmatrix} \end{aligned}$$

while

$$\begin{aligned} \mathbf{BA} &= \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 & 0 \\ -2 & -2 & 3 \\ 2 & 0 & -1 \end{bmatrix}. \end{aligned}$$

We see from this that **AB** and **BA** are not the same size (so can't be equal) unless **A** and **B** are "square" matrices of the same size.

3. AB and **BA** are the same size, but not equal.

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Then

$$\begin{aligned} \mathbf{AB} &= \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 0 + 1 \times 1 & 1 \times 1 + 1 \times 2 \\ 2 \times 0 + 0 \times 1 & 2 \times 1 + 0 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

but

$$\begin{aligned}
\mathbf{BA} &= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times 1 + 1 \times 2 & 0 \times 1 + 1 \times 0 \\ 1 \times 1 + 2 \times 2 & 1 \times 1 + 2 \times 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 5 & 1 \end{bmatrix}.
\end{aligned}$$

Q. Why make matrix multiplication so hard? Why not just multiply component-by-component as for adding and subtracting matrices?

A. If we defined matrix multiplication any other way it might be easier to work out but it wouldn't give the right answers to questions like our steelworks problem.

Multiplication of a vector by a matrix

An n-dimensional vector is the same thing $n \times 1$ matrix, and if we multiply this vector by an $m \times n$ matrix the result will be an $m \times 1$ matrix, which is the same thing as a m-dimensional vector. For example if

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{X} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

we can calculate $\mathbf{A} \times \mathbf{X}$, since the number of columns of $\mathbf{A}$ is 4 which is the same as the number of rows of $\mathbf{X}$. The result is a 3-dimensional vector.

$$\begin{aligned}
\mathbf{A} \times \mathbf{X} &= \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times 0 + 1 \times 1 + 2 \times 2 + 3 \times 3 \\ 2 \times 0 + 3 \times 1 + 4 \times 2 + 5 \times 3 \\ 3 \times 0 + 4 \times 1 + 5 \times 2 + 6 \times 3 \end{bmatrix} \\
&= \begin{bmatrix} 0 + 1 + 4 + 9 \\ 0 + 3 + 8 + 15 \\ 0 + 4 + 10 + 18 \end{bmatrix} \\
&= \begin{bmatrix} 14 \\ 26 \\ 32 \end{bmatrix}.
\end{aligned}$$

In general if $\mathbf{A}$ is a $m \times n$ matrix and $\mathbf{X}$ is a n-dimensional vector, then the vector $\mathbf{Y} = \mathbf{A} \times \mathbf{X}$ has dimension m and its components are given by

$$Y_i = \sum_j A_{ij} \times X_j.$$

If $\mathbf{Y} = \mathbf{A} \times \mathbf{X}$ in the example above, for instance,

$$\begin{aligned} Y_2 &= A_{21} \times X_1 + A_{22} \times X_2 + A_{23} \times X_3 + A_{24} \times x_4 \\ &= 2 \times 0 + 3 \times 1 + 4 \times 2 + 5 \times 3 \\ &= 26. \end{aligned}$$

2.3 The Leslie Matrix Model of Population Growth

In this section we apply matrices to the biological problem of population growth.

Suppose a population is divided into a number of age groups of *equal time span* T and let

$$\begin{aligned} P_1 &= \text{number of individuals aged from 0 to } T \\ P_2 &= \text{number of individuals aged from } T \text{ to } 2T \\ P_3 &= \text{number of individuals aged from } 2T \text{ to } 3T \\ &\vdots \\ P_k &= \text{number of individuals aged from } (k-1)T \text{ to } kT. \end{aligned}$$

Suppose we know what the numbers $P_1, P_2, P_3, \ldots, P_k$ are now and we want to work out what they will be at time T from now. Before we can do that there are two sets of numbers we need to know:

1. S_i = proportion of individuals in the $(i-1)$th age group who survive to the ith age group.
2. B_i = birth rate per individual in the ith age group.

For example, if we are working with a human population and $T = 10$ yrs then S_7 is the number of people now in their 60's who survive another 10 years (when they will be in their 70's) and B_3 is the total number of children born in the next 10 years to people now in their 30's divided by the total number of those people.

Let P'_i be the number of individuals in the ith age group after a time T. These numbers can be worked out as follows:

$$\begin{aligned} P'_1 &= B_1P_1 + B_2P_2 + \cdots + B_kP_k \\ P'_2 &= S_2P_1 \\ P'_3 &= S_3P_2 \\ &\vdots \\ P'_k &= S_kP_{k-1}. \end{aligned}$$

The reason is this: Firstly, P'_1, the number of individuals aged from 0 to T after time T, must all be new births, since those originally in this age group will have aged to the next age group or died. So P'_1 is the sum of the number of surviving births for each age group, which is got by multiplying the birth rate B_i for the age group by the number of individuals P_i in the age group. Secondly, the numbers P'_2 up to P'_k are exactly the numbers of individuals surviving from the preceding age group, which is the number in the preceding age group P_{i-1} multiplied by the proportion S_i of this age group which survives.

The equations for the numbers P_i' can be written in matrix form

$$\begin{bmatrix} P_1' \\ P_2' \\ P_3' \\ \vdots \\ P_k' \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & B_3 & \dots & B_k \\ S_2 & 0 & 0 & \dots & 0 \\ 0 & S_3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & S_k & 0 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_k \end{bmatrix}.$$

Let

$$\mathbf{P}' = \begin{bmatrix} P_1' \\ P_2' \\ P_3' \\ \vdots \\ P_k' \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ \vdots \\ P_k \end{bmatrix}$$

and

$$\mathbf{L} = \begin{bmatrix} B_1 & B_2 & B_3 & \dots & B_{k-1} & B_k \\ S_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & S_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & S_k & 0 \end{bmatrix}.$$

Then

$$\mathbf{P}' = \mathbf{L} \times \mathbf{P}.$$

The $k \times k$ matrix $\mathbf{L}$ is called the **Leslie matrix** of the population.

Example

Suppose a certain species of beetle lives for only three months. The proportion of beetles aged 0–1 months surviving to age 1–2 months is 50% and the proportion aged 1–2 months surviving to age 2–3 months is 40%. The beetles give birth only in their third month and produce on average 6 surviving young. If initially there are 1000 beetles in each of the age groups 0–1, 1–2 and 2–3, how many beetles will there be in each age group after 1 month?

Here we have 3 age groups and

$$T = 1\,\text{month}.$$

The initial population vector is

$$\mathbf{P} = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}.$$

The birth rates are

$$\begin{aligned} B_1 &= 0 \\ B_2 &= 0 \\ B_3 &= 6 \end{aligned}$$

and the survival rates are

$$\begin{aligned} S_2 &= 0.5 \\ S_3 &= 0.4, \end{aligned}$$

so the Leslie matrix for the beetle population is

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{P}' &= \mathbf{L} \times \mathbf{P} \\ &= \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} \\ &= \begin{bmatrix} 6000 \\ 500 \\ 400 \end{bmatrix}, \end{aligned}$$

which tells us that after one month there will be 6000 beetles aged 0–1 months, 500 beetles aged 1–2 months and 400 beetles aged 2–3 months.

We will find out more about Leslie matrices after we learn some more matrix algebra.

2.4 The Identity Matrix

The $n \times n$ matrix with ones along the diagonal and zeroes everywhere else is called the $n \times n$ **identity matrix** and denoted by $\mathbf{I}$:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}.$$

If $\mathbf{I}$ is the $n \times n$ identity matrix then

$$\mathbf{I} \times \mathbf{A} = \mathbf{A}$$

for every $n \times m$ matrix $\mathbf{A}$. To see why this is so first notice that multiplying an $n \times n$ matrix by an $n \times m$ matrix gives an $n \times m$ matrix, so $\mathbf{I} \times \mathbf{A}$ is an $n \times m$ matrix, the same size as $\mathbf{A}$. Now look at the (i, j) component of $\mathbf{I} \times \mathbf{A}$, which is the inner product of the ith row of $\mathbf{I}$ with the jth column of $\mathbf{A}$. The ith row of $\mathbf{I}$ is is zero except for the ith component which is

1, so the vectors in the inner product are

$$\begin{array}{c}\left[\begin{array}{cccccccccc} 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \end{array}\right] \\ \underbrace{\uparrow}_{i\text{th component}}\end{array} \quad \text{and} \quad \left[\begin{array}{c} A_{1j} \\ A_{2j} \\ \vdots \\ A_{i-1\,j} \\ A_{ij} \\ A_{i+1\,j} \\ \vdots \\ 0 \end{array}\right].$$

The only non-zero term in the inner product is $1 \times A_{ij}$ and therefore the (i, j) component of $\mathbf{I} \times \mathbf{A}$ is A_{ij}, which is also the (i, j) component of $\mathbf{A}$. Doing this for every i and j shows that every component of $\mathbf{I} \times \mathbf{A}$ is the same as the corresponding component of $\mathbf{A}$. So $\mathbf{I} \times \mathbf{A} = \mathbf{A}$.

You can work out similarly that if $\mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{B}$ is an $m \times n$ matrix then

$$\mathbf{B} \times \mathbf{I} = \mathbf{B}.$$

So *multiplication of any matrix by an identity matrix leaves the original matrix unchanged* (provided the matrices are the right sizes for the multiplication to be done).

A special case is that if $\mathbf{X}$ is an n-dimensional vector and $\mathbf{I}$ is the $n \times n$ identity matrix, then

$$\mathbf{I} \times \mathbf{X} = \mathbf{X}.$$

Another important special case is when $\mathbf{A}$ is a square $n \times n$ matrix and $\mathbf{I}$ is the $n \times n$ identity matrix. In this case the products $\mathbf{I} \times \mathbf{A}$ and $\mathbf{A} \times \mathbf{I}$ are both defined and

$$\mathbf{I} \times \mathbf{A} = \mathbf{A} \times \mathbf{I} = \mathbf{A}.$$

This is one of the few cases where changing the order does not change the product of the matrices.

2.5 Powers of Matrices

If $\mathbf{A}$ is a square $n \times n$ matrix then the product of $\mathbf{A}$ with itself is also a square $n \times n$ matrix which is denoted by $\mathbf{A}^2$:

$$\mathbf{A}^2 = \mathbf{A} \times \mathbf{A}.$$

We get higher powers of $\mathbf{A}$ in the same way:

$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A} \times \mathbf{A} \times \mathbf{A} = \mathbf{A} \times \mathbf{A}^2 \\ \mathbf{A}^4 &= \mathbf{A} \times \mathbf{A} \times \mathbf{A} \times \mathbf{A} = \mathbf{A} \times \mathbf{A}^3 \\ \mathbf{A}^5 &= \mathbf{A} \times \mathbf{A}^4, \quad \text{etc.} \end{aligned}$$

For example, if

$$\mathbf{A} = \left[\begin{array}{ccc} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 1 & 0 & 1 \end{array}\right]$$

then

$$\begin{aligned}
\mathbf{A}^2 &= \mathbf{A}\times\mathbf{A}\\
&= \begin{bmatrix}1&2&0\\0&3&1\\1&0&1\end{bmatrix}\times\begin{bmatrix}1&2&0\\0&3&1\\1&0&1\end{bmatrix}\\
&= \begin{bmatrix}1\times1+2\times0+0\times1 & 1\times2+2\times3+0\times0 & 1\times0+2\times1+0\times1\\ 0\times1+3\times0+1\times1 & 0\times2+3\times3+1\times0 & 0\times0+3\times1+1\times1\\ 1\times1+0\times0+1\times1 & 1\times2+0\times3+1\times0 & 1\times0+0\times1+1\times1\end{bmatrix}\\
&= \begin{bmatrix}1+0+0 & 2+6+0 & 0+2+0\\ 0+0+1 & 0+9+0 & 0+3+1\\ 1+0+1 & 2+0+0 & 0+0+1\end{bmatrix}\\
&= \begin{bmatrix}1&8&2\\1&9&4\\2&2&1\end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{A}^3 &= \mathbf{A}\times\mathbf{A}^2\\
&= \begin{bmatrix}1&2&0\\0&3&1\\1&0&1\end{bmatrix}\times\begin{bmatrix}1&8&2\\1&9&4\\2&2&1\end{bmatrix}\\
&= \begin{bmatrix}3&26&10\\5&29&13\\3&10&3\end{bmatrix}
\end{aligned}$$

(check this). Higher powers of $\mathbf{A}$ are calculated similarly.

Application to Leslie matrices

We can apply this to the Leslie matrix of the example in section 3

$$\mathbf{L} = \begin{bmatrix}0&0&6\\0.5&0&0\\0&0.4&0\end{bmatrix},$$

with initial population vector

$$\mathbf{P} = \begin{bmatrix}1000\\1000\\1000\end{bmatrix}.$$

Let $\mathbf{P}_n$ be the population vector after n months. Then

$$\begin{aligned}
\mathbf{P}_1 = \mathbf{L}\times\mathbf{P}, &\\
\mathbf{P}_2 = \mathbf{L}\times\mathbf{P}_1 &= \mathbf{L}\times\mathbf{L}\times\mathbf{P}\\
&= \mathbf{L}^2\times\mathbf{P},\\
\mathbf{P}_3 = \mathbf{L}\times\mathbf{P}_2 &= \mathbf{L}\times\mathbf{L}^2\times\mathbf{P}\\
&= \mathbf{L}^3\times\mathbf{P},\\
\text{etc.}&
\end{aligned}$$

In general

$$\mathbf{P}_n = \mathbf{L}^n \times \mathbf{P}.$$

Now

$$\begin{aligned}
\mathbf{L}^2 &= \mathbf{L} \times \mathbf{L} \\
&= \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times 0 + 0 \times 0.5 + 6 \times 0 & 0 \times 0 + 0 \times 0 + 6 \times 0.4 & 0 \times 6 + 0 \times 0 + 6 \times 0 \\ 0.5 \times 0 + 0 \times 0.5 + 0 \times 0 & 0.5 \times 0 + 0 \times 0 + 0 \times 0.4 & 0.5 \times 6 + 0 \times 0 + 0 \times 0 \\ 0 \times 0 + 0.4 \times 0.5 + 0 \times 0 & 0 \times 0 + 0.4 \times 0 + 0 \times 0.4 & 0 \times 6 + 0.4 \times 0 + 0 \times 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2.4 & 0 \\ 0 & 0 & 3 \\ 0.2 & 0 & 0 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{L}^3 &= \mathbf{L} \times \mathbf{L}^2 \\
&= \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 2.4 & 0 \\ 0 & 0 & 3 \\ 0.2 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}.
\end{aligned}$$

So we get

$$\begin{aligned}
\mathbf{P}_2 &= \mathbf{L}^2 \times \mathbf{P} \\
&= \begin{bmatrix} 0 & 2.4 & 0 \\ 0 & 0 & 3 \\ 0.2 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times 1000 + 2.4 \times 1000 + 0 \times 1000 \\ 0 \times 1000 + 0 \times 1000 + 3 \times 1000 \\ 0.2 \times 1000 + 0 \times 1000 + 0 \times 1000 \end{bmatrix} \\
&= \begin{bmatrix} 2400 \\ 3000 \\ 200 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{P}_3 &= \mathbf{L}^3 \times \mathbf{P} \\
&= \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} \\
&= \begin{bmatrix} 1200 \\ 1200 \\ 1200 \end{bmatrix}.
\end{aligned}$$

Another way of doing this last calculation is to note

$$\begin{aligned}\mathbf{L}^3 &= \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \\ &= 1.2 \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= 1.2 \times \mathbf{I},\end{aligned}$$

where $\mathbf{I}$ is the 3×3 identity matrix, so

$$\mathbf{P}_3 = \mathbf{L}^3 \times \mathbf{P} = 1.2 \times \mathbf{I} \times \mathbf{P} = 1.2 \times \mathbf{P},$$

using the fact that $\mathbf{I} \times \mathbf{P} = \mathbf{P}$. Hence

$$\mathbf{P}_3 = 1.2 \times \mathbf{P} = 1.2 \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1200 \\ 1200 \\ 1200 \end{bmatrix}.$$

Although matrix powers are useful in calculating future populations, it requires less work if we multiply the initial population repeatedly by $\mathbf{L}$. That is it is easier to compute, for example,

$$\mathbf{P}_3 = \mathbf{L}(\times\mathbf{L}(\times\mathbf{L} \times \mathbf{P}))$$

than is does to compute $\mathbf{L}^3$ and then

$$\mathbf{P}_3 = \mathbf{L}^3 \times \mathbf{P}.$$

2.6 Inverse Matrices

We now know how to add, subtract and multiply matrices. It would be nice to be able to divide them too. To divide numbers it is enough to be able to find $1/a$ for every number a, because

$$b \div a = b \times (1/a).$$

For example, $5 \div 2 = 5 \times 1/2 = 2.5$. The number $1/a$ can also be written as a^{-1} and is called the **reciprocal** or **inverse** of a. It satisfies

$$a^{-1} \times a = 1.$$

The $n \times n$ matrix that behaves like the number 1 is the identity matrix $\mathbf{I}$. So to divide $n \times n$ matrices we need, for every $n \times n$ matrix $\mathbf{A}$, an $n \times n$ **inverse matrix** $\mathbf{A}^{-1}$ with

$$\mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}.$$

Because changing the order of matrix multiplication may make a difference we should also ask for

$$\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$$

as a separate request.

Actually, not every number has a reciprocal — 0 doesn't because the division $1/0$ can't be done. With numbers 0 is the only exception, but *there are many square matrices that don't have an inverse* not just matrices with all their components 0. However, most square matrices do have an inverse, so it is well worth finding out about inverses.

For example, the inverse of the Leslie matrix

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

of section 3 is

$$\mathbf{L}^{-1} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 5/2 \\ 1/6 & 0 & 0 \end{bmatrix}.$$

Work out the products $\mathbf{L}^{-1} \times \mathbf{L}$ and $\mathbf{L} \times \mathbf{L}^{-1}$ and check they are both $\mathbf{I}$.

It is a fact that if $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size with $\mathbf{B} \times \mathbf{A} = \mathbf{I}$ then $\mathbf{A} \times \mathbf{B} = \mathbf{I}$ too. So working out only one of the two products above is enough to check that the second matrix is the inverse of $\mathbf{L}$. Of course, when $\mathbf{C} \neq \mathbf{I}$ it is not in general true that if $\mathbf{BA} = \mathbf{C}$ then $\mathbf{AB} = \mathbf{C}$. There are systematic methods for computing the inverse of a matrix, but they are beyond the scope of this course.

The inverse of a 2×2 matrix

Only for 2×2 matrices is it easy to tell whether there is an inverse and, if so, find it. Inverses of square matrices of any size can be found by computer, however.

If

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is a 2×2 matrix its **determinant** is the number

$$\det \mathbf{A} = ad - bc.$$

(We are using a, b, c and d here because they are easier to write than A_{11}, A_{12}, A_{21} and A_{22}. Sometimes $\det \mathbf{A}$ is written as $|\mathbf{A}|$.) A 2×2 matrix $\mathbf{A}$ does not have an inverse if $\det \mathbf{A} = 0$, but if $\det \mathbf{A} \neq 0$ it does have an inverse and it is given by

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \times \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

This formula makes sense only when $\det \mathbf{A} \neq 0$ because otherwise the fraction $1/0$ appears. Let's check that this is the inverse:

$$\begin{aligned} \mathbf{A}^{-1} \times \mathbf{A} &= \frac{1}{ad - bc} \times \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad - bc} \times \begin{bmatrix} d \times a + (-b) \times c & d \times b + (-b) \times d \\ (-c) \times a + a \times c & (-c) \times b + a \times d \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{ad-bc} \times \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \mathbf{I} \end{aligned}$$

as required. You could also check that $\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$, though, as before, working out the product one way round is all that is necessary to check that we have the inverse of $\mathbf{A}$.

Examples

Let

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}.$$

Then

$$\det \mathbf{A} = (-1) \times 2 - (-1) \times 2 = 0$$

and

$$\det \mathbf{B} = 1 \times 4 - 2 \times 1 = 2,$$

so $\mathbf{B}$ has an inverse but $\mathbf{A}$ does not. The inverse of $\mathbf{B}$ is

$$\begin{aligned} \mathbf{B}^{-1} &= \frac{1}{2} \times \begin{bmatrix} 4 & -2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ -1/2 & 1/2 \end{bmatrix}. \end{aligned}$$

Applications of inverse matrices

There are many situations where we know matrices $\mathbf{A}$ and $\mathbf{B}$ and we need to find a matrix $\mathbf{X}$ with

$$\mathbf{A} \times \mathbf{X} = \mathbf{B}$$

(i.e., we would like to divide $\mathbf{B}$ by $\mathbf{A}$). How can $\mathbf{X}$ be found? Multiplying both sides of the above equation by $\mathbf{A}^{-1}$ gives

$$\begin{aligned} \mathbf{A}^{-1} \times \mathbf{A} \times \mathbf{X} &= \mathbf{A}^{-1} \times \mathbf{B} \\ \mathbf{I} \times \mathbf{X} &= \mathbf{A}^{-1} \times \mathbf{B} \\ \mathbf{X} &= \mathbf{A}^{-1} \times \mathbf{B}. \end{aligned}$$

So $\mathbf{X} = \mathbf{A}^{-1} \times \mathbf{B}$. For this to work $\mathbf{A}$ must be a square matrix with an inverse.

A special case that is often met is an equation $\mathbf{A} \times \mathbf{X} = \mathbf{Y}$ with $\mathbf{A}$ a square $n \times n$ matrix and $\mathbf{X}$ and $\mathbf{Y}$ n-dimensional vectors. If $\mathbf{A}$ and $\mathbf{Y}$ are known and $\mathbf{A}$ has an inverse then

$$\mathbf{X} = \mathbf{A}^{-1} \times \mathbf{Y}.$$

In section 3 we had a Leslie matrix

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 6 \\ 0.5 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

and population vector

$$\mathbf{P} = \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix}.$$

We found the population vector a month later by

$$\mathbf{P}_1 = \mathbf{L} \times \mathbf{P}.$$

What if we had wanted to know the population vector a month before? This is easily found if we know the inverse of $\mathbf{L}$. If the population vector a month before was $\mathbf{P}_{-1}$, then

$$\mathbf{P} = \mathbf{L} \times \mathbf{P}_{-1}$$

because multiplying by the Leslie matrix updates the population vector from one month to the next. Multiplying both sides of this equation by the matrix $\mathbf{L}^{-1}$ gives

$$\mathbf{L}^{-1} \times \mathbf{P} = \mathbf{L}^{-1} \times \mathbf{L} \times \mathbf{P}_{-1} = \mathbf{I} \times \mathbf{P}_{-1} = \mathbf{P}_{-1},$$

so

$$\mathbf{P}_{-1} = \mathbf{L}^{-1} \times \mathbf{P}.$$

We found $\mathbf{L}^{-1}$ earlier, and with the vector $\mathbf{P}$ as above we get

$$\begin{aligned} \mathbf{P}_{-1} &= \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 5/2 \\ 1/6 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 1000 \\ 1000 \\ 1000 \end{bmatrix} \\ &= \begin{bmatrix} 2000 \\ 2500 \\ 166\frac{2}{3} \end{bmatrix}. \end{aligned}$$

By repeating this process we can obtain the population vectors for two months before, three months before, and so on.

It is important to realise that we could do the calculation above only because $\mathbf{L}$ had an inverse. If $\mathbf{L}$ did not have an inverse there would be no way to deduce what the past populations were from the present population.

Finding inverses

For a 2×2 matrix there is an easy formula for calculating the inverse. For larger matrices there is a way to calculate the inverse of a matrix, but it is beyond the scope of this unit and, in any case, finding the inverse of matrices bigger than 2×2 is best done by computer.

What can easily be done for any matrix, is to check that a given matrix is its inverse. For a matrix $\mathbf{A}$, to check that a matrix $\mathbf{B}$ is its inverse simply multiply $\mathbf{AB}$ and check that the result is the identity matrix $\mathbf{I}$.

2.7 Exercises

Let **A**, **B** and **C** be the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -3 \\ -1 & 0 & 0 \\ 3 & -3 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -3 \\ 3 & 2 & -2 \\ -4 & 0 & -1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} -3 & -2 & 2 \\ 1 & 3 & 5 \\ -2 & -1 & 0 \\ -1 & -1 & -2 \end{bmatrix}.$$

1. What size are these matrices?

2. Write down the components A_{12}, A_{23} and A_{41}.

3. Calculate $\mathbf{A} + \mathbf{B}$.

4. Calculate $2\mathbf{C}$.

5. Calculate $\mathbf{A} + \mathbf{B} - 2\mathbf{C}$.

6. Suppose

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 & 4 \\ 0 & 6 & 7 & 3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{C} = \begin{bmatrix} 3 & 2 & 1 & -2 \\ 4 & 8 & -3 & -2 \end{bmatrix}$$

and **B** is matrix such that

$$\mathbf{A} - \mathbf{B} = \mathbf{C}.$$

Find **B**.

In a study of mosquito populations the following figures were obtained:

		eggs	nos. of wrigglers	tumblers	mosquitoes
	1	2438	1104	358	210
Area	2	5068	2462	730	551
	3	1438	982	412	289

A year later the numbers had changed to

$$\begin{bmatrix} 3700 & 1813 & 428 & 236 \\ 6238 & 3115 & 789 & 608 \\ 1649 & 1319 & 438 & 309 \end{bmatrix}.$$

Let **A** be the matrix containing the first set of data, and **B** the matrix containing the second set of data.

7. What does the matrix $\mathbf{B} - \mathbf{A}$ represent? Calculate $\mathbf{B} - \mathbf{A}$.

8. Calculate the matrix $\frac{1}{2}(\mathbf{B} - \mathbf{A})$.

In 1958 Peek found the following distribution of moose in three different vegetative areas of Montana.

		bulls	cows	cows with calves
Vegetative	1	98	39	22
Area	2	24	15	15
	3	42	22	17

Let $\mathbf{A}$ be the matrix representing the above data, and let

$$\mathbf{B} = \begin{bmatrix} 55 & 43 & 11 \\ 19 & 53 & 40 \\ 14 & 38 & 20 \end{bmatrix}$$

be the matrix representing the corresponding data for 1959.

9. Combine matrices $\mathbf{A}$ and $\mathbf{B}$ so as to obtain a matrix representing the total number of moose seen in each category in each area during 1958 and 1959.

10. Combine matrices $\mathbf{A}$ and $\mathbf{B}$ so as to obtain a matrix showing the change in the number of moose seen in each category in each area between 1958 and 1959.

11. Combine matrices $\mathbf{A}$ and $\mathbf{B}$ to form a new matrix representing the average number of moose seen per year in each category in each area over the 1958–59 period.

A manufacturer has three machines that produce a certain type of goods. These machines are operated on two shifts per day. Suppose production on day 1 sees 300 units produced on machine 1, 217 on machine 2, and 64 on machine 3 in the first shift, and 271 units produced on machine 1, 201 produced on machine 2, and 83 on machine 3 in the second shift. For day 2 the figures are 284, 196, 58 for shift 1 and 288, 202, 69 for shift 2, respectively; and for day 3 the production figures are 184, 112, and 49 for shift 1, and 318, 213, and 73 for shift 2.

12. Write down a matrix that gives the production figures for each shift on each machine for day 1. Call this matrix $\mathbf{A}$.

13. Similarly, write down matrices that give production figures for each shift on each machine for days 2 and 3. Let these matrices be $\mathbf{B}$ and $\mathbf{C}$.

14. Calculate (i) $\mathbf{A}+\mathbf{B}$, (ii) $\mathbf{C}-\mathbf{B}$, (iii) $\frac{1}{2}(\mathbf{B}+\mathbf{C})$, (iv) $\frac{1}{3}(\mathbf{A}+\mathbf{B}+\mathbf{C})$, and (v) $\frac{2}{3}\mathbf{A}-\frac{1}{3}\mathbf{B}-\frac{1}{3}\mathbf{C}$.

15. Explain what each of the matrices in Exercise **14** represents. For the last matrix, it may be helpful to note that

$$\frac{2}{3}\mathbf{A} - \frac{1}{3}\mathbf{B} - \frac{1}{3}\mathbf{C} = \mathbf{A} - \frac{1}{3}(\mathbf{A}+\mathbf{B}+\mathbf{C}).$$

Let

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 & 2 \\ 1 & -3 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}.$$

16. Write down the sizes of these three matrices.

17. Which of the matrix products $\mathbf{A}\times\mathbf{A}$, $\mathbf{A}\times\mathbf{B}$, $\mathbf{A}\times\mathbf{C}$, $\mathbf{B}\times\mathbf{A}$, $\mathbf{B}\times\mathbf{B}$, $\mathbf{B}\times\mathbf{C}$, $\mathbf{C}\times\mathbf{A}$, $\mathbf{C}\times\mathbf{B}$, and $\mathbf{C}\times\mathbf{C}$ are defined?

18. Do all the matrix multiplications in Exercise **17** that are defined.

Refer to the data on moose populations used in Exercises 9 10 and 11 and further suppose that the average amount of vegetation consumed by a moose throughout a season is 18.5 units for a bull, 15 units for a cow, and 19 units for a cow with calves. This data can be represented by the vector

$$\mathbf{C} = \begin{bmatrix} 18.5 \\ 15 \\ 19 \end{bmatrix}.$$

19. With the matrix **A** given earlier, calculate **AC**. What does this represent?

20. By combining **A**, **B** and **C** in some operations, obtain a vector whose entries show the total amount of vegetation eaten by moose for each of the three vegetative areas over the 1958–59 period.

A manufacturer makes two types of products I and II, at two plants X and Y. In the manufacturing process three types of pollutants — sulphur dioxide (SO_2), carbon monoxide (CO), and particle matter — are produced. The quantity of pollutants produced per day (in kg) in the manufacture of each product can be tabulated in a matrix **A**:

$$\begin{array}{cc} & \begin{array}{ccc} SO_2 & CO & \text{particle} \end{array} \\ \mathbf{A} = & \begin{bmatrix} 150 & 50 & 100 \\ 200 & 25 & 150 \end{bmatrix} \begin{array}{l} \text{product I} \\ \text{product II} \end{array} \end{array}$$

To satisfy government regulations, the pollutants must be removed. The cost in dollars for removing each kg. of pollutant at each plant can be represented by the matrix **B**:

$$\begin{array}{cc} & \begin{array}{cc} \text{plant} & \text{plant} \\ X & Y \end{array} \\ \mathbf{B} = & \begin{bmatrix} 10 & 16 \\ 6 & 8 \\ 4 & 2 \end{bmatrix} \begin{array}{l} SO_2 \\ CO \\ \text{particle} \end{array} \end{array}$$

21. Calculate **AB**.

22. What do each of the components of the product **AB** represent?

A certain species of beetle requires one month to complete its juvenile (egg, pupa, larva) stages during which the survival rate is 90%. Adults live for a maximum of three months with a survival rate of 50% in each of the first two months. Each adult produces, on average, 30, 30 and 40 eggs during the first, second and third months respectively.

23. Write down the Leslie matrix for this population. (Hint: there are 4 age groups and $T = 1$ month.)

24. If the initial population vector is

$$\mathbf{P} = \begin{bmatrix} 0 \\ 20 \\ 20 \\ 20 \end{bmatrix}$$

calculate the population in each age group after one month.

25. Calculate the population in each age group after two, three and four months.

Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}.$$

26. Find $\mathbf{A}^2$.

27. Find $\mathbf{A}^3$ and $\mathbf{A}^4$.

28. Find $\mathbf{A}^{-1}$.

29. There are two ways to interpret $\mathbf{A}^{-2}$:

$$\text{(i)} \quad \mathbf{A}^{-2} = (\mathbf{A}^{-1})^2 \quad \text{(the square of the inverse of } \mathbf{A}\text{)},$$

and

$$\text{(i)} \quad \mathbf{A}^{-2} = (\mathbf{A}^{2})^{-1} \quad \text{(the inverse of the square of } \mathbf{A}\text{)}.$$

Check that that these give the same answer.

30. Check that inverse of the Leslie matrix in Exercise 23 is

$$\mathbf{L}^{-1} = \begin{bmatrix} 0 & \frac{10}{9} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ \frac{1}{40} & 0 & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}.$$

31. If the current population vector is

$$\mathbf{P} = \begin{bmatrix} 700 \\ 200 \\ 0 \\ 5 \end{bmatrix}$$

calculate the population vector one month ago.

32. Calculate the population vector two and three months ago.

Chapter 3

Functions

In this chapter we will be looking at common functional relationships between a pair of variables x and y. We will usually think of x as the **independent variable** and y as the **dependent variable**. This means that each value of x determines a definite value of y and is usually expressed by an equation

$$y = f(x)$$

indicating that y is a **function** of x. Usually a function is given by a mathematical formula. For example, suppose

$$y = x^2 + x - 1.$$

Given any value of x we can work out the corresponding value of y. The following table gives a few values.

x	-2	-1	0	1	2
y	1	-1	-1	1	5

These values can be plotted on a graph of y against x and the set of all possible values of x and y is represented by a smooth curve as in Figure 3.1.

3.1 The Equation Of A Straight Line

This section looks at functions whose graph is a straight line. These functions are called **linear functions** and take the form

$$y = bx + a,$$

where a and b are numbers whose values determine the position and slope of the straight line. For example if

$$y = 2x - 1$$

we can compute a table of values of x and y

x	-2	-1	0	1	2
y	-5	-3	-1	1	3

and join the points together to get a graph of y as a function of x as in Figure 3.2.

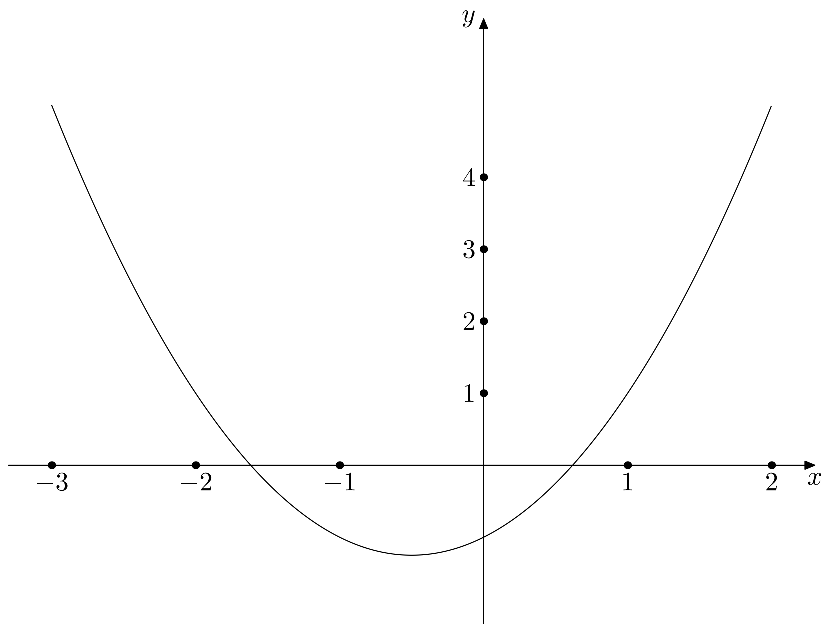

Figure 3.1: Graph of $y = x^2 + x - 1$

The **slope** of a straight line is the ratio of the change in y values to the change in x values for any points lying on the line:

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}.$$

If (x_1, y_1) and (x_2, y_2) are two points on the line, then

$$\text{change in } y = y_2 - y_1$$

and

$$\text{change in } x = x_2 - x_1$$

so

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since $y_1 = bx_1 + a$ and $y_2 = bx_2 + a$

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{(bx_2 + a) - (bx_1 + a)}{x_2 - x_1} \\ &= \frac{b(x_2 - x_1)}{x_2 - x_1} \\ &= b. \end{aligned}$$

So the slope of the straight line $y = bx + a$ is b. For example, $y = 2x - 1$ has slope 2. We can check this by taking a pair of points on the line, e.g., $(x_1, y_1) = (-1, -3)$ and $(x_2, y_2) = (1, 1)$

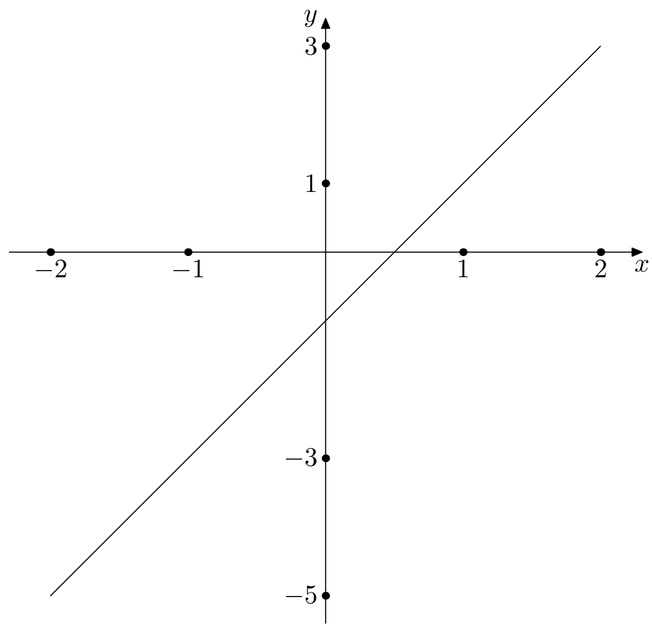

Figure 3.2: Graph of $y = 2x - 1$

(which were obtained from the previous table) and calculating

$$\text{slope} = (1 - (-3))/(1 - (-1)) = \frac{4}{2} = 2.$$

If the slope of a line is positive it slopes from lower left to upper right, if the slope is negative it slopes from lower right to upper left. The greater the slope, the steeper the straight line appears. See Figure 3.3.

It is often useful to know where a straight lines crosses the x and y axes. Points on the x axis have $y = 0$ and points on the y axis have $x = 0$. To find where a line cuts the x axis we set $y = 0$ in the equation $y = bx + a$

$$0 = bx + a$$

subtract a from both sides

$$bx = -a$$

and divide both sides by a

$$x = -\frac{a}{b}.$$

So the line $y = bx + a$ cuts the x axis where $x = -a/b$. To find where the line cuts the y axis set $x = 0$ in the equation of the line:

$$y = b \times 0 + a = a.$$

So the line $y = bx + a$ cuts the y axis at the point $y = a$. For example, the line $y = 2x - 1$ cuts the x axis at

$$x = -\frac{-1}{2} = \frac{1}{2}$$

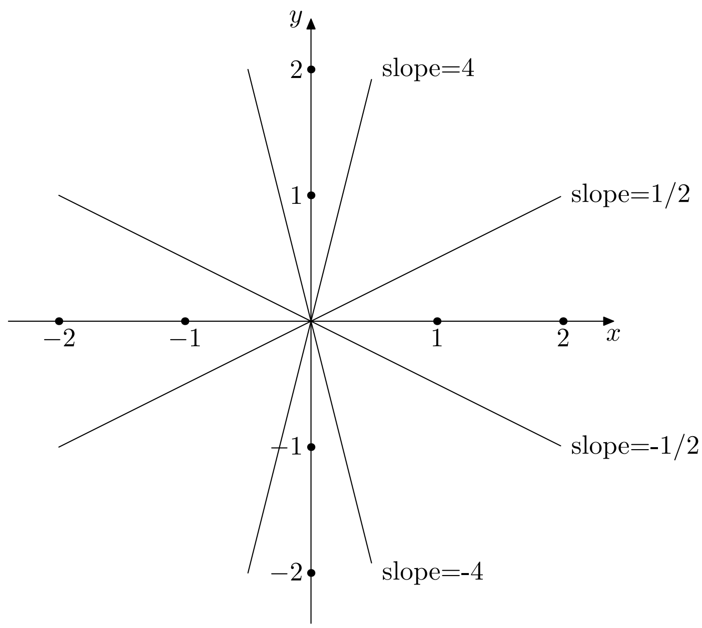

Figure 3.3: Slopes of Straight Lines

and the y axis at

$$y = -1$$

(see Figure 3.2).

We have seen how to work out the slope and intercepts of a straight line from its equation. In general we can work out anything we want to know about a line from its equation. So if a straight line is described to us in some other way, for example its slope and y-intercept, it will often be useful to find its equation, as from that we can derive any other information needed. A common way lines occur in experimental science is as an approximate fit to a set of data (either a fit by eye or a computer generated fit) and if we have such a fit, plotted on graph paper for example, we can find some points on the line. An important point to remember is that if we know **two** points on the line we know the whole line.

Example

Suppose we know two points (x_1, y_1) and (x_2, y_2) on the line. Then

$$b = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1}.$$

To find a take either of the points (x_1, y_1) or (x_2, y_2) and put its coordinates into the equation of the line. For example,

$$y_1 = bx_1 + a$$

which gives

$$a = -bx_1 + y_1$$

from which we can find a, since we know b already. Thus we proceed in two steps:

1. Use the two points to find the slope of the line.

2. Substitute one of the points into the equation for the line.

Take the two points $(x_1, y_1) = (-1, -3)$ and $(x_2, y_2) = (1, 1)$ and compute the equation of the straight line through these points. The formula for the slope is

$$b = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-3)}{1-(-1)} = \frac{4}{2} = 2$$

and substituting the first point into the equation for the line

$$\begin{aligned} y_1 &= bx_1 + a \\ -3 &= 2(-1) + a, \end{aligned}$$

so $a = -1$ and the equation of the line is

$$y = 2x - 1.$$

Of course, we saw earlier that a is the y intercept of the line, which may be an easier way of finding it. However we may not always be able to read off the y intercept from every graph whereas the method above always works.

Example

Another common situation is where we are given a point (x_1, y_1) on the line and the slope b of the line. To find the equation of the line we use the fact that if (x, y) is *any* point on the line then

$$\text{slope} = b = \frac{y - y_1}{x - x_1}$$

so

$$b(x - x_1) = y - y_1$$

or

$$y = bx - bx_1 + y_1.$$

The procedure here is to use the information to write down an equation for the slope of the line and to rearrange this equation into the standard form for the equation of a straight line.

Find the equation of the line through the point $(2, -3)$ having slope -2. The formula for the slope is

$$\text{slope} = b = \frac{y - y_1}{x - x_1}$$

and substituting values for b, x_1 and y_1 gives

$$-2 = \frac{y - (-3)}{x - 2}$$

from which it follows that

$$-2(x - 2) = y + 3$$

or

$$\begin{aligned} y &= -2x + 4 + (-3) \\ y &= -2x + 1 \end{aligned}$$

which is the equation of the line.

Example

A person on a strict diet plans to breakfast on cornflakes, milk, and a boiled egg. After allowing for the egg, his diet permits an additional 300 calories for this meal. If 1 ounce of milk contains 20 calories and 1 ounce of cornflakes (plus sugar) contains 160 calories, what is the relation between the number of ounces of milk and cornflakes that can be consumed?

Let x be the number of ounces of milk to be consumed, and y be the number of ounces of cornflakes. Then

$$20x + 160y = 300,$$

or, on dividing through by 160 and rearranging

$$y = -\frac{1}{8}x + \frac{15}{8},$$

which is the relation between the number of ounces of milk and cornflakes that can be consumed. Note that since we cannot have negative quantities consumed, we require that $x \geq 0$ and $y \geq 0$. The graph of $y = -1/8x + 15/8$ is shown in Figure 3.4.

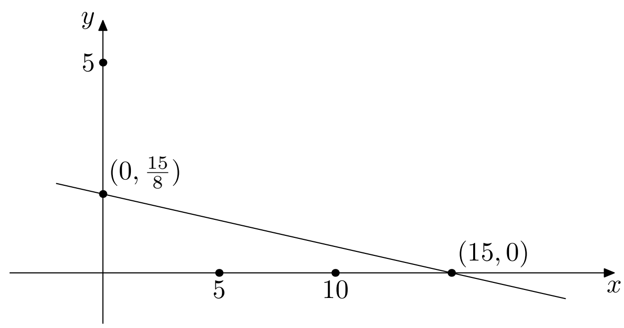

Figure 3.4: Graph of $y = -1/8x + 15/8$

If $x = 0$, then $y = 15/8$; and if $y = 0$, then $x = 15$. Although the straight line extends indefinitely, we are only interested in points on the line between $(0, 15/8)$ and $(15, 0)$. Any point on the line between these two points gives a solution to our problem (e.g. if $x = 5$, then $y = 5/4$; so one possibility is to consume 5 ounces of milk and $1\frac{1}{4}$ ounces of cornflakes).

3.2 Translations

Here **translation** means moving or shifting the graphs of functions.

Vertical translations

Consider an equation

$$y = f(x) \tag{3.1}$$

and compare this to the equation

$$y = f(x) + d. \tag{3.2}$$

For a given value of x the corresponding value of y in equation 3.2 is d greater than the value of y in equation 3.1.

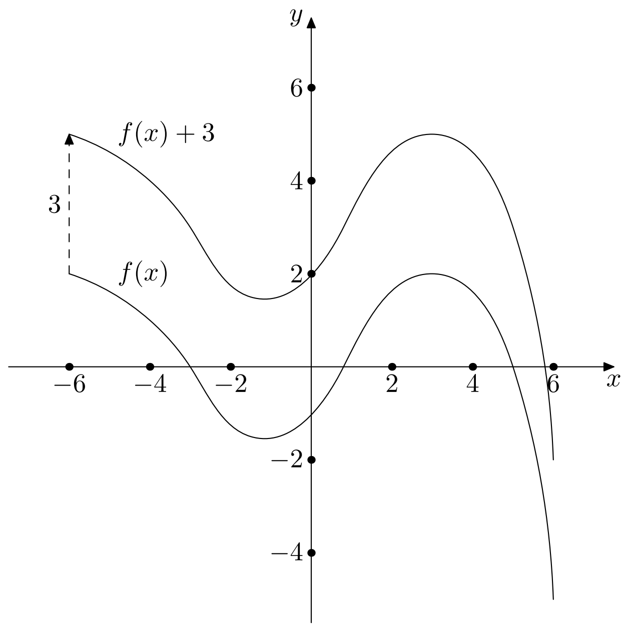

Figure 3.5: Vertical translation of a function.

Changing $f(x)$ to $f(x)+d$ shifts the graph of the function upwards a distance d.

(See Figure 3.5.)

To shift the graph downwards just take d negative.

Horizontal translations

Replacing x by $x+l$ in equation 3.1 gives

$$y = f(x+l). \tag{3.3}$$

This shifts the graph of $f(x)$ a distance l to the left, since the value of y in equation 3.3 corresponding to an x value is the same as the value of y in equation 3.1 corresponding to an x value larger by l.

Changing x to $x+l$ shifts the graph to the left a distance l.

(See Figure 3.6.)

To shift to the right take l negative.

3.3 Quadratic Functions

A function of the form

$$y = ax^2 + bx + c,$$

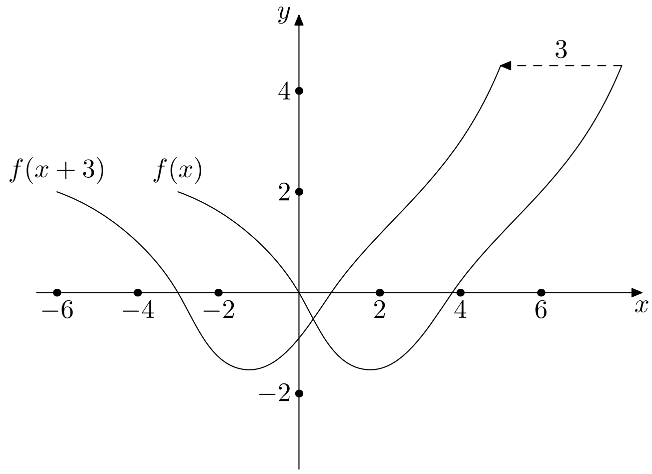

Figure 3.6: Horizontal translation of a function.

where a, b and c are any numbers and the highest power of x occurring is x^2, is called a **quadratic function**. We can take $a \neq 0$, since otherwise the function is **linear**. The graph of a general quadratic function can be obtained from the special case $y = ax^2$ by translations.

Special case $y = ax^2$

Here $b = 0$ and $c = 0$.

(i) $a > 0$. Then $y = 0$ when $x = 0$ and $y > 0$ when $x \neq 0$ (since all squares of non-zero numbers are positive). So the minimum value of y is 0 and occurs at $x = 0$. The graph is **concave up** with lowest point $(0, 0)$.

(ii) $a < 0$. Then $y = 0$ when $x = 0$ and $y < 0$ when $x \neq 0$. So the maximum value of y is 0 and occurs at $x = 0$. The graph is **concave down** with highest point $(0, 0)$.

Some graphs are shown in Figure 3.7.

General case $y = ax^2 + bx + c$

Start with $y = ax^2$. Choose $l = b/2a$ and shift left by l:

$$\begin{aligned} y &= a\left(x + \frac{b}{2a}\right)^2 \\ &= a\left(x^2 + 2\frac{b}{2a}x + \left(\frac{b}{2a}\right)^2\right) \\ &= ax^2 + bx + \frac{b^2}{4a}. \end{aligned}$$

This matches the coefficients of x^2 and x but the constant term is wrong. The constant term can be changed by a vertical translation.

$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \tag{3.4}$$

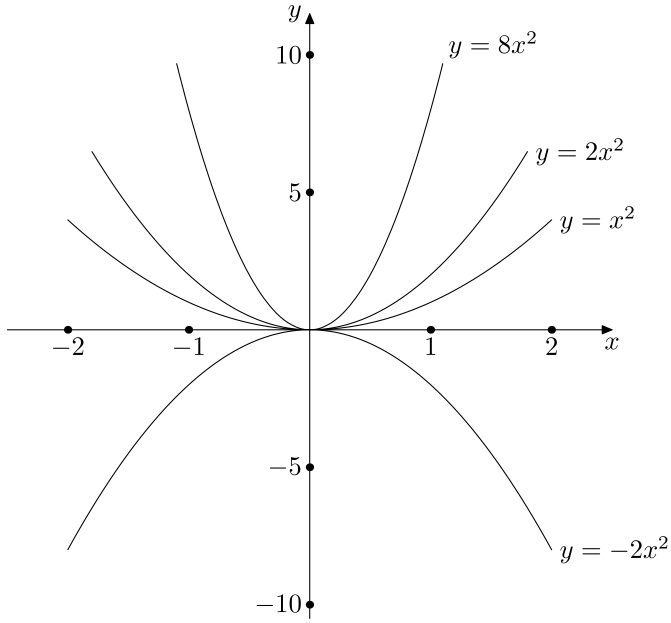

Figure 3.7: Graphs of the form $y = ax^2$.

$$= \quad ax^2 + bx + c.$$

The graph of $y = ax^2 + bx + c$ is obtained by translating the graph of $y = ax^2$ left by $b/2a$ and down by $(b^2/4a) - c$.

This process is called **completing the square**.

Axis of symmetry

The graphs of the functions $y = ax^2$ are symmetric about the y-axis, that is the line $x = 0$. Since the general quadratic $y = ax^2 + bx + x$ is obtained from $y = ax^2$ by shifting left by an amount $b/2a$, its graph is symmetric about the line $x = -b/2a$. The maximum or minimum value must be on this line.

(i) $a > 0$. The minimum value of y is $c - (b^2/4a)$ and occurs at $x = -b/2a$. The graph is concave up.

(ii) $a < 0$. The maximum value of y is $c - (b^2/4a)$ and occurs at $x = -b/2a$. The graph is concave down.

Some graphs are shown in Figure 3.8.

Intercepts

The graph cuts the y-axis where $x = 0$. So the y-intercept is

$$y = a \times 0^2 + b \times 0 + c = c.$$

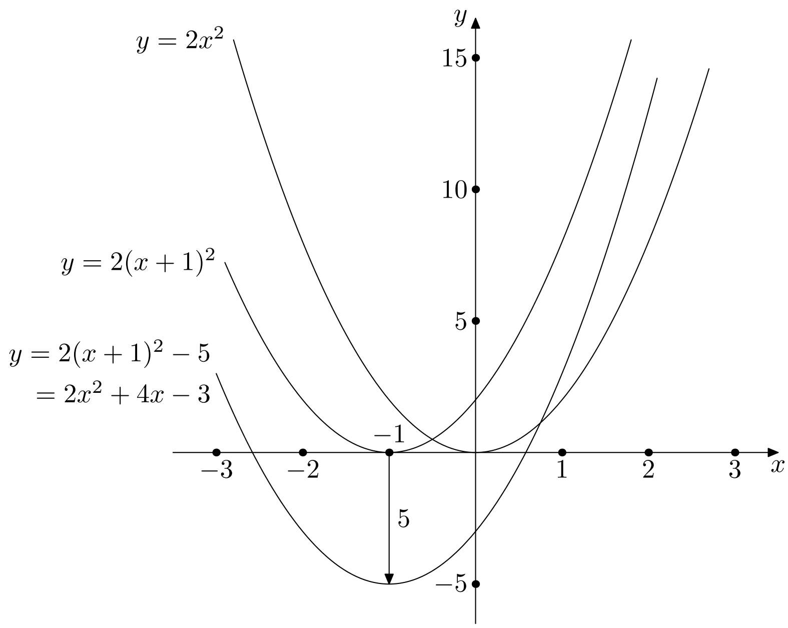

Figure 3.8: Some quadratics

The x-intercepts, that is the roots of the quadratic equation, are given by the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three cases to consider:

- (i) $b^2 - 4ac < 0$. Here the quantity inside the square root is negative and there are no roots. This means that the graph lies entirely above are below the x-axis.
- (ii) $b^2 - 4ac = 0$. In this case there is one root $x = -b/2a$. The graph just touches the x-axis and the root lies on the axis of symmetry of the quadratic.
- (iii) $b^2 - 4ac > 0$. In this case there are two roots.

Figure 3.9 shows the roots of some quadratics.

Sketching the graph of a quadratic

The procedure to sketch the graph of a quadratic function

$$y = ax^2 + bx + c$$

is as follows:

1. Look at the sign of the leading term ax^2 to determine whether the graph is concave up or concave down.

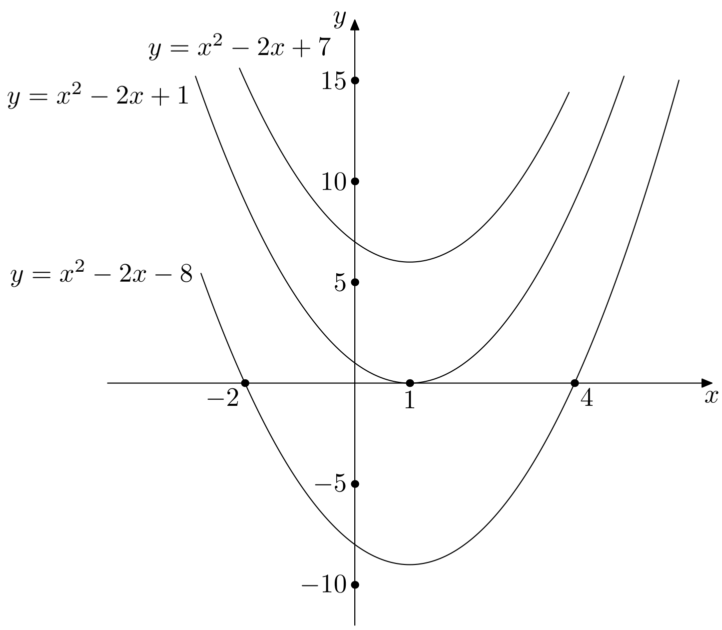

Figure 3.9: Roots of some quadratics

2. Determine the axis of symmetry $x = -b/2a$ and mark it as a vertical line on the graph. Note that the graph must be symmetric about this line and that the maximum or minimum value also occurs on this line.

3. Determine the y intercept $(= c)$.

4. Find the roots
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
and (if there are any roots) mark them on graph.

Example

A farmer has 200m of fencing with which to enclose a rectangular field. On one side of the field he can make use of a fence which already exists. What is the maximum area he can enclose?

Let the sides be of length x and y as shown in Figure 3.10, the side of length x being parallel to the already existing fence. Let the enclosed area be A. Then the total length of fencing is
$$x + 2y = 200$$
and the enclosed area is
$$A = xy.$$
Solving the first equation for y gives
$$y = 100 - \frac{x}{2},$$

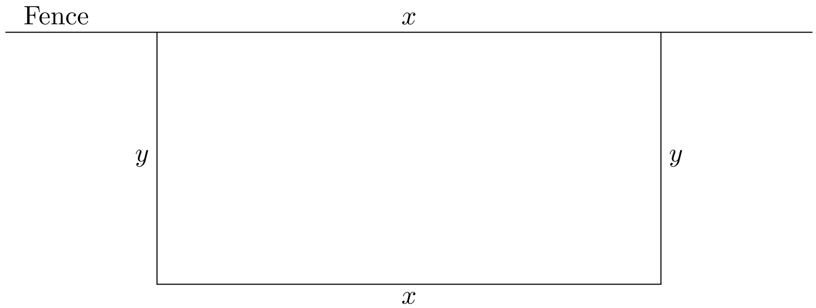

Figure 3.10:

and substituting into the formula for the area gives

$$\begin{aligned} A &= x(100 - \frac{x}{2}) \\ &= -\frac{x^2}{2} + 100x \end{aligned}$$

This is a quadratic equation for the area with $a = -1/2$, $b = 100$ and $c = 0$. Using the expression given earlier for the maximum of quadratic with $a < 0$ we find

$$A_{\text{max}} = c - \frac{b^2}{4a} = -\frac{10000}{-2} = 5000 \quad \text{square metres,}$$

which occurs when

$$x = \frac{-b}{2a} = 100 \quad \text{metres.}$$

3.4 Power Functions

If m is a positive whole number, the **power function**

$$y = x^m$$

makes y the result of multiplying m x's together. The number m is called the **exponent** of x. Here are some examples:

$$2^3 = 2 \times 2 \times 2 = 8, \quad 3^2 = 3 \times 3 = 9, \quad (1\tfrac{1}{2})^3 = 1\tfrac{1}{2} \times 1\tfrac{1}{2} \times 1\tfrac{1}{2} = 3\tfrac{3}{8},$$
$$(-2\tfrac{1}{3})^2 = (-2\tfrac{1}{3}) \times (-2\tfrac{1}{3}) = 5\tfrac{4}{9}, \quad 99^1 = 99.$$

Rules for combining powers

If we multiply m x's then multiply n x's then multiply the two answers, what we have done is multiply $m + n$ x's together. So

$$\boxed{x^m x^n = x^{m+n}.}$$

If we multiply m x's then multiply n copies of the answer, what we have done is multiply mn x's together. So

$$\boxed{(x^m)^n = x^{mn}.}$$

Exponents that are not positive whole numbers

We established the above two rules when m and n were positive whole numbers. We now use these rules to give meaning to power functions x^p when p is not a positive whole number. The rules are still true for these new power functions even when the exponents are not positive whole numbers.

The power function x^0: $1 \times x = x = x^1 = x^{0+1} = x^0 \times x^1 = x^0 \times x$. Hence

$$\boxed{x^0 = 1.}$$

The power function x^{-1}: $x^{-1}x = x^{-1}x^1 = x^{-1+1} = x^0 = 1$. Hence

$$\boxed{x^{-1} = \frac{1}{x}.}$$

Other negative exponents: $x^{-p} = x^{p\times(-1)} = (x^p)^{-1} = \dfrac{1}{x^p}$.

Special fractional exponents: $\left(x^{\frac{1}{p}}\right)^p = x^{\frac{1}{p}p} = x^1 = x$. Hence

$$\boxed{x^{1/p} = \sqrt[p]{x}.}$$

General fractional exponents: $x^{q/p} = x^{q\frac{1}{p}} = (x^q)^{\frac{1}{p}} = \sqrt[p]{x^q}$.

Example

Find the value of $y = x^{-3/2}$ when $x = 4$.

$$y = x^{-3/2} = \frac{1}{x^{3/2}} = \frac{1}{\sqrt{x^3}}.$$

Now putting $x = 4$ gives

$$y = \frac{1}{\sqrt{4^3}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

Warning

When x is negative there can be trouble with power functions whose exponents are not positive integers. For example, $(-3)^{1/2}$ should mean the square root of -3. But -3 has no square root since squares can't be negative. For this and similar reasons we shall always stick to positive values of x for such power functions.

Graphs of power functions with positive exponents

Here is a table of values for some power functions. Check the numbers on your calculator.

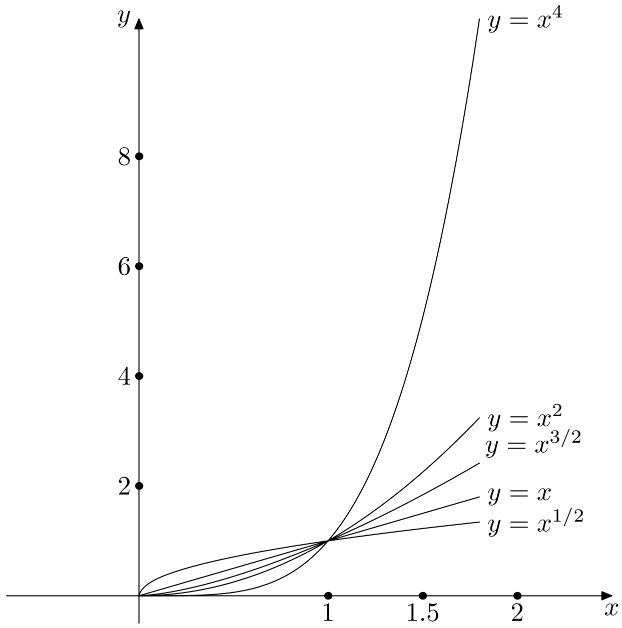

Figure 3.11: Graphs of power function

x	0	1/4	1/2	1	2	4
x^0	1	1.0000	1.0000	1.0000	1.0000	1.0000
$x^{1/2}$	0	0.5000	0.7071	1.0000	1.4142	2.0000
x^1	0	0.2500	0.5000	1.0000	2.0000	4.0000
$x^{3/2}$	0	0.1250	0.3536	1.0000	2.8284	8.0000
x^2	0	0.0625	0.2500	1.0000	4.0000	16.0000
x^4	0	0.0039	0.0625	1.0000	16.0000	256.0000

Figure 3.11 shows the graphs of these functions.

Some points to notice:

1. All of these functions (except x^0) increase with x.
2. All of the graphs pass through the point $(1, 1)$.
3. For $p > 1$ the steepness of the graph increases as x increases, while for $p < 1$ (but > 0) the steepness decreases as x increases.
4. For $x > 1$ the graphs for larger p lie above those with smaller p. For $0 < x < 1$ the graphs for larger p lie below those with smaller p.

Graphs of power functions with negative exponent

Here is a table of values for the power functions whose exponents are the negatives of the previous set. Since $x^{-p} = 1/x^p$ this table is easily computed from the previous one. Check the numbers.

x	1/4	1/2	1	2	4
$x^{-1/3}$	1.5874	1.2600	1.0000	0.7937	0.6300
$x^{-1/2}$	2.0000	1.4142	1.0000	0.7071	0.5000
x^{-1}	4.0000	2.0000	1.0000	0.5000	0.2500
x^{-2}	16.0000	4.0000	1.0000	0.2500	0.0625
x^{-4}	256.0000	16.0000	1.0000	0.0625	0.0039

Figure 3.12 shows the graphs of these functions.

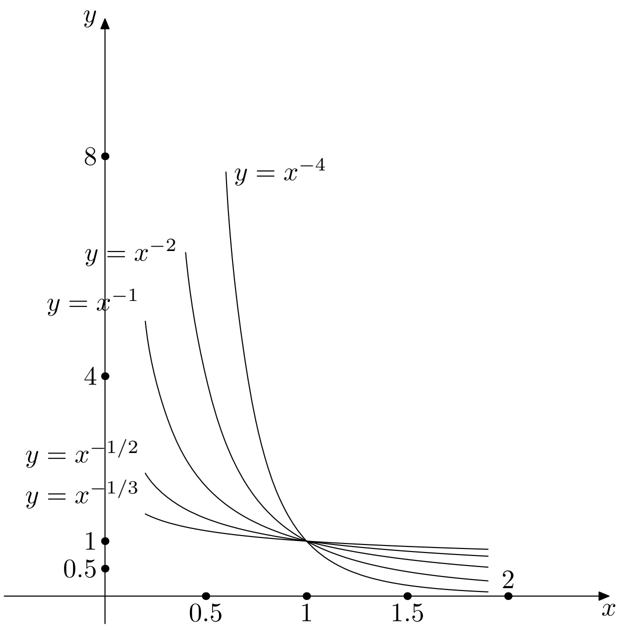

Figure 3.12: Graphs of power function with negative exponents.

Note that

1. These functions are not defined at $x = 0$.
2. All of these functions decrease with x.
3. All of the graphs again go through point $(1, 1)$.
4. As p increases the value of x^{-p} decreases for $x > 1$ and increases for $x < 1$.
5. All the graphs approach the x-axis as x becomes large.

3.5 Exponential Functions

In the section 4 we looked at the power functions

$$y = x^p$$

with fixed exponent and variable base. Now let's look at the **exponential** functions

$$y = a^x$$

with fixed base and variable exponent. We must take $a > 0$ here or there will be trouble with fractional powers of negative numbers. These functions are defined for all values of x, however.

Here is a table of values of some exponential functions. You can check the numbers on your calculator.

x	-10	-1	0.1	0	0.1	1	10
$(\frac{1}{2})^x$	1024	2	1.071	1	0.933	0.500	9.766×10^{-4}
$(\frac{3}{4})^x$	17.768	1.333	1.029	1	0.972	3/4	0.0563
1^x	1	1	1	1	1	1	1
2^x	9.766×10^{-4}	1/2	0.933	1	1.071	2	1024
5^x	1.024×10^{-7}	1/5	0.851	1	1.1746	5	9.766×10^{6}

Graphs of these functions are shown in Figure 3.13.

Note that

1. The functions with base > 1 increase with x — the larger the base the faster they increase.

2. The functions with base < 1 decrease with x — the smaller the base the faster they decrease.

3. All the graphs go through the point $(0, 1)$.

By thinking about what happens to the graphs as the base is gradually changed we see that there is some base $a > 0$ for which the graph has slope exactly 1 at the point $(0, 1)$; that is, the graph of a^x just touches the line $y = x + 1$ (which also has slope 1 and goes through $(0, 1)$). This special base is the irrational number

$$e = 2.71828182845\ldots.$$

The values of e^x for several values of x are tabulated below. (They can be found using the "exp" key of your calculator.)

x	-10	-1	0.1	0	0.1	1	10
e^x	4.540×10^{-5}	0.368	0.905	1	1.105	2.718	2.202×10^{4}

Sometimes the notation $y = \exp x$ is used (as on calculators) instead of $y = e^x$. Here "exp" is short for "exponential". The importance of the number e will become apparent when we study calculus , beginning in Chapter 6.

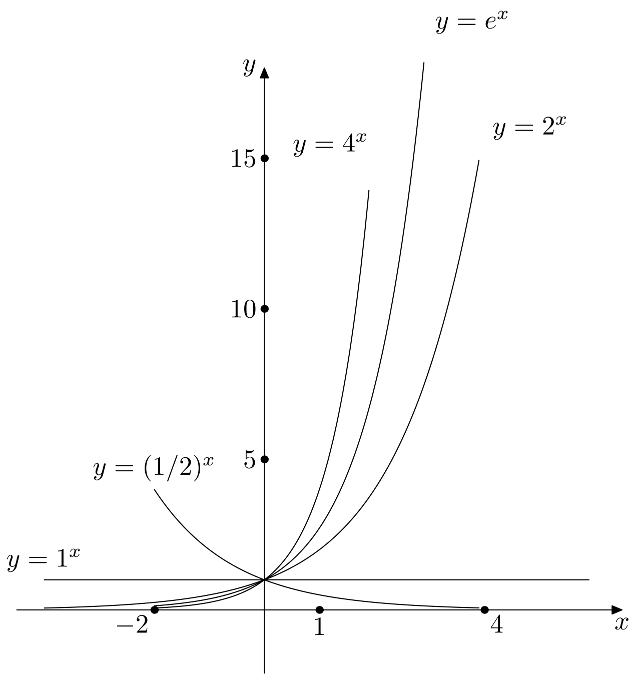

Figure 3.13: Graphs of exponentials.

3.6 Logarithmic Functions

If we solve the exponential function

$$y = a^x$$

for x the solution is given by the function

$$x = \log_a y$$

called the **logarithm of y to the base** a. The equivalence

$$y = a^x \quad \text{is equivalent to} \quad x = \log_a y$$

can be taken as the definition of the logarithm. Finding the logarithm of y to the base a is the same as finding the power to which a must be raised to give y. For example

$$64 = 4^3 \qquad \text{so} \qquad 3 = \log_4 64,$$

$$10000 = 10^4 \qquad \text{so} \qquad 4 = \log_{10} 10000$$

and

$$1/16 = 2^{-4} \qquad \text{so} \qquad -4 = \log_2 1/16.$$

The special notation "ln" is used for logarithms to the base e (it is derived from the words "natural logarithm"). So

$$y = e^x \qquad \text{is equivalent to} \qquad x = \ln y.$$

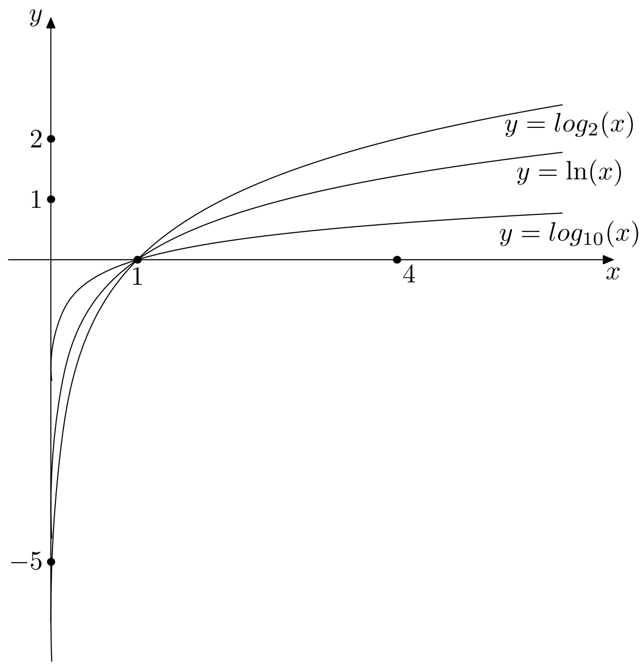

Figure 3.14: Graphs of logarithmic functions.

On calculators a plain "$\log x$" means "$\log_{10} x$" (but "$\log x$" is sometimes used to mean "$\ln x$" as well). In computer science and information theory logarithms to the base 2 are often used. Logarithms of negative numbers are NOT defined for any base.

Here is a table of values of some logarithmic functions.

x	1/10	1/2	1	2	10	100
$\log_2 x$	−3.3219	−1.0000	0.0000	1.0000	3.3219	6.6439
$\log_e x = \ln x$	−2.3026	−0.6931	0.0000	0.6931	2.3026	4.6052
$\log_{10} x = \log x$	−1.0000	−0.3010	0.0000	0.3010	1.0000	2.0000

Their graphs are shown in Figure 3.14.

Note that the graphs of logarithmic functions all go through the point $(1, 0)$.

Rules for logarithms of products and powers

Suppose $y_1 = a^{x_1}$ and $y_2 = a^{x_2}$. Then

$$y_1 y_2 = a^{x_1} a^{x_2} = a^{x_1 + x_2}$$

so

$$\log_a y_1 y_2 = x_1 + x_2 = \log_a y_1 + \log_a y_2.$$

$$\boxed{\log_a y_1 y_2 = \log_a y_1 + \log_a y_2}$$

Also if $y = a^x$ then $y^k = (a^x)^k = a^{kx}$ so

$$\log_a y^k = kx = k \log_a y.$$

$$\boxed{\log_a y^k = k \log_a y}$$

Taking $k = -1$ gives the useful rule

$$\boxed{\log_a(1/y) = -\log_a y.}$$

Here are some examples (in base e):

$$\ln 20 = \ln(2 \times 10) = \ln 2 + \ln 10 = 0.6931 + 2.3026 = 2.9957,$$

$$\ln(10^5) = 5 \times \ln 10 = 5 \times 2.3026 = 11.513.$$

3.7 Trigonometric Functions

This section reviews some basic facts about trigonometric functions. We begin with a quick summary of the foundations of trigonometry itself.

Angle measure

Mathematicians and other scientists usually measure angles in **radians** rather than degrees. The number of radians in an angle is the length of arc the angle cuts out of a circle of radius 1. Since the length of the complete circumference of the circle is $2\pi \times \text{radius} = 2\pi$,

$$\begin{aligned} 360\,\text{degrees} &= 2\pi\,\text{radians} \\ &= 6.283\ldots\ \text{radians}. \end{aligned}$$

Other useful conversions to know are

$$180^\circ = \pi\,\text{rad}, \quad 90^\circ = \tfrac{\pi}{2}\,\text{rad}, \quad 60^\circ = \tfrac{\pi}{3}\,\text{rad}, \quad 45^\circ = \tfrac{\pi}{4}\,\text{rad}, \quad 30^\circ = \tfrac{\pi}{6}\,\text{rad}.$$

From now on all angles will be measured in radians unless we explicitly state otherwise.

What are the trigonometric functions?

Suppose one angle of a right-angled triangle is x (radians!). Let the length of the side joining this angle to the right angle be b, the length of the hypoteneuse be c and the length of the remaining side be a. Then the functions **sin**, **cos** and **tan** are given by

$$\sin x = \frac{a}{c} \qquad \cos x = \frac{b}{c} \qquad \tan x = \frac{a}{b}$$

(see Figure 3.15).

Warning: When working out these functions on your calculator you must remember to put it into "rad" or "deg" mode depending on whether the angles are measured in radians or degrees. Think of the variable x as an angle, not a number.

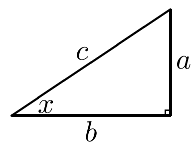

Figure 3.15: $\sin x = a/c$, $\cos x = b/c$, $\tan x = a/b$.

Here are some examples:

$$\tan(\pi/4) = 1,$$

$$\sin(\pi/6) = \frac{1}{2},$$

$$\sin(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}} = 0.7071\ldots,$$

$$\cos(\pi/6) = \frac{\sqrt{3}}{2} = 0.866\ldots, \quad \tan(\pi/6) = \frac{1}{\sqrt{3}} = 0.577\ldots,$$

$$\cos 0 = 1, \quad \sin 0 = \tan 0 = 0,$$

$$\sin(\pi/2) = 1, \quad \cos(\pi/2) = 0, \quad \tan(\pi/2) = \infty,$$

$$\sin(\pi/3) = \frac{\sqrt{3}}{2}, \quad \cos(\pi/3) = \frac{1}{2}, \quad \tan(\pi/3) = \sqrt{3} = 1.577\ldots.$$

Relations between trigonometric functions

By Pythagoras's theorem (discovered about 600 BC), the sides of the triangle in Figure 3.15 satisfy

$$a^2 + b^2 = c^2.$$

Divide both sides of the equation by c^2

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1,$$

rearrange

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

Thus

$$\boxed{\sin^2 x + \cos^2 x = 1.}$$

(Notice that $\sin^2 x$ means $\sin x \times \sin x$, and similarly for $\cos^2 x$.)

Also

$$\frac{a}{b} = \frac{a/c}{b/c}$$

so

$$\boxed{\tan x = \frac{\sin x}{\cos x}.}$$

Application

Standing at 20 meters from the halls of residence boiler house chimney we measure that the chimney top subtends an angle $51°32'$ above the horizontal. How high is the chimney?

Let the height be h meters. Then

$$\tan 51°32' = h/20$$

so

$$h = 20 \tan 51°32' = 20 \tan 51.53° = 20 \times 1.259 = 25.17\,\text{m}.$$

(Notice that since the angle is measured in degrees we need to use the "deg" mode of the calculator to work this out.)

Angles greater than a right angle

In a right-angled triangle the other two angles are both less than a right angle, so our definition of $\sin x$, $\cos x$ and $\tan x$ only works for $x \leq \frac{\pi}{2}$ (= one right angle).

We can also define these functions as follows. Let $\mathbf{v}$ be the line (of length 1) got by rotating the vector $(1, 0)$ anticlockwise through an angle θ about $\mathbf{0}$. Then

$$\begin{aligned} \cos\theta &= x\text{-coordinate of } \mathbf{v}, \\ \sin\theta &= y\text{-coordinate of } \mathbf{v}, \\ \tan\theta &= \frac{\sin\theta}{\cos\theta} = \frac{y\text{-coordinate of } \mathbf{v}}{x\text{-coordinate of } \mathbf{v}}. \end{aligned}$$

This is the same as the first definition when $0 \leq \theta \leq \pi/2$ (taking $c = 1$) but works for other values of θ too. For example,

$$\sin\frac{3\pi}{4} = 0.7071\ldots, \quad \cos\frac{3\pi}{4} = -0.7071\ldots, \quad \tan\frac{3\pi}{4} = -1,$$

$$\sin\frac{5\pi}{4} = -0.7071\ldots, \quad \cos\frac{5\pi}{4} = -0.7071\ldots, \quad \tan\frac{5\pi}{4} = 1.$$

When the line $\mathbf{v}$ is rotated anticlockwise through a right angle its new y-coordinate becomes the same as its previous x-coordinate and its new x-coordinate is minus its previous y-coordinate. So

$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta \quad \text{and} \quad \cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta.$$

The first of these equations (when we replace θ by x) tells us that the graph of $\cos x$ is the same as the graph of $\sin x$ shifted left by $\pi/2$.

Periodicity of trigonometric functions

Adding 2π radians to θ rotates the line $\mathbf{v}$ through a complete revolution about $\mathbf{0}$, leaving its x and y coordinates as they were. So

$$\cos(\theta + 2\pi) = \cos\theta \qquad \text{and} \qquad \sin(\theta + 2\pi) = \sin\theta.$$

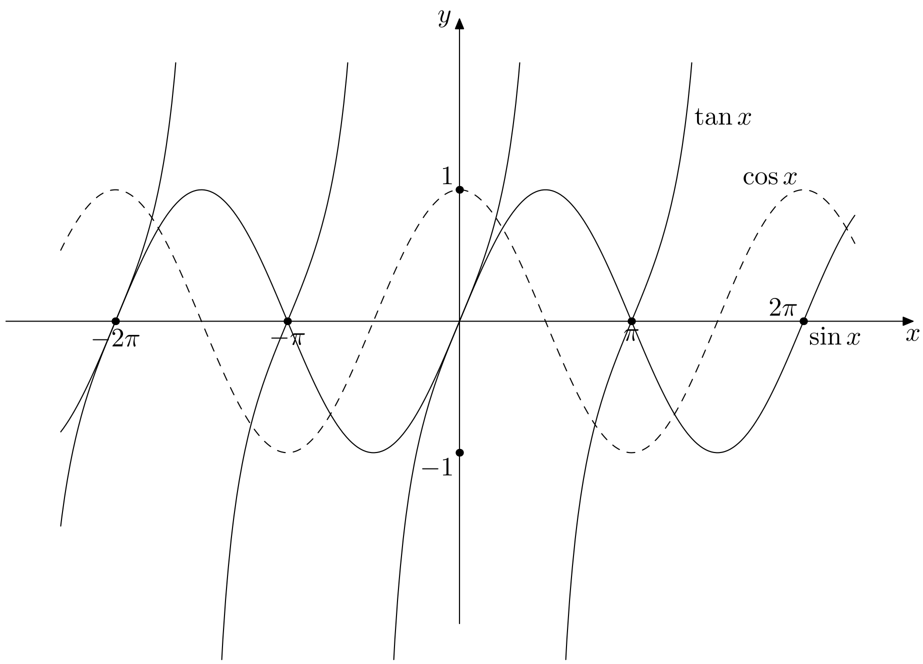

Figure 3.16: Trigonometric functions

Also

$$\tan(\theta + 2\pi) = \frac{\sin(\theta + 2\pi)}{\cos(\theta + 2\pi)} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

This means (replacing θ by x) that adding 2π to x does not change the value of $\sin x$, $\cos x$ or $\tan x$. We say that $\sin x$, $\cos x$ and $\tan x$ are **periodic functions** with **period** 2π. The periodicity of sin and cos makes them very useful for describing repetitive phenomena, which occur a great deal in biology as well as in all other sciences.

The graphs of $\sin x$, $\cos x$ and $\tan x$ are shown in Figure 3.16.

Amplitude, Frequency and Phase

A **periodic function** is a function $f(x)$ with

$$f(x + l) = f(x)$$

for some l. Such a function repeats itself at intervals of length l along the x-axis (because when it is shifted left by l it has the same values). The length l is called the **period** of the function. We have already seen that $\sin x$, $\cos x$ and $\tan x$ each have period 2π.

Using the sin function we can form the 3-parameter family of periodic functions

$$y = A\sin(\omega x + \phi).$$

The coefficient $A > 0$ is the **amplitude** of the function, $\omega > 0$ is the **angular frequency** and ϕ is the **phase**. We now look at the significance of these parameters and the effect they have on how to sketch the graph of the function.

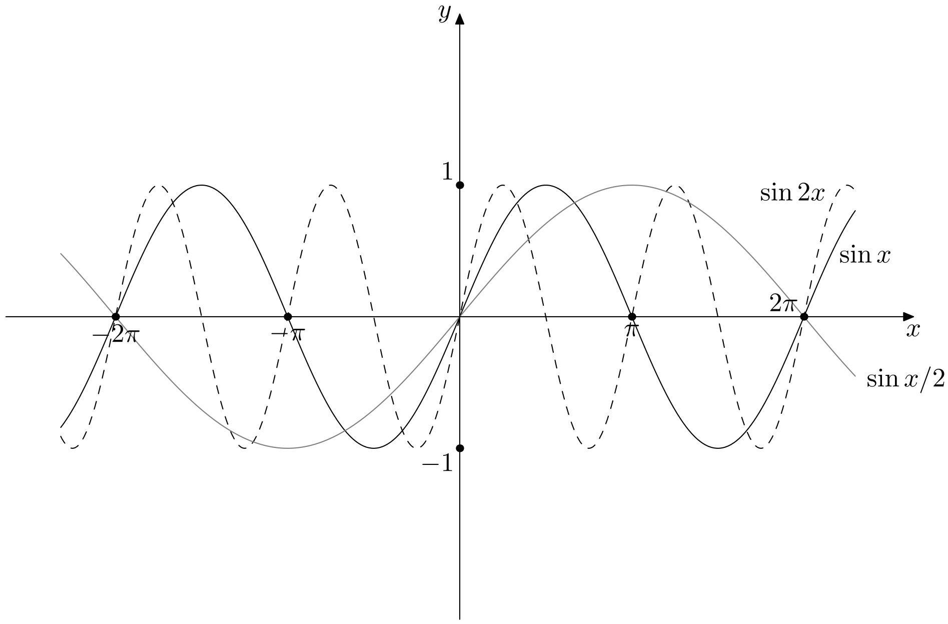

Figure 3.17: The functions $\sin(x)$, $\sin(2x)$, $\sin(x/2)$.

Angular frequency The angular frequency ω is related to the period of the function. As x runs from 0 to $2\pi/\omega$, ωx runs from 0 to 2π — a full cycle of the sin function — so the function $y = A\sin(\omega x + \phi)$ has period $2\pi/\omega$.

Let's start with the special case $A = 1$, $\phi = 0$. Then the function is

$$y = \sin(\omega x).$$

When $\omega < 1$ the period $2\pi/\omega$ is greater than 2π and this function looks like $\sin x$ but more spread out. When $\omega > 1$ the period $2\pi/\omega$ is less than 2π and the function looks like $\sin x$ but more bunched up. See Figure 3.17.

Phase Keep $A = 1$ and look at the effect of the phase ϕ. The function

$$y = \sin(\omega x + \phi)$$

Has the same graph as $\sin(\omega x)$ except that it is shifted left by an amount ϕ/ω. This is because

$$\omega(x + \frac{\phi}{\omega}) = \omega x + \phi.$$

Note that $\cos x$ is one of these functions with phase $\pi/2$, since we have seen that

$$\sin(x + \frac{\pi}{2}) = \cos x.$$

See Figure 3.18.

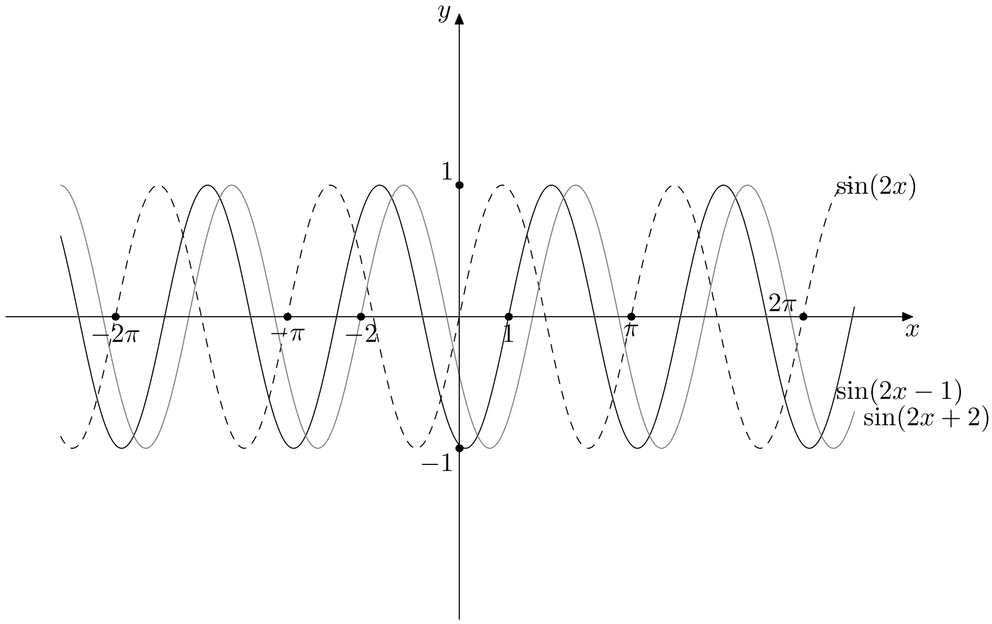

Figure 3.18: The functions $\sin(2x)$, $\sin(2x-1)$, $\sin(2x+2)$.

Amplitude The amplitude A is the height of the crests of the periodic function. The function

$$y = A\sin(\omega x + \phi)$$

is obtained from $y = \sin(\omega x + \phi)$ by multiplying by A, which stretches the function in the y direction if $A > 1$ and shrinks the function towards the x-axis if $A < 1$. See Figure 3.19.

Physical illustration

Suppose a string of length A has one end at the origin and a weight tied to the other end. The weight is swung round in a circle anticlockwise with angular velocity ω radians/sec., starting from the point where the string makes an angle ϕ with the x-axis. Let's work out the y-coordinate of the weight t seconds after starting. At that time the angle between the string and the x-axis is $\phi + \omega t$ radians, so the y-coordinate of the weight is $A\sin(\omega t + \phi)$ — one of our family of simple periodic functions.

In the equation

$$y = A\sin(\omega t + \phi)$$

all the parameters have simple physical meanings:

y	is the position of the shadow of the weight on the y-axis,
amplitude A	is the length of the string,
angular frequency ω	is the angular velocity of the weight, and
phase ϕ	is the starting angle.

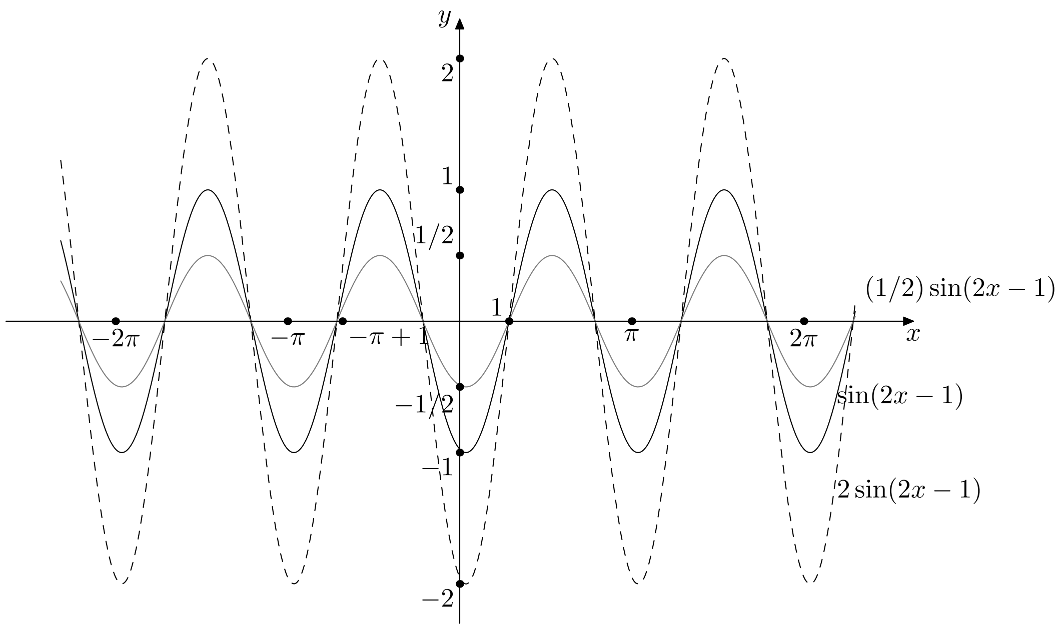

Figure 3.19: The functions $\sin(2x-1)$, $2\sin(2x-1)$, $(1/2)\sin(2x-1)$.

This illustrates that the functions $\sin x$ and $\cos x$ (sometimes called the "circular functions") are very closely related to motion in a circle.

Graphing trigonometric functions

The following steps allow any trigonmetric function of the form

$$y = A\sin(\omega x + \phi)$$

or

$$y = A\cos(\omega x + \phi)$$

to be graphed easily:

1. Start by drawing a sin or cos curve without scaling the x-axis .
2. Determine the period $2\pi/\omega$. Now the x-axis can be scaled since the period is the distance for which the graph goes through a complete cycle.
3. Move the y-axis to the right by ϕ/ω. This is the same as shifting the graph left by this amount.
4. Scale the y-axis so that the maximum height of the graph is A. Turn the graph upside down if A is negative.

Try this!

Other periodic functions

The functions

$$y = A\sin(\omega x + \phi)$$

are the simplest periodic functions. Much more complicated periodic functions can arise, for example in the readout from an electrocardiogram. An important branch of mathematics called *Fourier analysis* is concerned with how complicated periodic functions can be built up from these simple ones. The most important conclusion is that any function with period ω can be built up by adding together a collection of these simple periodic functions (perhaps infinitely many of them) whose angular frequencies are integer multiples of ω.

3.8 Combining Functions

So far we have looked at a number of basic functions: linear, quadratic, powers, exponential, logarithmic and trigonometric. There are many ways they can be combined to form the more complicated functions that occur in practice.

Multiples of a function

Given a function $f(x)$ and a number c we can form a new function $cf(x)$ by multiplying f by c. For example, if $f(x) = x^{\frac{1}{2}}$ and $c = \frac{1}{4}$, then

$$cf(x) = \frac{1}{4}x^{\frac{1}{2}}.$$

Multiplying by c stretches out the graph of $f(x)$ in the y direction if $|c| > 1$ and squashes it down if $|c| < 1$. When c is negative the graph gets reflected in the x-axis too. See Figure 3.20.

Sums of functions

Given two functions $f(x)$ and $g(x)$ we can form a new function $f(x) + g(x)$ by adding them. It is not hard to get the graph of $f(x)+g(x)$ from the graphs of $f(x)$ and $g(x)$: for each value of x add the y values given by f and g. See Figure 3.21.

Differences of functions

The difference $f(x) - g(x)$ of the functions $f(x)$ and $g(x)$ is similarly defined by subtracting the functions. See Figure 3.21.

Products of functions

From two functions $f(x)$ and $g(x)$ we can form their product $f(x)g(x)$ by multiplying them. When sketching the graph of a product it is useful to bear in mind the following facts abouts products of positive numbers:

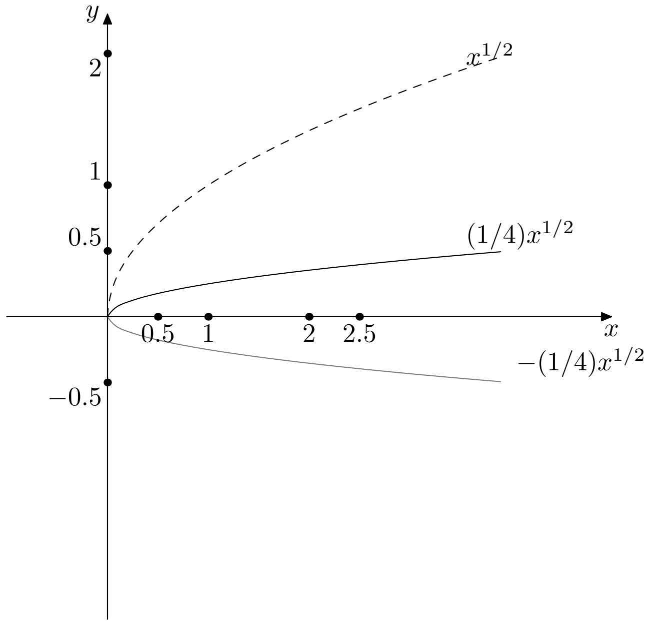

Figure 3.20: Graphs of $x^{1/2}$, $(1/4)x^{1/2}$ and $(-1/4)x^{1/2}$.

- the product of two numbers greater than 1 is greater than either of them,
- the product of a number greater than 1 and a number less than 1 lies between them,
- the product of two numbers less than 1 is smaller than either of them.

We also need to take into account the signs of $f(x)$ and $g(x)$.

Quotients

Similarly we can form the quotient $f(x)/g(x)$ of two functions $f(x)$ and $g(x)$ by dividing them. In this case there is a new kind of behaviour: since we can't divide by 0 we must pay special attention to the the points where $g(x) = 0$. At such points the function $f(x)/g(x)$ is not defined (at least when $f(x)$ is not also 0) and near such points the function values tend to plus or minus infinity. A good example of such behaviour is provided by the function $y = \tan(x)$, which can be understood as the quotient of $f(x) = \sin(x)$ and $g(x) = \cos(x)$ (See Figure 3.16).

Composite functions

We have looked at various ways of combining a pair of functions to produce a new function, in each case by performing an arithmetic operation on the y-values, or *output*, of the two functions. Thus we produce the function $f \pm g$ by taking the sum or difference of y-values of $f(x)$ and $g(x)$ for the same x. Similarly we produce functions $f \times g$ and $\frac{f}{g}$ by taking the product or quotient of the y-values for the same x. It would seem then that we combine functions in the same ways that we combine individual pairs of numbers, but there is another type of combination of functions that goes beyond arithmetic. If we recall that a function is a mathematical formula that has an independent variable, x (which we may call the *input* of

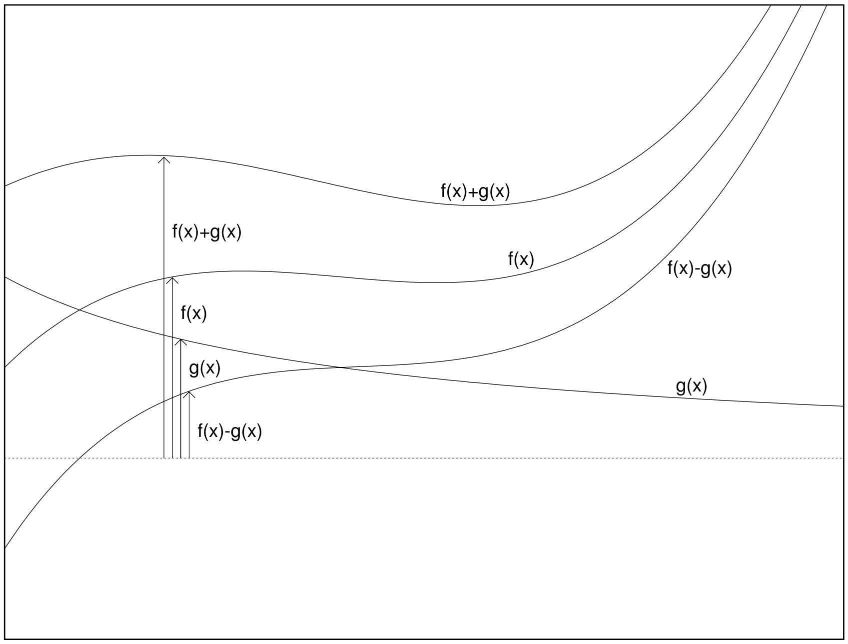

Figure 3.21: Sums and differences of functions

the function), and a dependent variable, y (which we may call the *output* of the function), then what happens if we **compose** two functions by taking the output of $f(x)$ to be the input of g? The output of this combined process then becomes $g(f(x))$, which is what we mean by the **composition** of f and g. In practice, the letter "x" in $g(x)$ becomes a space into which we put the whole formula for $f(x)$ in order to obtain a new formula for $g(f(x))$.

For example, the composite of the functions

$$g(x) = \sqrt{x} \qquad \text{and} \qquad f(x) = e^x$$

is

$$g(f(x)) = \sqrt{e^x}.$$

(Think of $g(\ \ ) = \sqrt{\quad}$, then just write the formula e^x in the empty space under the square root sign).

As another example, the composite of

$$y = x^2 \qquad \text{and} \qquad y = \sin x$$

is

$$y = (\sin x)^2.$$

We obtain the graph of this function by first drawing the graph of $\sin x$ and then for each x squaring the corresponding value of y. See Figure 3.22.

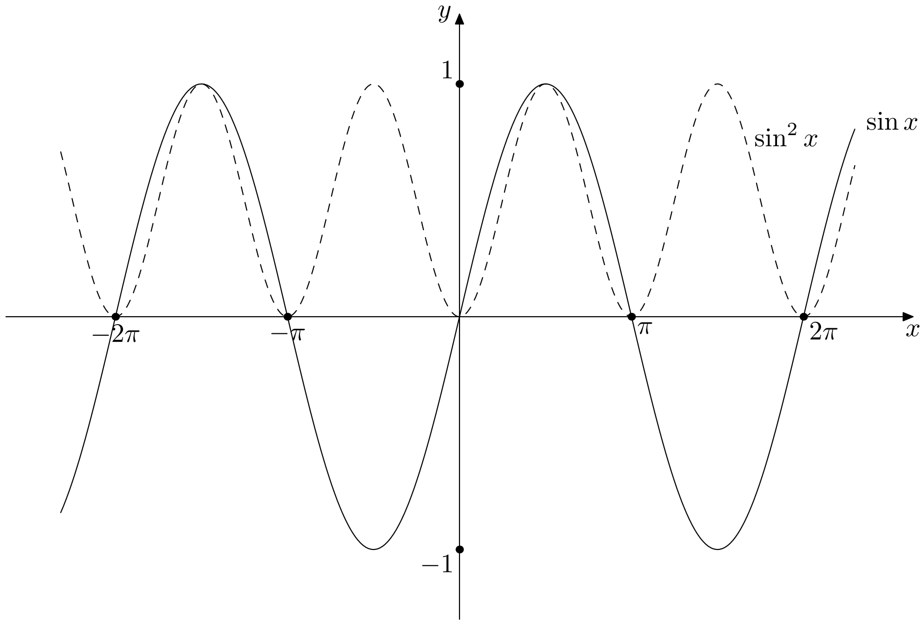

Figure 3.22: Graphs of $\sin(x)$ and $\sin^2(x)$.

Important warning

Notice that (like the order of multiplying matrices) it is important *in which order* we compose functions, since composing them in a different order usually gives a different result. For example, if we return to $g(x) = \sqrt{x}$ and $f(x) = e^x$, recall that $g(f(x)) = \sqrt{e^x}$, but

$$f(g(x)) = f(\sqrt{x}) = e^{\sqrt{x}}\ .$$

Clearly $f(g(x)) \neq g(f(x))$, since $f(g(1)) = e^1 = e$, whereas $g(f(1)) = \sqrt{e^1} = \sqrt{e} \neq e$.

Inverse functions

A special type of composition occurs when

$$f(g(x)) = x\ .$$

This can only happen if the equation $y = g(x)$ can be turned around to make x the subject, so that we get $x = f(y)$, then

$$f(g(x)) = f(y) = x\ .$$

For example, consider the linear function $y = 3x - 1$, which we can solve for x by writing

$$y + 1 = 3x \quad \text{and hence} \quad x = \frac{1}{3}(y+1)\ .$$

Now if $g(x) = 3x - 1$ we can write $f(y) = \frac{1}{3}(y+1)$ and

$$f(g(x)) = \frac{1}{3}(g(x)+1) = \frac{1}{3}(3x-1+1) = x\ .$$

But it is also true that

$$g(f(y)) = 3f(y) - 1 = 3(\frac{1}{3}(y+1)) - 1 = y + 1 - 1 = y \ .$$

In this case we say that f is the **inverse function** of g, and vice versa. We have already seen another important example of this with exponential and logarithm functions. Recall

$$y = e^x \quad \text{if and only if} \quad x = \ln(y) \ ,$$

so we can write

$$\ln(e^x) = x \quad \text{but also} \quad e^{\ln(y)} = y \ .$$

Thus $\ln(x)$ is said to be the *inverse function* of e^x, and vice versa. In the same way, any logarithm function $\log_a(x)$ is the inverse of the exponential function a^x, and vice versa.

3.9 Exercises

1. Sketch on the one graph the following straight lines:

(i) $y = 3x + 1$,

(ii) $y = 3x - 1$,

(iii) $y = -3x + 1$,

(iv) $y = -3x - 1$.

2. Find the equations of the straight lines satisfying:

(i) slope $1/5$ and y-intercept 7 ,

(ii) slope -8 and y-intercept 0 .

3. Find the equation of the straight line joining the points $(2, 7)$ and $(4, 3)$.

4. Find the equations of the straight lines in Figure 3.23.

5. If $3x + 2y = 6$, express y as a function of x.

6. A patient on a liquid diet has the choice of prune juice and orange juice to satisfy his daily requirement of thiamine, which is 1 mg. 1 oz of prune juice contains 0.05 mg of thiamine, and 1 oz of orange juice contains 0.08 mg of thiamine. Let his daily consumption be x oz of prune juice and y oz of orange juice. What is the relationship between x and y that exactly satisfies his thiamine requirement? Sketch this relationship on a pair of axes, labelling on the graph at least three points that exactly satisfy the thiamine requirement.

7. The respiratory rate y of a normal person, measured in breaths/minute, is a linear function of the partial pressure x of carbon dioxide in the lungs. When a normal person inhales air from a cylinder containing 2% CO_2, the partial pressure is found to be 40 torr and respiratory rate is 14 breaths/minute. When inhaling from a cylinder containing 6% CO_2, the partial pressure is 50 torr and the respiratory rate is 20 breaths/minute. (Note that 1 torr = pressure of 1 mm of mercury.)

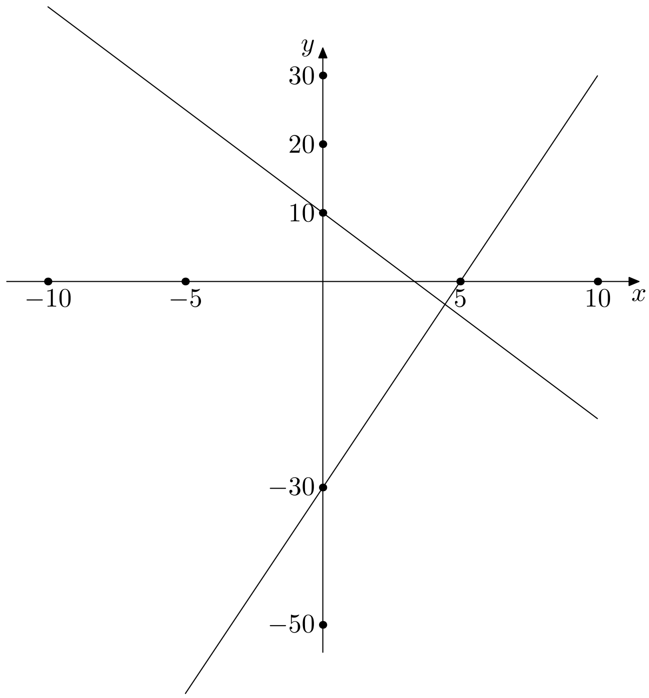

Figure 3.23: Diagram for Exercise 4.

(i) Draw a straight line ($y = bx + a$) through these two points, and from your graph find a and b.

(ii) What is the physical significance of b?

(iii) If the partial pressure of CO_2 is 45 torr, what is the respiratory rate?

(iv) As for (iii), but a partial pressure of 10 torr. Any comments on the result?

(v) Do you think a straight line relationship provides a good description of the link between respiratory rate and partial pressure of CO_2 in the lungs? Give reasons for your answer.

8. On one set of axes, sketch the graphs of $y = 2x^2$, $y = 2x^2 + 2$, and $y = -2x^2 + 2$.

9. Sketch the graphs of the following quadratic functions:

(i) $y = x^2 + x + 1$,

(ii) $y = -x^2 + 1$,

(iii) $y = 4x^2 - 4x + 1$.

In each case find the maximum or minimum values of the function and the roots of the function.

10. The yield of apples from each tree in an orchard is $x(500 - 5x)$ lb, where x is the density with which the trees are planted (i.e. the number of trees per acre). Find the value of x that makes the total yield per acre a maximum. What is this maximum yield?

11. Without using a calculator compute the following numbers: (i) 8^2, (ii) 8^1, (iii) 8^0, (iv) $8^{2/3}$, (v) 8^{-1}, (vi) 8^{-2}, (vii) $8^{-1/3}$.

12. Sketch on the one set of axes the graphs of

(i) $y = x^4$,

(ii) $y = x^{1/3}$,

(iii) $y = x^{-1}$.

13. On one set of axes, sketch the graphs of $y = 2^x$, $y = 3^x$, and $y = 3^{-x}$.

14. Radioactive substances decay according to a law of the form $y = Ab^{-t}$, where y is the amount of the substance remaining at time t, and A and b are constants with $A > 0$ and $b > 1$.

(i) What is the initial amount of substance?

(ii) Sketch the graph of y against time.

(iii) What happens to y as t becomes large?

15. The yield of crop that can be expected from the application of fertilizer at different rates can often be described by a curve of the form $y = a - b\,e^{-cx}$, where a, b, and c are constants. Suppose $a = 100$, $b = 50$, and $c = 0.1$.

(i) Sketch the graph of y for $0 \leq x \leq 20$.

(ii) What will happen to y as x becomes large?

(iii) Do you think the curve suitably represents the expected yield for very large rates of fertilizer application? Give reasons.

16. Assuming only that $\ln 2 = 0.693$ and $\ln 3 = 1.099$, calculate $\ln 6$, $\ln 64$, $\ln\left(\frac{1}{128}\right)$ and $\ln\left(\frac{8}{9}\right)$.

17. In a chemical solution let $[H^+]$ denote the concentration of hydrogen ions measured in moles/litre. The pH of the solution is defined as pH$= -\log\,[H^+]$.

(i) Find the pH of solutions in which

(a) $[H^+] = (3.1)(10^{-4})$,

(b) $[H^+] = (0.21)(10^{-9})$.

(ii) A solution of lemon juice has a pH of 3. What is its hydrogen ion concentration?

18. The Richter magnitude of an earthquake is defined as the base 10 logarithm of the strength of the quake (as measured by the output of a seismograph).

(i) How many times stronger than an earthquake of magnitude 3 is an earthquake of magnitude 6?

(ii) The Japanese earthquake of 1923 was 4 times stronger than the San Francisco earthquake of 1906. The Japanese quake had a magnitude of 8.9 on the Richter scale. What was the magnitude of the San Francisco earthquake?

(iii) In 1972 there was an earthquake in Nicaragua of magnitude 6.23. How many times stronger was the 1923 Japanese earthquake?

19. The elevation of the summit of a mountain measured from a base point 7 km away is $23°$. What is the height in meters of the summit above the base point.

20. Without using a calculator write down the values of the following trigonometric functions: (i) $\sin(\pi/6)$, (ii) $\tan(\pi/4)$, (iii) $\cos(2\pi/3)$, (iv) $\tan(7\pi/4)$, (v) $\cos\pi$, (vi) $\cos(3\pi/2)$, (vii) $\sin(-\pi/2)$, (viii) $\cos 0$.

21. Sketch the graphs of the following functions:

(i) $y = \cos^2(x)$,

(ii) $y = \frac{1}{x+1}$,

22. The relationship between the density of plants, S (plants/hectare), and yield of dry matter from a plant, y (cubic metres), is given by $y = \frac{1}{\alpha+\beta S}$, where α and β are constants. For Douglas-fir trees $\alpha = 0$ and $\beta = 0.0039$.

(i) Sketch the curve for $0 < S \leq 5000$.

(ii) What volume of wood would you expect to obtain per plant if

(a) $S = 2450$,

(b) $S = 1260$,

(c) $S = 5$ plants/hectare?

(iii) In order to obtain 0.5 cubic metres of wood from each tree, how many trees/hectare should be planted?

(iv) Would you expect the given relation between y and S to hold for very large S?, for very small S? Explain your answer.

Chapter 4

Solving Equations

4.1 Single Equations

There are many circumstances where we need to find a value of x that solves an equation like

$$f(x) = c.$$

This is easy to do from the graph of $f(x)$: just draw the horizontal line $y = c$ and note the value of x where it meets the graph. It is important to keep in mind the possibilities that the equation may have more than one solution (when the line meets the graph in more than one place) or it may not have any solution (when the line misses the graph altogether). For example, the equation

$$\cos x = 1$$

has an infinite number of solutions $x = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$, while the equation

$$\cos x = -3$$

has no solutions.

We now look at exact solutions of certain common types of equations.

Linear equations: $bx + a = c$

Do the same thing to both sides of the equation till you get x on its own.
Subtract a from both sides

$$bx = c - a$$

divide both sides by b

$$x = \frac{c-a}{b}.$$

Quadratic equations: $ax^2 + bx + c = 0$

Use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There may be two solutions, one solution or none.

(This time we didn't put a constant on the right hand side. If we had it could simply have been subtracted off again, the only effect being to change the value of c on the left hand side.)

Power functions: $ax^p = c$

Again, do the same thing to both sides to get x on its own.
Divide both sides by a

$$x^p = \frac{c}{a}$$

take p-th roots of both sides

$$x = \sqrt[p]{\frac{c}{a}}.$$

Exponentials: $ae^{kx} = c$

Same recipe.
Divide both sides by a

$$e^{kx} = \frac{c}{a}$$

Take logs of both sides

$$kx = \ln\left(\frac{c}{a}\right)$$

divide both sides by k

$$x = \frac{1}{k}\ln\left(\frac{c}{a}\right).$$

[It is useful to remember that $\ln y = x \Leftrightarrow x = e^y$.]

Trigonometric functions

An example is

$$A\sin(\omega x + \phi) = c.$$

This has no solutions if $c > A$ or $c < -A$ (since the sin function only takes values between -1 and 1). When $-A \leq c \leq A$ there are infinitely many solutions but they occur in a regular pattern, so are easily found once we have found one. (For example, the function has period $2\pi/\omega$ so if x is a solution then so is $x + 2\pi n/\omega$ for every whole number n.) One solution of the equation

$$y = \sin x$$

is given by the function

$$x = \arcsin y.$$

(The relation between the sin and arcsin functions is exactly like that between the exponential and logarithmic functions.) All scientific calculators have a key for the arcsin function, although it is also sometimes called the inverse sin and written $\sin^{-1}$.

The equation can now be solved like this:
divide both sides by A

$$\sin(\omega x + \phi) = \frac{c}{A}$$

apply the arcsin function to both sides

$$\omega x + \phi = \arcsin\left(\frac{c}{A}\right)$$

solve this linear equation for x

$$x = \frac{1}{\omega}\left(\arcsin\left(\frac{c}{A}\right) - \phi\right).$$

The inverse functions to cos and tan are 'arccos' and 'arctan' (sometimes called '$\cos^{-1}$' and '$\tan^{-1}$'). So we have

$$\begin{aligned} x = \arcsin y &\implies y = \sin x, \\ x = \arccos y &\implies y = \cos x, \\ x = \arctan y &\implies y = \tan x. \end{aligned}$$

4.2 Simultaneous Equations

In the previous section we looked at how to solve a single equation $f(x) = c$ for a single unknown x. Now let's solve *two* equations for *two* unknowns:

$$y = f(x) \tag{4.1}$$

$$y = g(x). \tag{4.2}$$

We want to find values of x and y that make both these equations true at once. This is the same thing as finding a point where the graphs $y = f(x)$ and $y = g(x)$ meet. Of course, the graphs may not meet in only one point — they may meet in many points or in no points at all. Finding all solutions of the equations is the same as finding all points where the graphs meet.

The thing to do with sets of equations that have more than one unknown is to combine the equations in such a way as to "eliminate" one of the unknowns, giving ourselves a smaller number of equations and unknowns to deal with. For example, if (4.1) and (4.2) are both true then

$$f(x) = g(x); \tag{4.3}$$

we have merged the two equations into one and eliminated the variable y. Equation (4.3) has only one unknown, so can be solved to find x, and then this value of x can be substituted into either (4.1) or (4.2) to give y.

Example

$$y = 2x^2 + x + 1 \tag{4.4}$$

$$y = x^2 + 2x + 3. \tag{4.5}$$

Eliminating y gives

$$2x^2 + x + 1 = x^2 + 2x + 3,$$

which simplifies to

$$x^2 - x - 2 = 0.$$

Now the quadratic equation formula gives

$$\begin{aligned} x &= \frac{1 \pm \sqrt{1+8}}{2} \\ &= \frac{1+\sqrt{9}}{2} \text{ or } \frac{1-\sqrt{9}}{2} \\ &= 2 \text{ or } -1. \end{aligned}$$

Substituting these values in (4.4) gives

$$y = 2 \times 2^2 + 2 + 1 = 11$$

or

$$y = 2 \times (-1)^2 + (-1) + 1 = 2.$$

So there are two points of intersection (x, y) of the graphs (4.4) and (4.5): $(x, y) = (2, 11)$ and $(x, y) = (-1, 2)$. You should substitute both values in (4.5) as a check on the correctness of the solutions.

Pairs of linear equations

From now on we shall stick to the easiest case of simultaneous equations when the functions f and g in (4.1) and (4.2) are both linear (so their graphs are straight lines). Suppose the equations are

$$y = ax + b \tag{4.6}$$
$$y = cx + d. \tag{4.7}$$

Eliminating y gives

$$ax + b = cx + d$$

which simplifies to

$$(a - c)x = d - b.$$

If $a \neq c$ we can divide both sides by $a - c$ to get

$$x = \frac{d-b}{a-c},$$

and substituting this value of x into the first equation gives

$$y = a\left(\frac{d-b}{a-c}\right) + b = \frac{ad-bc}{a-c}.$$

This method doesn't work when $a = c$, but then the lines have the same slope so are parallel and do not meet.

Linear equations and matrices

Suppose we have two linear equations in the form

$$\begin{aligned} ax + by &= e \\ cx + dy &= f. \end{aligned}$$

(The main difference between these and (6), (7) is that we have put the constant terms by themselves on one side instead of y by itself on one side. Also we allow y to have coefficients different from 1.) These equations can be written in matrix form as

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$

where $\mathbf{A}$ is the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and $\mathbf{x}$ and $\mathbf{y}$ are the vectors

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

If $\mathbf{A}$ has an inverse we can multiply both sides by it to get the solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}.$$

(Knowing the vector $\mathbf{x}$ is the same as knowing the numbers x and y.) Remember that $\mathbf{A}$ always has an inverse provided that $\det \mathbf{A} = ad - bc \neq 0$.

As an example, suppose we wish to find the point of intersection of the lines $y = 2x + 3$ and $y = x + 1$. Rewrite the equations as

$$\begin{aligned} 2x - y &= -3 \\ x - y &= -1, \end{aligned}$$

which have the matrix form

$$\begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}.$$

The inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$$

is

$$\mathbf{A}^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

and the point of intersection is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\mathbf{y} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix},$$

that is

$$x = -2, \quad y = -1.$$

Check this answer by putting these values of x and y into the original equations. (*Never* believe you have found the solution until you've tried it in the equations!)

4.3 Exercises

Solve the following equations:

1. $2x + 3 = 5$.

2. $x^2 + 3x - 2 = 0$.

3. $x^{3/2} = 1.5$.

4. $-3e^{2x} = -6$.

5. $2\sin 3x = 1$.

6. Find the point of intersection of the straight lines

$$y = 3x + 6$$

and

$$y = x - 5.$$

7. Find the points of intersection of the straight line

$$y = -2x + 2$$

with the quadratic

$$y = x^2 - 6x + 6.$$

Chapter 5

Differentiation

5.1 Definition of the Derivative

Differential calculus is the study of the changes that occur in one quantity when other quantities on which it depends change. For instance, the change in the growth of a culture of bacteria with each additional hour, or the change in crop yield which occurs with each additional kg of added fertilizer.

More particularly, we shall be interested in an instantaneous rate of change. For example, suppose a car is traveling from a point A to a point B. Then the average rate of change of distance is:

$$\frac{\text{the distance between } A \text{ and } B}{\text{time taken to get from } A \text{ to } B}.$$

This quantity is more generally called the average speed of the car. However, if the car hits a wall on the way, then the average speed of the car tells us nothing about the force of the impact. What we need to know is the speed of the car at the instant the car hits the wall, i.e. we need to know the instantaneous rate of change of distance.

Suppose $y = f(x)$ is a function of x, and suppose we want to find the instantaneous rate of change of y with respect to x at a particular point x_0. We take a point $x_0 + \Delta x$ that is nearby to x. (Here the symbol Δ is the Greek letter delta, and is used to indicate a small change. Hence Δx means a small change in the quantity x.) Then the average rate of change of y with respect to x over the interval from x_0 to $x_0 + \Delta x$ is given by

$$\frac{(\text{change in } y)}{(\text{change in } x)} = \text{slope of the straight line from} (x_0, f(x_0)) \text{to} (x_0 + \Delta x,\ f(x_0 + \Delta x)).$$

Therefore the average rate of change of y with respect to x is

$$\frac{f(x_0 + \Delta x) - f(x_0)}{(x_0 + \Delta x) - x_0} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

For example, suppose $y = f(x) = x^2$, and that the particular point is $x_0 = 2$. Then the average rate of change of y with respect to x over the interval from 2 to $2 + \Delta x$ is given by

$$\begin{aligned}
\frac{(2 + \Delta x)^2 - 4}{(2 + \Delta x) - 2} &= \frac{4 + 4\Delta x + (\Delta x)^2 - 4}{\Delta x} \\
&= \frac{4\Delta x + (\Delta x)^2}{\Delta x} \\
&= 4 + \Delta x.
\end{aligned}$$

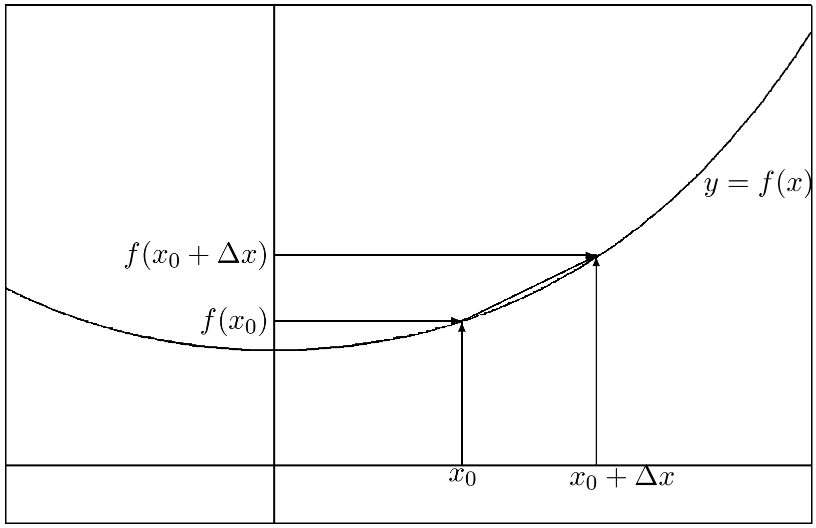

Figure 5.1: Definition of the derivative

Thus if $\Delta x = 1$, then the average rate of change of y over the interval from $x = 2$ to $x = 3$ is $x + \Delta x = 4 + 1 = 5$. For the interval from $x = 2$ to $x = 2.1$ ($\Delta x = 0.1$), the average rate of change is $4 + 0.1 = 4.1$; from 2 to 2.01 it's 4.01; from 2 to 2.001 it's 4.001; and so on. We can see that as Δx approaches 0 the values of the average rate of change are approaching 4. Thus it seems reasonable to say that the instantaneous rate of change of $f(x) = x^2$ at $x = 2$ is 4. This value of 4 is called the "derivative" of $f(x) = x^2$ with respect to x at $x = 2$.

In general, if $y = f(x)$ is any function, then **the derivative of y with respect to x**, written as $\frac{dy}{dx}$ or $f'(x)$, is defined as

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The symbol $\lim\limits_{\Delta x \to 0}$ stands for 'the limit as Δx approaches 0', so the derivative is the limit of the average rates of change as the interval widths get smaller and smaller. $\frac{dy}{dx}$ or $f'(x)$ is the instantaneous rate of change of y with respect to x at the point x, and its value is also the slope of the tangent line to the graph of $y = f(x)$ at the point $(x, f(x))$.

Examples

1. Find $\frac{dy}{dx}$ for $y = f(x) = x^2$, and use the result to find $f'(-1)$ and $f'(3)$. What is the equation of the line tangent to the graph of $y = x^2$ at $x = 4$?

By definition,

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}
\end{aligned}$$

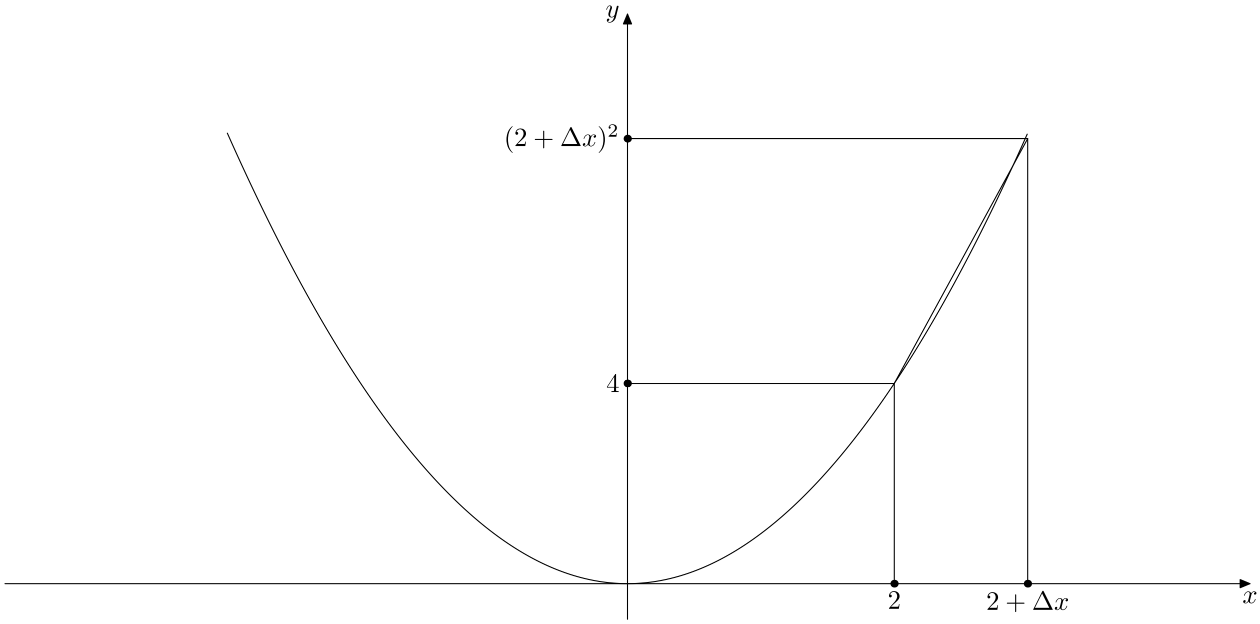

Figure 5.2: Derivative of x^2

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2x + \Delta x \\
&= 2x
\end{aligned}
$$

Thus if $y = x^2$, then $\frac{dy}{dx} = 2x$; or, in alternate notation, if $f(x) = x^2$, then $f'(x) = 2x$. In particular, $f'(-1) = -2$ and $f'(3) = 6$. We next find the tangent line at $x = 4$. This line has slope $f'(4) = 8$, and passes through $(4, 16)$ (since $x = 4$ implies $y = 4^2 = 16$). Thus the equation of the tangent line is $y - 16 = 8(x - 4)$, or $y = 8x - 16$. See Figure 5.3.

2. Suppose $y = f(x) = c$, where c is a constant. Then y never changes, and so we would expect the instantaneous rate of change to be 0. This can be proved using the definition of the derivative, i.e.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0.$$

Hence the derivative of a constant function is zero.

3. Suppose $y = f(x) = bx + a$. The graph of $y = f(x)$ is a straight line with slope b. Thus for any triangle we have that the rate of change of y over the interval from x_1 to x_2 is $\frac{y_2 - y_1}{x_2 - x_1} = b$. Thus we would expect that $f'(x) = b$.

Again this can be proved using the definition of the derivative. We have

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{[b(x + \Delta x) + a] - (bx + a)}{\Delta x}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{bx + b\Delta x + a - bx - a}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{b\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} b \\
&= b.
\end{aligned}
$$

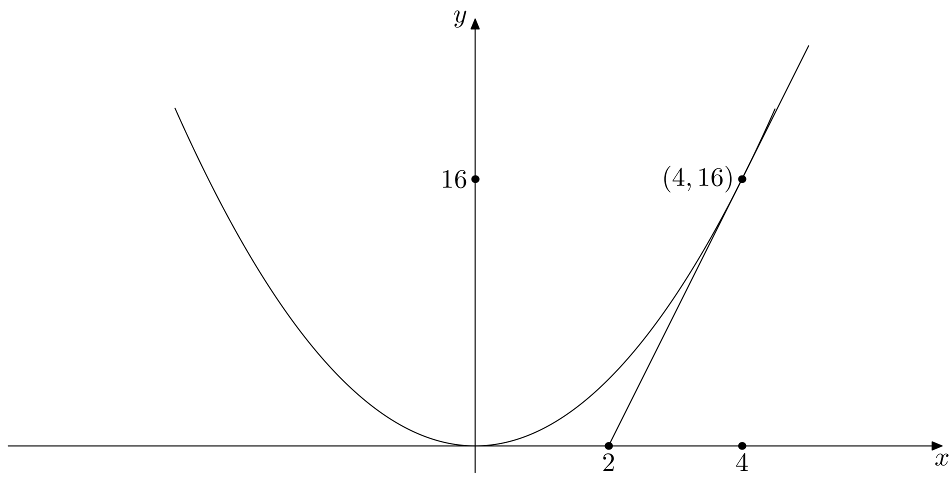

Figure 5.3: Tangent to x^2 at $x = 4$

4. Find the derivative of $y = f(x) = x^3$.

We have

$$
\begin{aligned}
\frac{d(x^3)}{dx} &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2 + 3x\Delta x + (\Delta x)^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + (\Delta x)^2) \\
&= 3x^2.
\end{aligned}
$$

Actually, it can be shown that for any value of n (not only for positive integers)

$$\frac{d(x^n)}{dx} = nx^{n-1}.$$

For example,

$$\frac{d(x^7)}{dx} = 7x^6,$$

$$\frac{d(\sqrt{x})}{dx} = \frac{d(x^{\frac{1}{2}})}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

and

$$\frac{d(1/x^4)}{dx} = \frac{d(x^{-4})}{dx} = -4x^{-4-1} = -4x^{-5} = \frac{-4}{x^5}.$$

[**Note:** The material in this section on differentiation using deltas and limits is included to explain the definition of the derivative and show how how it can be calculated from first principles. You will not be required to perform these calculations in the examination.]

5.2 Rules for Differentiation

At this point we have discussed the definition of a derivative and found the derivatives of some particular functions. Now we introduce some general rules for computing derivatives. The first two rules were derived in the previous section.

1. If c is a constant then
$$\frac{d(c)}{dx} = 0,$$
i.e. the derivative of a constant is zero.

2. For any power of x we have the "Power Rule"
$$\frac{d(x^n)}{dx} = nx^{n-1}.$$
Recall from the previous section that the linear function $y = bx + a$ satisfies $\frac{dy}{dx} = b$. A special case of this expression corresponds to $y = x$ (i.e., $b = 1$ and $a = 0$), so that $\frac{dy}{dx} = 1$. Another way of thinking about this case is to write $y = x^1$ and apply the Power Rule :
$$\frac{d(x^1)}{dx} = 1.x^{1-1} = 1.x^0 = 1 \ .$$
Bringing back rule 1 about the derivative of a constant, we now notice that
$$\frac{d(bx+a)}{dx} = b.\frac{dx}{dx} + \frac{d(a)}{dx} = b.1 + 0 = b \ .$$
This is actually a special case of two general rules of differentiation. One is about the effect on $\frac{dy}{dx}$ of multiplying a function $f(x)$ by a constant, the other describes how to go about differentiating a function that is a sum or difference of two simpler functions. We present these as rules 3 and 4 below.

3. If c is a constant and $y = f(x)$ is a function then
$$\frac{d(cy)}{dx} = c\frac{dy}{dx},$$
i.e. the derivative of a constant times a function is the constant times the derivative of the function. For example,
$$\frac{d(3x^2)}{dx} = 3\frac{d(x^2)}{dx} = (3)(2x) = 6x,$$
and
$$\frac{d(-\frac{9}{4}x^{-\frac{7}{2}})}{dx} = -\frac{9}{4}\frac{d(x^{-\frac{7}{2}})}{dx} = (-\frac{9}{4})(-\frac{7}{2}x^{-\frac{9}{2}}) = \frac{63}{8}x^{-\frac{9}{2}}.$$

4. If u and v are two functions of x, then

$$\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx},$$

i.e. the derivative of the sum or difference of two functions is the sum or difference of their derivatives. For example,

$$\frac{d(x^2 - 3x)}{dx} = \frac{d(x^2)}{dx} - \frac{d(3x)}{dx} = \frac{d(x^2)}{dx} - 3\frac{d(x)}{dx} = 2x - 3.$$

This rule also extends to more than two functions. For example,

$$\begin{aligned}\frac{d}{dx}(4 + 2x - x^2 + x^{-2}) &= \frac{d(4)}{dx} + \frac{d(2x)}{dx} - \frac{d(x^2)}{dx} + \frac{d(x^{-2})}{dx} \\ &= 0 + 2 - 2x - 2x^{-3} \\ &= 2 - 2x - \frac{2}{x^3}.\end{aligned}$$

Taken together, rules 3 and 4 describe the "linearity" property of the derivative. If we combine them in the following form

$$\frac{d(b.u + v)}{dx} = b.\frac{du}{dx} + \frac{dv}{dx} \ ,$$

we can see how it is related to the special case of differentiating the straight line function $y = bx + a$.

5. Again let u and v be two functions of x. To differentiate their product $u.v$, let's first consider the special case of two power functions, $u = x^m$ and $v = x^n$, where m and n are two different numbers. In this case we can write $y = u.v = (x^m)(x^n) = x^{m+n}$, and hence the power rule tells us

$$\frac{dy}{dx} = (m + n)x^{m+n-1} = m.x^{m+n-1} + n.x^{m+n-1}$$

$$= m.x^{m-1}.x^n + n.x^{n-1}.x^m = \frac{du}{dx}.v + \frac{dv}{dx}.u \ .$$

In general, for **any** two functions $u = f(x)$ and $v = g(x)$ the **product rule** says that

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Here are a couple of examples: (1) to differentiate

$$f(x) = (x + 1)(x^2 - 3)$$

we let

$$u(x) = x + 1 \qquad \text{and} \qquad v(x) = x^2 - 3.$$

Thus $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 2x$, and so

$$\frac{d}{dx}((x+1)(x^2-3)) = (x+1)(2x) + (x^2-3)(1) = 3x^2+2x-3.$$

(2) To differentiate

$$f(x) = (x^2+3x-2)(4x-7x^4)$$

we let

$$u(x) = x^2+3x-2 \qquad \text{and} \qquad v(x) = 4x-7x^4.$$

Thus $\frac{du}{dx} = 2x+3$ and $\frac{dv}{dx} = 4-28x^3$, and so

$$\begin{aligned}\frac{d}{dx}((x^2+3x-2)(4x-7x^4)) &= (x^2+3x-2)(4-28x^3)+(4x-7x^4)(2x+3)\\ &= -42x^5-105x^4+56x^3+12x^2+24x.\end{aligned}$$

6. To differentiate the quotient of two functions u and v, let's use the same model: $u = x^m$ and $v = x^n$. Now let $y = \frac{u}{v} = x^{m-n}$, and hence the power rule tells us that

$$\begin{aligned}\frac{dy}{dx} = (m-n)x^{m-n-1} &= m.x^{m-n-1} - n.x^{m-n-1}\\ &= \frac{m.x^{m-1}}{x^n} - \frac{x^m.n.x^{n-1}}{x^{2n}}\\ &= \frac{m.x^{m-1}.x^n - x^m.n.x^{n-1}}{x^{2n}}\\ &= \frac{\frac{du}{dx}.v - u.\frac{dv}{dx}}{v^2}\ .\end{aligned}$$

In general, for **any** two functions, $u = f(x)$ and $v = g(x)$, with $y = \frac{u}{v}$, the **quotient rule** says that

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

For example to differentiate

$$f(x) = \frac{2x+3}{x-4}$$

we let

$$u(x) = 2x+3 \qquad \text{and} \qquad v(x) = x-4.$$

Then

$$\frac{d}{dx}\left(\frac{2x+3}{x-4}\right) = \frac{(x-4)(2)-(2x+3)(1)}{(x-4)^2} = \frac{-11}{(x-4)^2}.$$

Thus far all our derivatives have been with respect to x. However, this doesn't have to be the case. For instance, if $u = 3t^2-4t$, then it makes sense to talk about $\frac{du}{dt}$. In fact, $\frac{du}{dt} = 6t-4$. Similarly, if $y = 4u^2+8u^{-3}$, then $\frac{dy}{du} = 8u-24u^{-4}$, and so on.

Let us next consider the function $y = (3x^2+11x-7)^{14}$. We could, of course, already find its derivative, but to multiply $3x^2+11x-7$ times itself fourteen times in order to find the derivative using the rules we already have would be a great deal of work. To handle this sort of example, it turns out to be much better to think in terms of **composition** of functions (cf. section 3.8). In other words, if we took $f(v) = v^{14}$, where $v = g(x) = 3x^2 + 11x - 7$, we could write the composition

$$f(g(x)) = (3x^2 + 11x - 7)^{14}$$

and think about how to find the derivative of $f(g(x))$ in terms of the derivatives of $f(x)$ and $g(x)$. For this let's go back once more to our models $u = f(v) = v^m$ and $v = g(x) = x^n$. Notice first that if we write

$$y = v^m = f(v) = f(g(x)) = (x^n)^m = x^{nm} \ ,$$

then, using the power rule, we have

$$\frac{dy}{dx} = nm.x^{nm-1} = nm.x^{nm-n+n-1}$$

$$= nm.x^{nm-n}.x^{n-1} = m(x^n)^{m-1}.n.x^{n-1} = m.v^{m-1}.\frac{dv}{dx}$$

$$= (\frac{dy}{dv})(\frac{dv}{dx}) \ .$$

7. In general, for **any** functions $u = f(v)$ and $v = g(x)$, which compose to give a function $y = f(g(x))$, we have the **chain rule**, which says that

$$\frac{dy}{dx} = (\frac{dy}{dv})(\frac{dv}{dx}).$$

More precisely, this can also be written in the form

$$\frac{d(f(g(x)))}{dx} = f'(g(x)).g'(x) \ .$$

Notice that $g'(x)$ is the same thing as $\frac{dv}{dx}$, but the expression we obtain for $\frac{dy}{dv}$ needs to have the v replaced by the full expression for $g(x)$. This is why it is more precise to write $f'(g(x))$ instead of $\frac{dy}{dv}$.

Thus to find the derivative of $y = (3x^2 + 11x - 7)^{14}$, we let $v = 3x^2 + 11x - 7$, so that $y = v^{14}$ and $\frac{dv}{dx} = 6x + 11$. Then

$$\frac{dy}{dx} = 14v^{13}.(6x + 11) = 14(3x^2 + 11x - 7)^{13}(6x + 11).$$

Examples

Quite complicated functions can be differentiated using the rules given above.
1. If $y = (2x^2 - 3)^4(2x + 9)^2$, then by the product rule

$$\frac{dy}{dx} = (2x^2 - 3)^4\frac{d}{dx}(2x + 9)^2 + (2x + 9)^2\frac{d}{dx}(2x^2 - 3)^4.$$

Now to find the derivative of $(2x+9)^2$, let $u = 2x+9$. Then

$$\frac{d}{dx}(2x+9)^2 = \frac{d}{dx}(u^2) = \frac{d}{du}(u^2)\frac{du}{dx} = (2u)(2) = 4(2x+9).$$

To find the derivative of $(2x^2-3)^4$, let $u = 2x^2-3$. Then

$$\frac{d}{dx}(2x^2-3)^4 = \frac{d}{dx}(u^4) = \frac{d}{du}(u^4)\frac{du}{dx} = (4u^3)(4x) = 16x(2x^2-3)^3.$$

Thus we arrive at

$$\begin{aligned}\frac{dy}{dx} &= (2x^2-3)^4 4(2x+9) + (2x+9)^2 16x(2x^2-3)^3 \\ &= 4(2x^2-3)^3(2x+9)[(2x^2-3)+4x(2x+9)] \\ &= 4(2x^2-3)^3(2x+9)(10x^2+36x-3).\end{aligned}$$

2. To differentiate the function

$$f(x) = \frac{(x^2-3)^3}{(2x+7)^2}$$

we use the quotient rule

$$\begin{aligned}\frac{d}{dx}\left(\frac{(x^2-3)^3}{(2x+7)^2}\right) &= \frac{(2x+7)^2\frac{d}{dx}\left((x^2-3)^3\right) - (x^2-3)^3\frac{d}{dx}\left((2x+7)^2\right)}{(2x+7)^4} \\ &= \frac{(2x+7)^2 3(x^2-3)^2 2x - (x^2-3)^3 4(2x+7)}{(2x+7)^4} \\ &= \frac{2(2x+7)(x^2-3)^2[3x(2x+7)-2(x^2-3)]}{(2x+7)^4} \\ &= \frac{2(x^2-3)^2(4x^2+21x+6)}{(2x+7)^3}.\end{aligned}$$

5.3 Higher Derivatives

Just as we found the derivative, or **first derivative** of a function $y = f(x)$, we can go on and find the derivative of the function $y = f'(x)$. This is called the **second derivative** of $y = f(x)$, and is written as $f''(x)$ or $\frac{d^2y}{dx^2}$.

Examples

1. For

$$f(x) = 4x^5,$$

then

$$\frac{dy}{dx} = 20x^4,$$

and so

$$\frac{d^2y}{dx^2} = 80x^3.$$

2. For

$$f(x) = (2x^2 + 4x - 7)(2x - 3),$$

then

$$f'(x) = (2x^2 + 4x - 7)(2) + (2x - 3)(4x + 4) = 12x^2 + 4x - 26.$$

Hence

$$f''(x) = 24x + 4.$$

If we can differentiate a function to obtain third, fourth, fifth, etc. derivatives, then these are written as $f'''(x),\ f^{(4)}(x),\ f^{(5)}(x), \ldots$, or as $\frac{d^3y}{dx^3},\ \frac{d^4y}{dx^4},\ \frac{d^5y}{dx^5}, \ldots$.

Note that one interpretation of the second derivative is as an acceleration. That is, if $s(t)$ denotes the distance travelled from a reference point by a moving object at time t, then $s'(t)$ is the velocity of the object (= instantaneous rate of change of distance), and $s''(t)$ is the acceleration of the object (= instantaneous rate of change of velocity).

5.4 Derivatives of Exponential, Logarithmic and Trigonometric Functions

Logarithmic and Exponential Functions

We introduced the functions $y = \log_a(x)$ in section 3.6, and particular mention was made of the logarithm *base* e, which is called the **natural logarithm**

$$y = \ln(x) \ .$$

But so far it has been kept secret why the logarithm base e should be any more natural than any other base. The full answer to this puzzle will be revealed in chapter 7, when we will explain why it is true that

$$\text{if } \ y = \ln(x) \ \text{ then } \ \frac{dy}{dx} = \frac{1}{x} \qquad (*).$$

Accepting this fact on faith for the time being allows us to draw some useful conclusions about differentiation of other exponential and logarithmic functions. Recall first that for any function $y = f(x)$, and any point x_0, the derivative $f'(x_0)$ (i.e., $\frac{dy}{dx}$ evaluated at the point x_0) corresponds to the slope of the straight line which is tangent to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$. This slope is interpreted as the instantaneous ratio of a small change in the y-variable to a small change in the x-variable for the function $y = f(x)$. We also saw in chapter 3 that it is often possible to turn around the relation $y = f(x)$, and solve for x as a function of y, written $x = f^{-1}(y)$. The derivative $\frac{dx}{dy}$ would then measure the inverse ratio, corresponding to a small change in x over a small change in y. We summarise this relationship in the form

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad (\dagger).$$

This applies especially to the exponential function $y = e^x$, since we also have the inverse relation $x = \ln(y)$. We can make the following statement by just switching the roles of x and y in (*) above :

$$\text{if } \ x = \ln(y) \ \text{ then } \ \frac{dx}{dy} = \frac{1}{y} \ .$$

But combining this with (†) also tells us that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{1}{y}} = y \ .$$

In summary,

$$\text{if} \quad y = e^x \quad \text{then} \quad \frac{dy}{dx} = y = e^x \ .$$

This is a very important fact : when you differentiate $y = e^x$, the function **stays the same**. To repeat,

$$\frac{d(e^x)}{dx} = e^x \ .$$

We can now go further and try to differentiate the general exponential function $y = a^x$ for some fixed base other than e. Notice that if we take the natural log of both sides of this equation, we get

$$\ln(y) = \ln(a^x) = x\ln(a) \ ,$$

using the properties of logarithms. If we write $x\ln(a)$ in the form $c.x$, where c is a constant number corresponding to $\ln(a)$, then it is easy to see that if we now take the derivative, we get

$$\frac{d(x\ln(a))}{dx} = \frac{d(c.x)}{dx} = c = \ln(a) \ ,$$

just by using one of our basic rules for differentiation from section 2. On the other hand, to find $\frac{d(\ln(y))}{dx}$ we need to make a slightly cunning use of the chain rule. Notice that if we write $v = \ln(y)$ and $y = f(x)$ (it doesn't matter that $f(x) = a^x$ in our present situation), the chain rule tells us that

$$\frac{dv}{dx} = (\frac{dv}{dy})(\frac{dy}{dx}) \ .$$

Once again, if we just use v and y instead of y and x in the original statement of (*) above, we see that $\frac{dv}{dy} = \frac{1}{y}$, so we can combine our results to get

$$\frac{d(\ln(y))}{dx} = (\frac{1}{y})(\frac{dy}{dx})$$

$$= \frac{d(x\ln(a))}{dx} = \ln(a) \ .$$

But now to make $\frac{dy}{dx}$ the subject of the equation, we just multiply through by y to conclude with the rule for the derivative of $y = a^x$:

$$\frac{dy}{dx} = \ln(a).y = \ln(a).a^x \ .$$

In particular, when $a = e$, we have $\ln(a) = \ln(e) = 1$, and so we get back to the original formula for the derivative of e^x.

Examples

1. To differentiate

$$y = e^{x^2+2x}$$

we use the chain rule with $u = x^2 + 2x$, so $y = e^u$. Therefore

$$\frac{dy}{dx} = (\frac{dy}{du})(\frac{du}{dx}) = e^u(2x+2) = (2x+2)e^{x^2+2x}.$$

2. For

$$y = x^2 e^{2x+3}$$

the product rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d(x^2)}{dx}\, e^{2x+3} + x^2 \frac{d(e^{2x+3})}{dx} \\ &= 2x\, e^{2x+3} + x^2\, e^{2x+3}(2) \\ &= 2x\, e^{2x+3}(x+1). \end{aligned}$$

3. To differentiate

$$y = 10^{2x^2+3}$$

the chain rule (let $u = 2x^2 + 3$) gives

$$\begin{aligned} \frac{dy}{dx} = (\frac{dy}{du})(\frac{du}{dx}) &= (\ln(10).10^u)(\frac{d}{dx}(2x^2+3)) \\ &= (\ln(10).10^{2x^2+3})(4x)\ . \end{aligned}$$

Now let's come back to general logarithmic functions $y = \log_a(x)$ for any fixed base a other than e. Recall that we can turn this equation around to write $x = a^y$, and hence we can write

$$\frac{dx}{dy} = \ln(a).a^y\ ,$$

just by switching the roles of x and y in the rule for exponentials. But now we get the general formula for differentiation of logs :

$$\frac{dy}{dx} = \frac{1}{\ln(a).a^y} = (\frac{1}{\ln(a)})\frac{1}{x}\ .$$

Note that when $a = e$, we again get $\ln(a) = \ln(e) = 1$, and hence the formula becomes $\frac{dy}{dx} = \frac{1}{x}$, which was our starting point for $y = \ln(x)$, and so we have come full circle.

Examples

1. To differentiate

$$y = \ln(3x^2 + 4x - 7)$$

we use the chain rule with $u = 3x^2 + 4x - 7$, so $y = \ln u$. Therefore

$$\frac{dy}{dx} = \frac{dy}{du}\frac{dy}{dx} = \left(\frac{1}{u}\right)(6x+4) = \frac{6x+4}{3x^2+4x-7}.$$

2. To differentiate

$$y = \ln(3x^2 + 4x - 7)^4$$

we use the chain rule with $u = (3x^2 + 4x - 7)^4$. Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \left(\frac{1}{u}\right) 4(3x^2 + 4x - 7)^3(6x + 4) \\ &= \left(\frac{1}{3x^2 + 4x - 7)^4}\right)(4(3x^2 + 4x - 7)^3(6x + 4)) \\ &= \frac{4(6x + 4)}{3x^2 + 4x - 7}. \end{aligned}$$

An alternate approach is to first use the log rules before differentiating: first we write

$$y = \ln(3x^3 + 4x - 7)^4 = 4\ln(3x^2 + 4x - 7).$$

Then

$$\begin{aligned} \frac{dy}{dx} &= (4)\left(\frac{1}{3x^2 + 4x - 7}\right)(6x + 4) \\ &= \frac{4(6x + 4)}{3x^2 + 4 - 7}. \end{aligned}$$

3. For

$$y = \log_{10}(3x^2 + 4x - 7)$$

we use the chain rule (let $u = 3x^2 + 4x - 7$):

$$\begin{aligned} \frac{dy}{dx} = (\frac{dy}{du})(\frac{du}{dx}) &= (\frac{1}{\ln(10)})(\frac{1}{u})\frac{d}{dx}(3x^2 + 4x - 7) \\ &= (\frac{1}{\ln(10)})(\frac{1}{3x^2 + 4x - 7})(6x + 4). \end{aligned}$$

Trigonometric Functions

The derivatives of $y = \sin(x)$ and $y = \cos(x)$ are given by

$$\frac{d}{dx}(\sin x) = \cos x$$

and

$$\frac{d}{dx}(\cos x) = -\sin x.$$

Like the fact that $\frac{d(e^x)}{dx} = e^x$, these formulae tell us something very important, but the reason why they are true will be discussed in the next chapter. We will therefore ask the reader to take these results also on faith for a little while. Using these results, we can find the derivatives of other functions which involve the trigonometric functions.

Examples

1. For

$$y = \frac{\sin x}{\cos x}$$

we use by the quotient rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x}.
\end{aligned}$$

2. To differentiate

$$y = \sin(2x+3)$$

we use the chain rule with $y = \sin u$, $u = 2x+3$,

$$\frac{dy}{dx} = \cos(2x+3)\frac{d}{dx}(2x+3) = 2\cos(2x+3).$$

3. To differentiate

$$y = \sin^2 x$$

recall that $y = \sin^2 x = (\sin x)^2$. Thus by the chain rule,

$$\frac{dy}{dx} = 2\sin x \frac{d}{dx}(\sin x) = 2\sin x \cos x.$$

4. For

$$y = \sin(x^2)$$

the chain rule with $y = \sin u$, $u = x^2$ gives

$$\frac{dy}{dx} = \cos(x^2)\frac{d}{dx}(x^2) = 2x\cos(x^2).$$

5. Differentiating

$$y = \sin^2 x + \cos^2 x$$

we get

$$\begin{aligned}
\frac{d}{dx}(\sin^2 x + \cos^2 x) &= \frac{d}{dx}(\sin x)^2 + \frac{d}{dx}(\cos x)^2 \\
&= 2\sin x \frac{d}{dx}(\sin x) + 2\cos x \frac{d}{dx}(\cos x) \\
&= 2\sin x \cos x - 2\cos x \sin x \\
&= 0.
\end{aligned}$$

(This result isn't surprising, since we know that $\sin^2 x + \cos^2 x = 1$ for all values of x, and the derivative of the constant 1 is 0.)

6. To differentiate

$$y = e^{2x}\sin(2x^2+2)$$

we use the product rule combined with the chain rule

$$\begin{aligned}\frac{dy}{dx} &= e^{2x}\frac{d}{dx}(\sin(2x^2+2)) + \sin(2x^2+2)\frac{d}{dx}(e^{2x}) \\ &= e^{2x}\cos(2x^2+2)4x + \sin(2x^2+2)2\,e^{2x} \\ &= 2\,e^{2x}(2x\cos(2x^2+2)+\sin(2x^2+2)).\end{aligned}$$

7. Differentiating

$$y = x^2\cos(3x-4)$$

is another application of the product rule,

$$\begin{aligned}\frac{dy}{dx} &= x^2\frac{d}{dx}(\cos(3x-4)) + \cos(3x-4)\frac{d}{dx}(x^2) \\ &= -3x^2\sin(3x-4) + 2x\cos(3x-4).\end{aligned}$$

5.5 Exercises

1. A population of bacteria is introduced to a nutrient medium. Suppose that the weight of the population in milligrams changes according to the formula

$$P(t) = 50 + \frac{100t}{21+t^2}$$

where t is the time measured in hours. Determine the average rate of growth of the population during the five hour period starting at $t = 2$ hours.

2. Find the average rate of change of

$$y = f(x) = 2 - 3x + 2x^2$$

over the interval from $x = 1$ to $x = 1 + \Delta x$, and hence find $f'(1)$.

3. Use the definition of the derivative to find $\frac{dy}{dx}$ if $y = 2 - 3x + 2x^2$.

4. If

$$y = f(x) = 3 + 4x - 3x^2$$

use the definition of the derivative to find the slope and equation of the tangent to the graph of $y = f(x)$ at the point where $x = 3$.

5. Find the derivatives of the following functions:

(a) $y = 10x^6$

(b) $y = 3x^8 + 6x^5 - 4x^2 + 17$

(c) $y = t^{7.5}$

(d) $y = \dfrac{x^5 + 2x^3 - 3}{x^3}$

(e) $u = \dfrac{5}{t^3} + 4t^2$

(f) $y = \sqrt[6]{x^7}$

(g) $v = (x^2 + 3x)(2x - 4)$

6. Find the equation of the tangent to the curve $y = 5x^2 + \frac{2}{x}$ at the point where $x = 1$.

7. Find $\frac{dy}{dx}$ in the following cases:

(a) $y = (x^2 - 4x + 3)(x^3 + 4)$

(b) $y = 7(x^4 + 3x^3 + 1)^{\frac{1}{3}}$

(c) $y = 3\sqrt{x^2 + 7}$

(d) $y = \dfrac{x^2 - 3}{3x^3 + 4}$

(e) $y = \dfrac{2x^2 + 1}{\sqrt{x}}$

8. Let x be the size of a certain population of predators and y be the size of the population of prey upon which they feed. As functions of time t, $x = t^2 + 4$ and $y = 2t^2 - 3t$. Let u be the ratio of the number of prey to each predator. Find the rate of change of u with respect to time.

9. The mass of a growing cell at time t is given by

$$m(t) = (\frac{1}{3}t + m_0^{\frac{1}{3}})^3$$

where m_0 is a constant.

(i) By putting $t = 0$ in the above formula, give an interpretation of the meaning of m_0.

(ii) What is the instantaneous rate of change of m at time t?

(iii) What is the instantaneous rate of change of m at times $t = 0$ and $t = 1$?

(iv) Can the mass of the cell decrease? Give a reason.

10. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for

$$y = 3x^5 + 7x^3 - 4x^2 + 12.$$

Without doing the actual calculation, what can you say about $\frac{d^6y}{dx^6}$?

11. Find the following higher derivatives:

(a) Find $f'''(t)$ if $f(t) = \frac{t-1}{t+1}$.

(b) Find $\frac{d^2y}{dt^2}$ for $y = \sqrt{t^2 + 1}$.

12. If the distance S travelled in time t is given by $S = t(3 - t)$,

(a) at what time is the velocity zero?

(b) what is the acceleration when the velocity equals zero?

13. Find $\frac{dy}{dx}$ for the following functions:

(a) $y = e^{x^2}$

(b) $y = \dfrac{e^x}{x}$

(c) $y = \dfrac{e^{\sqrt{x}}}{e^x}$

(d) $y = \ln(x^4)$

(e) $y = (\ln x)^5$

(f) $y = e^x \ln x$

(g) $y = x^2 \ln(x^2 + x)$

(h) $y = \ln\left(\dfrac{e^x\sqrt{x-1}}{x^3}\right)$ (Hint: Simplify y before differentiating.)

14. A population grows according to the formula

$$y = p(1 - c\,e^{-kt})^3$$

where p, c, and k are constants. Find the rate of growth of the population at general time t.

15. Differentiate the following functions with respect to x:

(a) $y = \sin(3x)$

(b) $y = \dfrac{\cos x}{x}$

(c) $y = \sin^2(\sqrt{x})$

(d) $y = 5\sqrt{x^3 + 1}\cos x$

(e) $y = e^{\cos(2x)}$

(f) $y = \ln(\cos x)$

(g) $y = \dfrac{1 + \cos x}{\sin x}$

16. Find the second derivatives of the following functions:

(a) $y = x \sin x$

(b) $y = \sin^2 x$

17. A population of bacteria is growing in such a way that its weight after a time t hours is given by

$$w = e^t(2 - \cos t)$$

where w is measured in g. Find the rate of growth of the population at $t = 0$, $t = \frac{\pi}{2}$, and $t = 2\pi$. Can the weight of the population ever decrease?

Chapter 6

Maxima and Minima

6.1 Stationary Points

An important application of differentiation is to find the maximum or minimum values of a function. For instance, we might want to find the amount of drug administered to a person that gives a maximum reaction, or the amount of fertilizer applied to a crop that gives a maximum yield, or the time when a bacterial population is a maximum.

Let us start by considering the graph of a function (Figure 6.1):

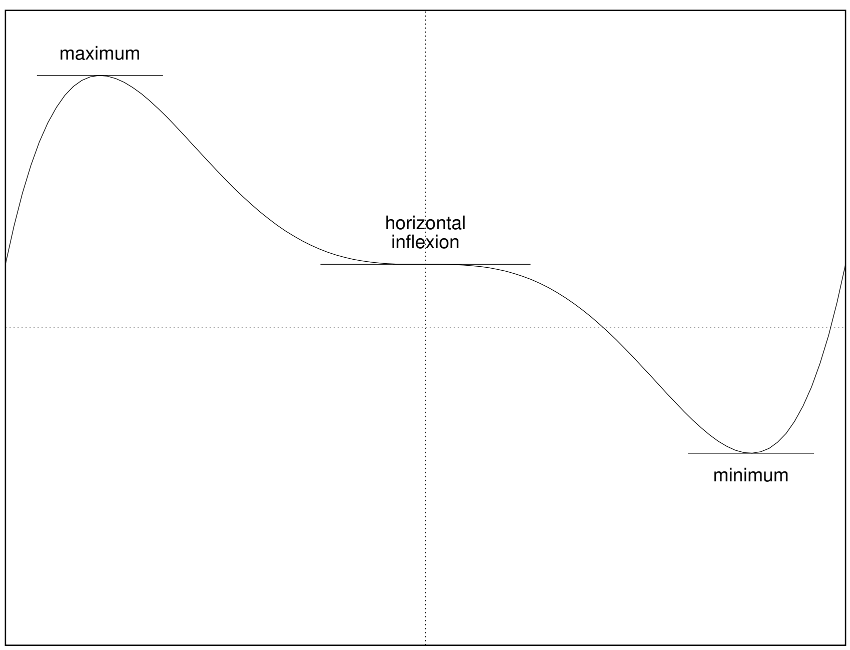

Figure 6.1: Stationary Points

As can be seen from the graph, at a maximum or minimum point the tangent is horizontal, and so the slope of the tangent is zero. Thus for a maximum or minimum point we have $f'(x) = 0$. However, there is another type of point where $f'(x) = 0$. This is called a point of horizontal inflexion. A typical example of a point of horizontal inflexion is the point $x = 0$ for the function $y = x^3$.

To find the maximum or minimum points of a function $y = f(x)$, we must find all values of x for which $f'(x) = 0$. Such points are called **stationary points**. Before we go on to discuss a more formal procedure for classifying stationary points as maxima, minima, or inflexions, however, let's first look at a simple case that will serve as our intuitive model.

Example: Recall that the quadratic function $y = ax^2 + bx + c$ describes a parabola, for which the vertex can be treated as a stationary point, since the tangent line at the vertex is always horizontal. To convince ourselves that this is true, we need only compute $\frac{dy}{dx} = f'(x) = 2ax + b$ and observe that

$$2ax + b = 0 \quad \text{when} \quad x = \frac{-b}{2a} \ .$$

Since this is indeed the formula for the x-coordinate of the vertex (cf. section 3.3), we see that the tangent really is horizontal there (and nowhere else). Now recall that the question of whether the vertex is a maximum or minimum is really equivalent to asking whether the parabola is concave up or down - which in turn boils down to whether the leading coefficient a in the equation is a negative or a positive number.

There are two ways to develop this last observation into a useful test for maxima and minima. One way is to observe that if $a > 0$ (vertex a minimum) then the slope of the straight line $y = 2ax + b$ is positive, and so the y-value of this function changes from negative to zero to positive as the x-value passes through $\frac{-b}{2a}$, going from left to right. Conversely, if $a < 0$ (vertex a maximum) then the slope of the line is negative, and the value of $2ax + b$ changes from positive to zero to negative as x passes through $\frac{-b}{2a}$ from left to right. We state this first approach to maxima and minima in a more general way as follows.

First Derivative Test for Classifying Stationary Points

Suppose c is a stationary point of $y = f(x)$, so that $f'(c) = 0$.

(i) $x = c$ is a **local maximum point** of $y = f(x)$ if $f'(x)$ changes sign from positive to negative as x changes from just below c to just above c.

(ii) $x = c$ is a **local minimum point** of $y = f(x)$ if $f'(x)$ changes sign from negative to positive as x changes from just below c to just above c.

(iii) $x = c$ is a horizontal **inflexion point** of $y = f(x)$ if $f'(x)$ doesn't change sign as x changes from just below c to just above c.

Examples.

(i) Find and classify all stationary points of $f(x) = x^4 - 4x^3 + 7$.

Solution. For a stationary point, $0 = f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$. This equation is satisfied if and only if either the factor $4x^2$ is zero or the factor $x - 3$ is zero, so either $x = 0$ or $x = 3$. Each of these values corresponds to a stationary point on the graph of $y = x^4 - 4x^3 + 7$, for which we also need the y-values. Thus $f(0) = 7$ and $f(3) = 3^4 - 4.3^3 + 7 = 81 - 108 + 7 = -20$, which means that the stationary points have coordinates $(0, 7)$ and $(3, -20)$.

Consider first the stationary point $(0,7)$. If x is just less than 0, then $x^2 > 0$ and $x-3<0$. Therefore $f'(x) < 0$. On the other hand, if x is just greater than 0, then again $x^2 > 0$ and $x-3<0$. Thus $f'(x)$ doesn't change sign, and so $(0,7)$ is a point of inflexion.

Next consider the stationary point $(3,-20)$. If x is just less than 3, then $x^2 > 0$ and $x-3<0$. Therefore $f'(x)<0$. On the other hand, if x is just greater than 3, then $x^2>0$ and $x-3>0$. Therefore $f'(x)>0$. Thus $f'(x)$ changes sign from negative to positive, and so $(3,-20)$ is a local minimum point.

(ii) The size of a bacteria population that is introduced to a nutrient grows according to the formula

$$N(t) = 5,000 + \frac{30,000t}{100+t^2},$$

where the time t is measured in hours. Determine the maximum size of this population.

Solution. Using the quotient rule we see that

$$\frac{dN}{dt} = \frac{30,000(100+t^2) - 30,000t(2t)}{(100+t^2)^2} = \frac{30,000(100-t^2)}{(100+t^2)^2}.$$

Thus $\frac{dN}{dt} = 0$ if and only if the numerator is zero, i.e., when $t^2 = 100$, or $t = \pm 10$. Since the problem requires t to be positive, we can ignore $t=-10$. Therefore $t=10$ is the only stationary point of N.

Now if $t<10$, then $100-t^2>0$, and so $\frac{dN}{dt}>0$. If $t>10$, then $100-t^2<0$, and so $\frac{dN}{dt}<0$. Thus at $t=10$ we see that $\frac{dN}{dt}$ changes from being positive to being negative. This means $t=10$ gives a maximum value for N. Hence $N(10) = 5,000 + \frac{300,000}{100+100} = 6,500$ is the maximum size of the bacteria population, and this occurs after 10 hours.

(iii) After a drug has been administered, its reaction at time t is given by

$$R(t) = t^2\, e^{-t}.$$

At what time is the reaction a maximum?

Solution. Using the product rule,

$$\frac{dR}{dt} = (2t)\, e^{-t} + t^2\, e^{-t}(-1) = t\, e^{-t}(2-t).$$

Now we can assume $t>0$, and also the exponential function is never zero. Thus $\frac{dR}{dt}=0$ only at $t=2$. Suppose t is just less than 2. Then $2-t>0$, and so $\frac{dR}{dt}>0$. On the other hand, if t is just greater than 2, then $2-t<0$, and $\frac{dR}{dt}<0$. Therefore $t=2$ is a maximum point for R, and so this is when the reaction of the drug is a maximum.

Remark: At this point it is appropriate to point out that the terms "maximum" and "minimum" as we have used them are strictly local in nature. The first derivative test (like the second derivative test, which we will meet shortly) usually only tells us whether a stationary point is the highest or lowest point in some small "neighbourhood" of the graph of a function. It does **not** tell us whether the graph of a given function has a maximum or minimum value overall. Some extra information is usually needed to answer this "global" question. In the next section, we will apply our understanding of stationary points to give a heuristic justification for the claim that

$$\frac{d(\sin(x))}{dx} = \cos(x) \ .$$

6.2 Derivative of sine and cosine Revisited

At the outset we will mention two facts which are needed in our effort to understand how to differentiate $y = \sin(x)$. One is the "double-angle formula" from basic trigonometry

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \ .$$

The other is a special limiting ratio, the value of which is plausible enough to believe, but requires some careful geometric analysis to prove, so we will merely state this fact without proof,

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \ .$$

Looking back at the graphs of $\sin(x)$ and $\cos(x)$ in figure 3.16 (chapter 3), you can see that, for example, $y = \sin(x)$ has many (in fact infinitely many) stationary points. The maxima and minima occur at all the odd multiples

$$x = (2k+1)\frac{\pi}{2}, \qquad k = 0, \pm 1, \pm 2, \dots \ ,$$

and hence $\frac{dy}{dx} = 0$ at each of these points. In fact, the places where the derivative of $\sin(x)$ vanishes correspond exactly to the values of x where $\cos(x)$ vanishes. We will take this as a first piece of evidence that $\cos(x)$ and the derivative of $\sin(x)$ are really the same thing. Our next observation, from looking carefully at the graph of $y = \sin(x)$, is that the tangent lines at various points on the graph have their steepest slopes at all x where $\sin(x) = 0$, i.e., all x corresponding to multiples

$$k\pi, \qquad k = 0, \pm 1, \pm 2, \dots \ .$$

These points are therefore the places where $\frac{dy}{dx}$ has its maxima and minima. Now $y = \cos(x)$ also takes its maximum and minimum y-values, namely $y = \pm 1$, at these same values of x. If we can show that the derivative of $\sin(x)$ is also ± 1 in these places, then we will rest our case. In fact, it will be sufficient to show that the derivative takes the value 1 at $x = 0$ and -1 at $x = \pi$, since the periodic behaviour of $\sin(x)$ will repeat these two values at x equal to every alternate multiple of π.

Now, if $f(x) = \sin(x)$, going back to the definition of the derivative, we see that

$$f'(0) = \lim_{\Delta x \to 0} \frac{\sin(0 + \Delta x) - \sin(0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = 1 \ ,$$

using the limit ratio above, with Δx instead of θ. On the other hand,

$$f'(\pi) = \lim_{\Delta x \to 0} \frac{\sin(\pi + \Delta x) - \sin(\pi)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sin(\pi)\cos(\Delta x) + \cos(\pi)\sin(\Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cos(\pi)\sin(\Delta x)}{\Delta x}$$

$$= \cos(\pi). \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} = -1.1 = -1 \ ,$$

this time making use of the double-angle formula, the facts that $\sin(\pi) = 0$ and $\cos(\pi) = -1$, and bringing back our limit ratio at the end.

From this we conclude that

$$\frac{d(\sin(x))}{dx} = \cos(x) \ .$$

The argument for showing that $\frac{d(\cos(x))}{dx} = -\sin(x)$ is very similar, and can be left as an exercise.

6.3 Second Derivative Test for Classifying Stationary Points

Returning briefly to our model of the general parabola $y = ax^2 + bx + c$, recall that the vertex is a maximum if the parabola is concave down ($a < 0$) and a minimum if the parabola is concave up ($a > 0$). We have already tried to relate this to the slope of the straight line $y = 2ax + b$, corresponding to the first derivative $\frac{dy}{dx}$. Another point of view is to notice that the second derivative

$$\frac{d^2y}{dx^2} = 2a \ ,$$

hence we may say that the vertex is a maximum if $\frac{d^2y}{dx^2} < 0$ at $x = \frac{-b}{2a}$, and a minimum if $\frac{d^2y}{dx^2} > 0$ at $x = \frac{-b}{2a}$.

The corresponding statement for a general function $y = f(x)$ is called the **second derivative test**, and is often (though not always) quicker to use than the first derivative test:

Suppose c is a stationary point of $y = f(x)$, so that $f'(c) = 0$.

(i) If $f''(c) < 0$, then c is a local maximum point of $y = f(x)$.

(ii) If $f''(c) > 0$, then c is a local minimum point of $y = f(x)$.

(iii) If $f''(c) = 0$, then no conclusion can be drawn.

Example. Find and classify all stationary points of $f(x) = x^3 - 3x + 4$.
Solution.

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) \qquad \text{and} \qquad f''(x) = 6x.$$

The stationary points occur when $x^2 - 1 = 0$, or $x = \pm 1$ (hence $f(1) = 2$ and $f(-1) = 6$). Since $f''(1) = 6 > 0$, by the second derivative test $(1, 2)$ is a local minimum. Since $f''(-1) = -6 < 0$, by the second derivative test $(-1, 6)$ is a local maximum.

6.4 Exercises

1. Find and classify all stationary points of the following functions:

(a) $f(x) = 2x^3 + 3x^2 - 12x - 15$

(b) $f(x) = x\, e^{-2x}$

2. During the course of an epidemic, the proportion of the population infected after a time t is equal to $\frac{t^2}{5(1+t^2)^2}$, where t is measured in months, and the epidemic starts at $t = 0$. Find the maximum proportion of population that becomes infected.

3. A bacteria colony consisting of 10,000 bacteria is treated with a drug. After t hours have elapsed, the number of bacteria present is given by

$$N(t) = 10,000(1 + \frac{t}{5} - \ln(1 + t)).$$

Find any stationary points of $N(t)$, and discuss the meaning of your result.

4. When a person coughs, the radius R of the main air passage in the lungs decreases. The velocity V of air through the air passage during the cough is

$$V = hR^2(R_0 - R),$$

where R_0 is the initial radius (i.e. the radius before the cough) and h is a constant. What is the value of the radius that gives the maximum velocity of air through the passage during the cough?

5. During the course of an epidemic (different from that in Exercise # 2), the number of people infected at time t is given by

$$N = At^{\frac{5}{2}} e^{-3t},$$

where A is a constant. Find the value of t at which the number of infected people is a maximum, and also find the maximum value of N.

Chapter 7

Integration

7.1 Indefinite Integration

Suppose the velocity $v(t)$ of a moving object at time t is known, and that we want to find $s(t)$, the distance travelled up to time t. In order to find $s(t)$ we have to answer the question: 'What is the function whose derivative is $v(t)$?' In other words, we have to reverse the process of differentiation. This reversal of the process of differentiation is also needed if we are given the rate $p'(t)$ at which a population is growing, and wish to calculate $p(t)$, the population size at time t; or if we are given the rate at which a certain product is being produced in a chemical reaction, and wish to know the amount of product formed up to a certain

The reversal of the process of differentiation is called **integration**. Suppose $f(x)$ is a given function, and $F(x)$ is a function whose derivative is $f(x)$, i.e.

$$\frac{dF}{dx} = f(x).$$

Then $F(x)$ is called an **indefinite integral** or **anti- derivative** of $f(x)$. For example, $\frac{x^2}{2}$ is an indefinite integral of x since

$$\frac{d}{dx}\left(\frac{x^2}{2}\right) = \frac{1}{2}(2x) = x.$$

The shorthand way of writing

"$F(x)$ is an indefinite integral of $f(x)$"

is to write

$$\int f(x)\,dx = F(x).$$

Here the symbol $\int$ is called the 'integral sign', and $f(x)$ is the **integrand**. The symbol dx tells us that the integration is with respect to x. For example,

$$\int x^2\,dx = \frac{x^3}{3} \quad \text{and} \quad \int t^2\,dt = \frac{t^3}{3}$$

(this second integral is with respect to t). We have said that

$$\int x^2\,dx = \frac{x^3}{3},$$

but it is also true that

$$\int x^2\,dx = \frac{x^3}{3} + 1$$

(since the derivative of $\frac{x^3}{3} + 1$ is x^2). Likewise, it is true that

$$\int x^2\,dx = \frac{x^3}{3} - 17.$$

More generally,

$$\int x^2\,dx = \frac{x^3}{3} + c$$

for any fixed value of c. This holds for any indefinite integral, and so if

$$\frac{dF}{dx} = f(x) \quad \text{we have} \int f(x)\,dx = F(x) + c,$$

where c is an arbitrary constant (the **constant of integration**).

We may summarise with the remark that, in effect, "differentiation **undoes** what integration does to a function", and vice versa. More precisely,

$$\frac{d}{dx}\int f(x)dx = f(x)\ , \quad \text{and}$$

$$\int \frac{df}{dx}dx = f(x) + C\ .$$

Notice that the second combination doesn't quite return to its starting point $f(x)$, because indefinite integration creates constants C, while differentiation always destroys constants.

Standard Integrals

The following standard integrals should be known:

1. $\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + c$ for $n \neq -1$.
2. $\displaystyle\int e^x dx = e^x + c$
3. $\displaystyle\int e^{kx} dx = \frac{1}{k}e^{kx} + c$
4. $\displaystyle\int \cos\,x\,dx = \sin\,x + c$
5. $\displaystyle\int \sin\,x\,dx = -\cos\,x + c$
6. $\displaystyle\int \cos\,kx dx = \frac{1}{k}\sin\,kx + c$
7. $\displaystyle\int \sin\,kx dx = -\frac{1}{k}\cos\,kx + c$

All these standard integrals can be proved by showing the derivative of the function on the right-hand side is equal to the integrand on the left- hand side. For example, formula (7) is correct since

$$\frac{d}{dx}\left(-\frac{1}{k}\cos(kx)\right) = -\frac{1}{k}\frac{d}{dx}(\cos(kx)) = \left(-\frac{1}{k}\right)(-\sin(kx))(k) = \sin(kx).$$

Properties of integrals

In working out integrals, we shall also make use of the following rules:

1. $\int kf(x)dx = k\int f(x)dx$ when k is constant.
2. $\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$
3. $\int (f(x)-g(x))dx = \int f(x)dx - \int g(x)dx$

These rules follow from the corresponding rules for differentiation.

Examples

1.
$$\begin{aligned}\int 5\sin(7x)\,dx &= 5\int \sin(7x)\,dx\\ &= 5(-\frac{1}{7}\cos(7x))+c\\ &= -\frac{5}{7}\cos(7x)+c.\end{aligned}$$

2.
$$\begin{aligned}\int\left(\frac{t^2+3}{t}-2\,e^{5t}\right)dt &= \int\left(t+\frac{3}{t}-2\,e^{5t}\right)dt\\ &= \int t\,dt + 3\int\frac{1}{t}\,dt - 2\int e^{5t}\,dt\\ &= \frac{t^2}{2}+3\ln(t)-\frac{2}{5}\,e^{5t}+c.\end{aligned}$$

3. We note that there are no rules which give simple ways of integrating either the product or quotient of functions. If there is a product to be integrated, one possible way of performing the integration is to multiply out the integrand. For instance,

$$\begin{aligned}\int\left(x-\frac{4}{x^2}\right)^2 dx &= \int\left(x^2-\frac{8}{x}+\frac{16}{x^4}\right)dx\\ &= \int x^2\,dx - 8\int\frac{1}{x}\,dx + 16\int x^{-4}\,dx\\ &= \frac{x^3}{3}-8\ln(x)-\frac{16}{3}x^{-3}+c.\end{aligned}$$

Examples

1. The rate of growth of a colony of fruit flies at time $t \geq 1$ is equal to $\frac{10(t+2)}{t}$. When $t = 1$ there are 20 flies in the colony. Calculate the number of flies at a general value of $t > 1$.

   Solution. Let the population of flies at time t be given by $P(t)$. Then

$$P'(t) = \frac{10(t+2)}{t},$$

   so

$$\begin{aligned} P(t) &= \int \frac{10(t+2)}{t}\, dt \\ &= 10 \int \left(1 + \frac{2}{t}\right) dt \\ &= 10 \int 1\, dt + 20 \int \frac{1}{t}\, dt \\ &= 10t + 20\ln(t) + c. \end{aligned}$$

   In this case we can give c a specific value since we know $P(1) = 20$. Substituting $t = 1$ in the formula for $P(t)$ gives

$$20 = P(1) = 10(1) + 20\ln(1) + c = 10 + c,$$

   so $C = 10$. Thus the formula for $P(t)$ is

$$P(t) = 10t + 20\ln(t) + 10 = 10(1 + t + 2\ln(t)).$$

2. During daylight hours the velocity of a migrating goose is given by

$$v(t) = 20 - \frac{t}{3} \text{ mph},$$

   where t is the time measured in hours and starts at dawn with $t = 0$. How many miles has the goose travelled up to time t? How far does the goose fly in a 12-hour day?

   Solution. Let $s(t)$ denote the distance in miles travelled by the goose up to time t. Then

$$v(t) = s'(t) = 20 - \frac{t}{3},$$

   so

$$s(t) = \int (20 - \frac{t}{3})\, dt = \int 20\, dt - \frac{1}{3}\int\, dt = 20t - \frac{t^2}{6} + c.$$

   Now $s(0) = 0$, so

$$0 = 20(0) - \frac{(0)^2}{6} + c = c.$$

   Hence

$$s(t) = 20t - \frac{t^2}{6}.$$

   In particular, the distance travelled in a 12-hour day is

$$s(12) = (20)(12) - \frac{(12)^2}{6} = 216 \text{ miles}.$$

7.2 Definite Integration

Suppose that $f(x) \geq 0$ for $a \leq x \leq b$.

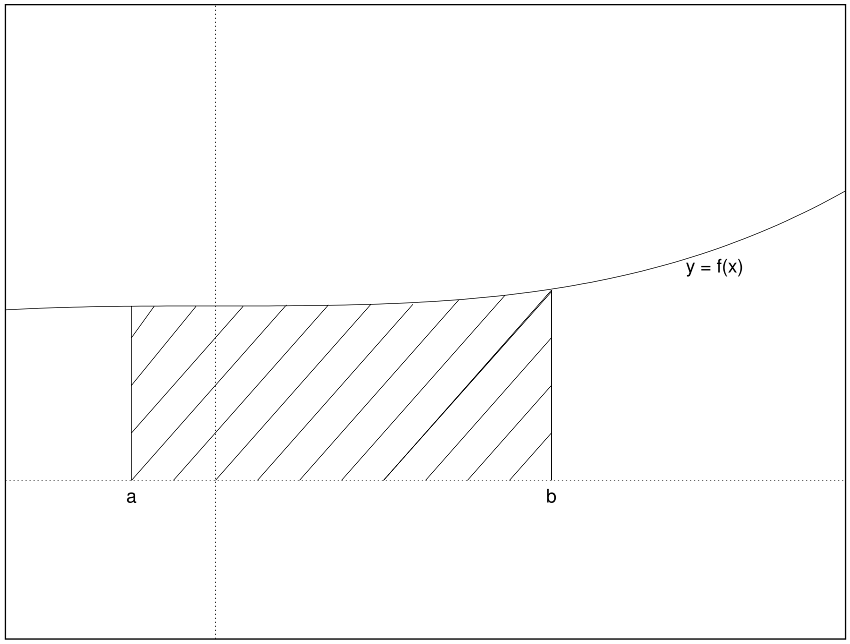

Figure 7.1: The Area under a curve.

The area between the graph of $y = f(x)$ and the x-axis for $a \leq x \leq b$ will be denoted by

$$\int_a^b f(x)\,dx,$$

(see figure 7.1.) This quantity is called a **definite integral**, and the numbers a and b are called the **limits of integration**. The question remains how to find this area.

The problem of finding $\int_a^b f(x)\,dx$ can be solved by using the **Fundamental Theorem of Calculus**. This theorem says that if $F(x)$ is any indefinite integral of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

(We note that the symbol $[F(x)]_a^b$ is often used to denote $F(b) - F(a)$.)

Examples

1. Find the area between the graph of $y = x$ and the x-axis for $0 \leq x \leq 5$.

   Solution.

   The area is given by

$$\int_0^5 x\,dx = \left[\frac{x^2}{2}\right]_0^5 = \frac{5^2}{2} - \frac{0}{2} = \frac{25}{2}.$$

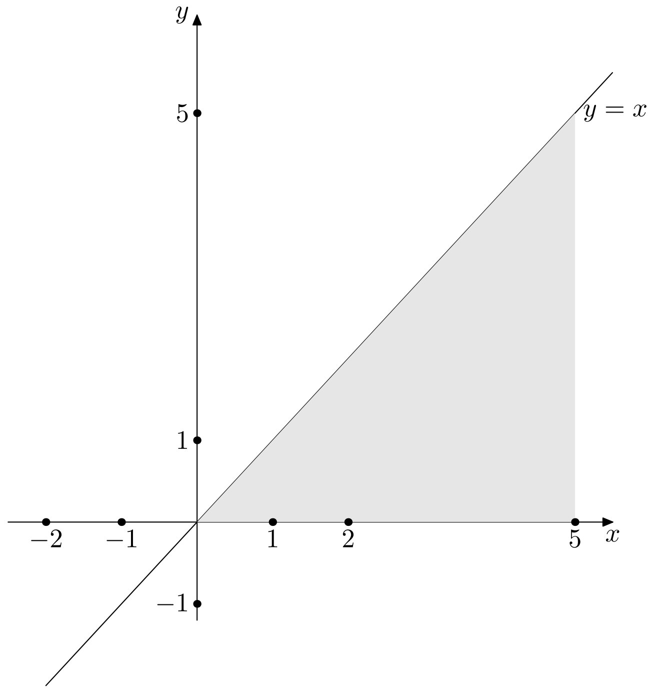

Figure 7.2: Area under $y = x$ from $x = 0$ to $x = 5$.

Note that if any other indefinite integral is used, the answer for the area remains the same. For example, suppose we take

$$\frac{x^2}{2} + 17 = \int x\,dx$$

then

$$\int_0^5 x\,dx = \left[\frac{x^2}{2} + 17\right]_0^5 = \left(\frac{25}{2} + 17\right) - \left(\frac{0}{2} + 17\right) = \frac{25}{2}.$$

Thus when evaluating definite integrals, the constant of integration doesn't matter.

Also note that in this example the required area is that of a triangle with base 5 and height 5. Hence the area is equal to

$$\frac{1}{2}\text{ (base) (height)} = \frac{1}{2}(5)(5) = \frac{25}{2},$$

which agrees with the result obtained by integration.

2. Find the area between the graph of $y = 2x^2 + x + e^{-2x}$ and the x-axis for $1 \le x \le 3$.

Solution.

$$\begin{aligned}
\text{Area} \quad = \int_1^3 (2x^2 + x + e^{-2x})\,dx &= \left[\frac{2}{3}x^3 + \frac{x^2}{2} - \frac{1}{2}\,e^{-2x}\right]_1^3 \\
&= \left(\frac{2}{3}(3)^3 + \frac{1}{2}(3)^2 - \frac{1}{2}\,e^{-6}\right) - \left(\frac{2}{3} + \frac{1}{2} - \frac{1}{2}\,e^{-2}\right) \\
&= \frac{64}{3} + \frac{1}{2}\,e^{-2} - \frac{1}{2}\,e^{-6}.
\end{aligned}$$

7.3 More on Areas

Definite integrals can be calculated even if the function f is not always non-negative for $a \leq x \leq b$. For example,

$$\int_{-1}^{3} (x^2 - 4)\,dx = \left[\frac{x^3}{3} - 4x\right]_{-1}^{3} = \left(\frac{3^3}{3} - 12\right) - \left(-\frac{1}{3} + 4\right) = -\frac{20}{3}.$$

See Figure 7.3.

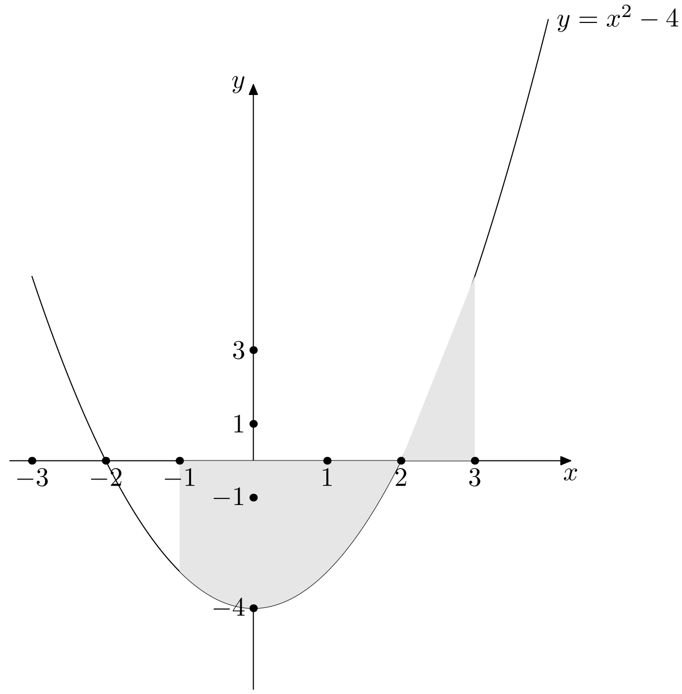

Figure 7.3: Areas for a function with negative values.

However, in this case the value doesn't represent an area. This is because $x^2 - 4$ is negative for some of the interval $-1 \leq x \leq 3$. The answer of $-\frac{20}{3}$ represents the shaded are, with area below the x-axis counted as being negative. If we want the actual shaded area, with all areas being counted as positive, we have to calculate $\int_2^3 (x^2 - 4)\,dx$ and then subtract $\int_{-1}^{2} (x^2 - 4)\,dx$, i.e.

$$\begin{aligned}
\text{shaded area} &= \left[\frac{x^3}{3} - 4x\right]_2^3 - \left[\frac{x^3}{3} - 4x\right]_{-1}^{2} \\
&= \left\{(9 - 12) - \left(\frac{8}{3} - 8\right)\right\} - \left\{\left(\frac{8}{3} - 8\right) - \left(-\frac{1}{3} + 4\right)\right\} \\
&= \frac{34}{3}
\end{aligned}$$

Definite integrals can also be used to find the area between two curves, (see Figure 7.4).

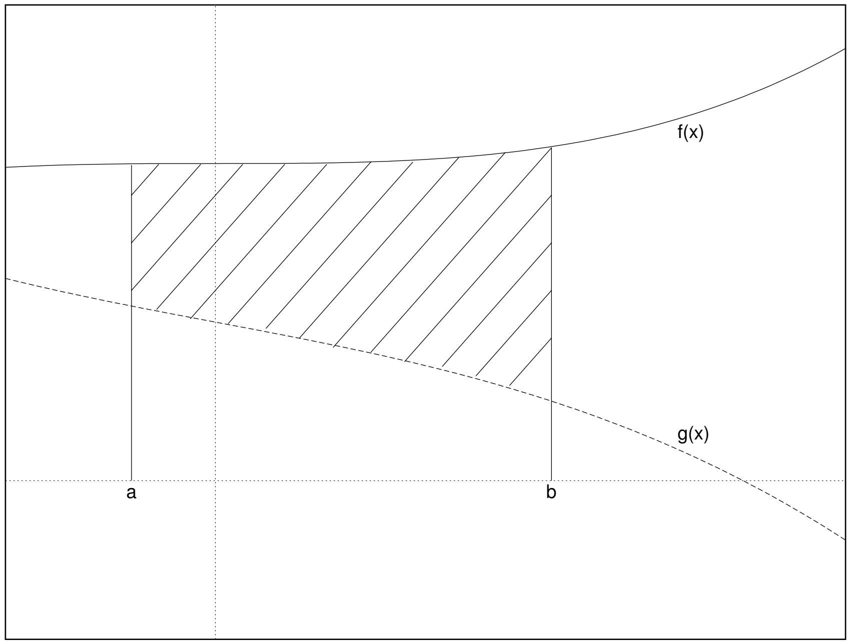

Figure 7.4: The area between two curves.

Suppose $f(x) \geq g(x)$ for $a \leq x \leq b$. Then the area between the curves is given by

$$\text{Area} = \int_a^b (f(x) - g(x))\, dx.$$

Example. Find the area between the graphs of $y = x^2$ and $y = 3x$ for $1 \leq x \leq 3$ (Figure 7.5).
Solution. We first see which function is the greater. The graphs of the functions intersect when $x^2 = 3x$. Thus $0 = x^2 - 3x = x(x-3)$, so $x = 0$ or 3. For $0 \leq x \leq 3$, we have $3x \geq x^2$; and so the area between the graphs for $1 \leq x \leq 3$ is

$$\int_1^3 (3x - x^2)\, dx = \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_1^3 = \left(\frac{27}{2} - 9\right) - \left(\frac{3}{2} - \frac{1}{3}\right) = \frac{10}{3}.$$

7.4 Where does the "natural" logarithm really come from?

By this stage of our tour through calculus, we have met the "Fundamental Theorem" that relates the concept of definite integration (involving area calculation as a special case) to the problem of "differentiating backwards", or anti-differentiation. Our most useful basic rule in the game of anti-differentiation is the reverse power rule (cf. section 7.1), which says

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} \ + \ C \ .$$

This rule applies, like the ordinary power rule for forwards differentiation, to a power function x^n for any real number n, **except** $n = -1$, since in this case the fraction $\frac{1}{n+1}$ isn't defined. So how do we find $\int \frac{1}{x} dx$?

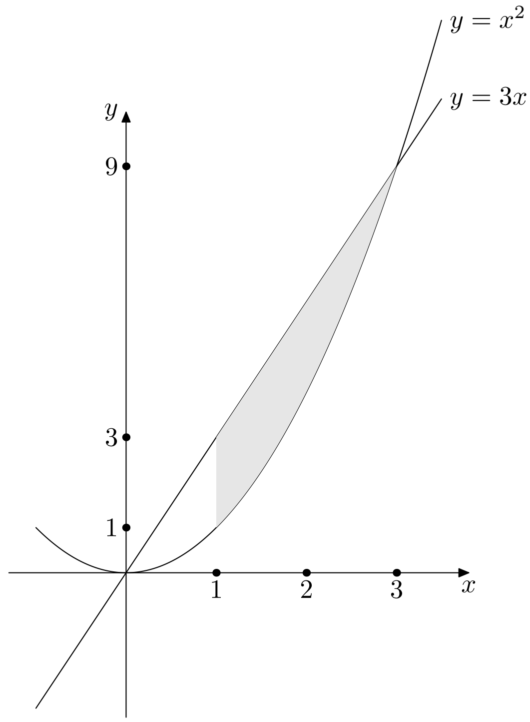

Figure 7.5: The area between $y = x^2$ and $y = 3x$, $1 \leq x \leq 3$.

The Fundamental Theorem of Calculus tells us that every reasonable function has an indefinite integral. Just choose a convenient number for which the function $\frac{1}{x}$ is well-defined, say $x = 1$, and define

$$F(b) \; := \; \int_1^b \frac{1}{x} dx \; .$$

It may come as a surprise that this is actually the true definition of $\ln(b)$, for any number $b > 0$. We write

$$\ln(b) \; := \; \int_1^b \frac{1}{x} dx \; .$$

The natural logarithm function was introduced back in chapter 3, but its place among the logarithms has until now been something of a mystery, especially because its base, the irrational number $e = 2.718....$, seems to have come from nowhere. But if this is how $\ln(b)$ is really defined, what does it have to do with logarithms, and why is its base e ? We will answer this question by showing that the formula $\int_1^b \frac{1}{x} dx$ has properties just like a logarithm, though this isn't obvious when you look at the definition.

(1) We begin by recalling that any logarithm must have the property

$$\log_a(1) = 0 \; .$$

If we set $b = 1$ in our definition, we see that

$$\ln(1) = \int_1^1 \frac{1}{x} dx = 0 \; ,$$

simply because the definite integral has no width if we integrate from 1 to 1.

(2) Another basic property of all logarithms is that

$$\log_a(a) = 1 \ .$$

In other words, the "base" of a logarithm is the particular number whose logarithm equals 1 (e.g., $\log_{10}(10) = 1$, $\log_2(2) = 1$). This is actually how we *define* e : it is the particular number for which

$$\ln(e) = \int_1^e \frac{1}{x} dx = 1 \ .$$

We can work out the value of this number to any order of accuracy, but the fact is that its decimal expansion never terminates or repeats itself, which is why we call it an "irrational" number.

(3) It is necessary that

$$\log_a(b.c) = \log_a(b) + \log_a(c) \ ,$$

$$\log_a(\frac{b}{c}) = \log_a(b) - \log_a(c) \ ,$$

$$\log_a(b^n) = n.\log_a(b) \ .$$

Showing that our definition of $\ln(b)$ really satisfies these properties is a bit more challenging, but we will carry it out for the first of these relations, and indicate that a similar argument will do the job for each of the other two. Recall from section 7.1 that for any function $f(x)$

$$\frac{d}{dx}(\int f(x)dx) = f(x) \ ,$$

and in particular, our definition tells us that

$$\frac{d(\ln(x))}{dx} = \frac{d}{dx}(\int \frac{1}{x} dx) = \frac{1}{x} \ .$$

Now write $v = b.u$, so that

$$\ln(v) = \ln(b.u) = \int_1^{b.u} \frac{1}{x} dx \ .$$

Then we apply the chain rule, treating b as a constant, and u and v as variables :

$$\frac{d(\ln(b.u)}{du} = (\frac{d(\ln(v))}{du})$$

$$= (\frac{d(\ln(v))}{dv})(\frac{dv}{du}) = \frac{1}{v}(\frac{d(b.u)}{du})$$

$$= (\frac{1}{bu}).b = \frac{1}{u} \ .$$

This tells us that

$$\frac{d(\ln(bu))}{du} = \frac{d(\ln(u))}{du} \ ,$$

but what does it mean when two functions, $f(u) = \ln(bu)$ and $g(u) = \ln(u)$ have the same derivative? Clearly the function $f(u) - g(u)$ has derivative equal to zero for all u, so it must be equal to a constant, C. Hence

$$\ln(bu) = \ln(u) + C \ .$$

This relation is true for any u, so let's see what it says for $u = 1$:

$$\ln(b) = \ln(1) + C = C \ ,$$

since $\ln(1) = 0$. In conclusion

$$\ln(bu) = \ln(u) + \ln(b) \ ,$$

which verifies the basic property of logarithms via a shrewd use of the chain rule.

We can write our definition of $\ln(x)$ as an indefinite integral also:

$$\int \frac{1}{x} dx = \ln(x) + C \ ,$$

provided the variable x is always understood to be positive. Like all logarithms, $\ln(x)$ only makes sense for $x > 0$, and this is reflected in the definition, since the function $\frac{1}{x}$ has a vertical asymptote at $x = 0$.

7.5 The Average Value of a Function

The quantity

$$\overline{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

can be thought of as the **average value** of $f(x)$ on the interval $a \leq x \leq b$. For example, if f is positive for $a \leq x \leq b$, then $\overline{f}$ is the height of the rectangle whose base is the interval $a \leq x \leq b$, and whose area is the same as the area under the graph of f. See Figure 7.6.

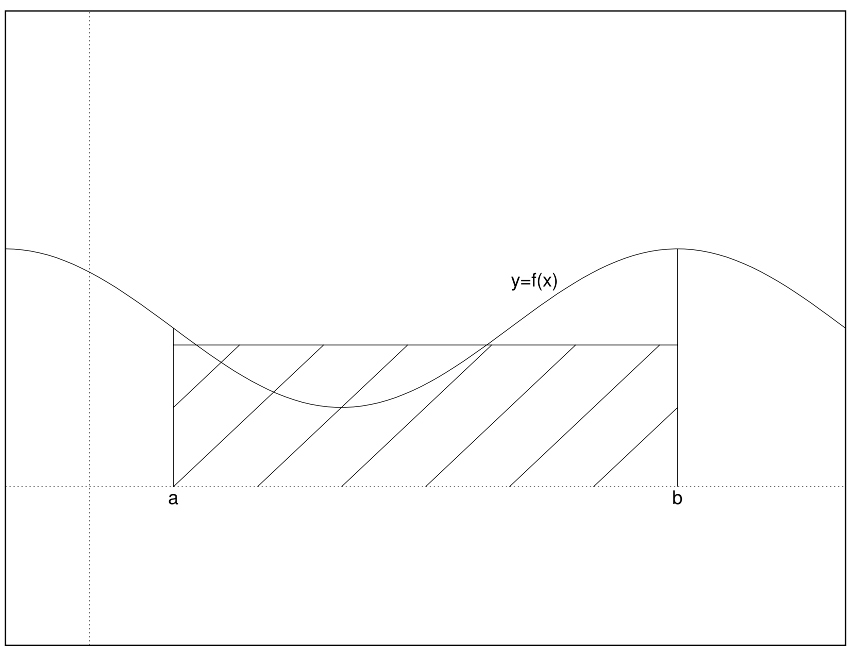

Figure 7.6: The average value of a function

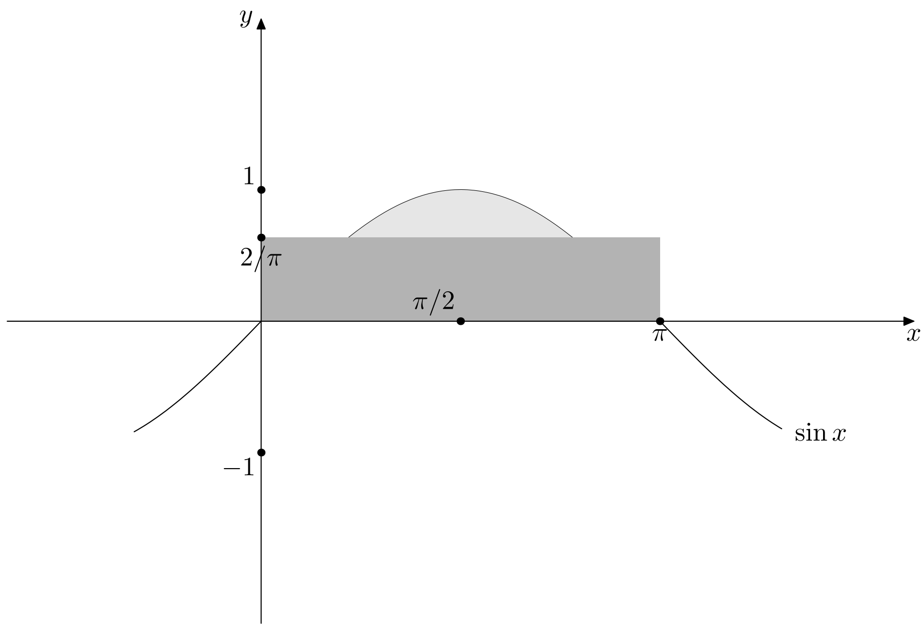

Figure 7.7: The average value of $\sin(x)$, $0 \leq x \leq \pi$.

Example. Find the average value of $f(x) = \sin(x)$ for $0 \leq x \leq \pi$.
Solution. See Figure 7.7.

$$\begin{aligned}
\overline{f} &= \frac{1}{\pi - 0} \int_0^\pi \sin(x)\, dx \\
&= \frac{1}{\pi} [-\cos(x)]_0^\pi \\
&= \frac{1}{\pi} \{-\cos(\pi) - (-\cos(0))\} \\
&= \frac{1}{\pi} (-(-1) - (-1)) \\
&= \frac{2}{\pi}
\end{aligned}$$

7.6 Exercises

1. Find the indefinite integrals with respect to x of the following functions:

(a) x^7

(b) $\sqrt{x}$

(c) $\dfrac{1}{\sqrt{x}} + \dfrac{3}{x^2}$

(d) $2\, e^{-x/2}$

(e) $3\sin\left(\frac{x}{2}\right) + 2\cos(2x)$

(f) $x^7 + 7x + \dfrac{x}{7} + \dfrac{7}{x}$

2. Find the indefinite integrals of the following functions with respect to the variable involved:

(a) $\dfrac{(2t+1)^2}{3t}$

(b) $\dfrac{2y^3 + 7y^2 - 6y + 9}{3y}$

(c) $(x+1)(2x-1)$

(d) $\dfrac{(t-t^2)^2}{t\sqrt{t}}$

3. Match each entry in the first column with an entry in the second column.

(1a)	$\int(3x+1)^3\,dx$	(2a)	$\frac{1}{6}(3x+1)^2 + C$
(1b)	$\int(3x+1)\,dx$	(2b)	$\frac{1}{2}(3x+1)^2 + C$
(1c)	$\int 3(3x+1)\,dx$	(2c)	$\frac{1}{4}(3x+1)^4 + C$
(1d)	$\int 3(3x+1)^3\,dx$	(2d)	$\frac{1}{12}(3x+1)^4 + C$

4. A stone is dropped from a cliff. After t seconds its speed is $32t$ feet per second. How far did it travel in the first 4 seconds?

5. A bullet is fired vertically upward. Its initial velocity is 1000 feet per second. Let h be the height after t seconds. A law of physics tells us that $\frac{d^2h}{dt^2}$ (i.e. the acceleration of the bullet) is always equal to -32. When does the bullet reach its maximum altitude? What is the maximum altitude? How long before the bullet returns to earth?

6. A patient's heart rate t minutes after the completion of exercise is approximately $150 - 3t$ beats per minute, for $0 \le t \le 20$. Each beat pumps 0.07 litres of blood through the heart. How much blood flows through the heart during the first 20 minutes after exercise?

7. In each of the following find the area bounded by the curve $y = f(x)$ and the x-axis for the given interval:

(a) $y = \cos(x),\ 0. \le x \le \frac{\pi}{2}$

(b) $y = e^{2x} + 7,\ -2 \le x \le 2$

(c) $y = 3\sqrt{x} + \frac{1}{x},\ 1 \le x \le 8$

8. For each of the following find the definite integral of the function $y = f(x)$ over the given interval, and also find the area between the curve $y = f(x)$ and the x-axis for the given interval. (Note: Sketches of the graphs of the functions will assist in answering the second part of this question.)

(a) $y = -x^2,\ 0 \le x \le 3$

(b) $y = x^3,\ -2 \le x \le 2$

(c) $y = \sin(x),\ 0 \le x \le 3\pi$

(d) $y = x^2 - 3x + 2,\ 0 \le x \le 4$

9. Find the area between the following pairs of curves over the given intervals:

(a) $y = 2x^2,\ y = 5x,\ 1 \le x \le 2$

(b) $y = e^x,\ y = x^2,\ 0 \le x \le 1$

10. Suppose a patient is administered a certain dose of a drug at time $t = 0$, and the resulting blood concentration of the drug at time t is given by $f(t)$. If t_1 and t_2 are two fixed times, then the area under the graph of $f(t)$ for $t_1 \le t \le t_2$ (which equals $\int_{t_1}^{t_2} f(t)\,dt$) is regarded as a measure of how effective the drug is over the time interval from t_1 to t_2 (i.e. the bigger the area, the more effective the drug).

(i) Suppose a doctor has a choice of administering equal doses of two drugs, A and B. If the blood concentration of drug A at time t is given by $10\,e^{-t}$, and of drug B is given by $10\,e^{-t/2} - 8\,e^{-2t}$, which drug is the more effective over the time interval from $t = 0$ to $t = 1$?

(ii) As for (i), but over the interval from $t = 0$ to $t = 5$.

(iii) What is the average concentration of each drug over the time interval?

Chapter 8

Differential Equations

8.1 Exponential Growth and Decay

For two related variables x and y:

- If we know how y varies with x we can work out the rate of change of y with respect to x (by differentiating).
- If we know how the rate of change of y varies with x we can work out y (by integrating).

  Another common situation is:
- We know how the rate of change of y varies with y.

Exponential growth

Example

Let $y(t)$ be the population of a country at time t. Let B be the annual birth rate per head of population and let D be the annual death rate per head. The statistics B and D are usually nearly constant and known quite accurately. How will the population $y(t)$ change with time?

$$\begin{aligned} \text{No. of births per year} &= By \\ \text{No. of deaths per year} &= Dy \end{aligned}$$

So rate of change of population per year is

$$By - Dy = ky,$$

where $k = B - D$ is called the **specific growth rate** of the population and assumed to be constant. Thus the rate of change of the population y with respect to time t is

$$\boxed{\frac{dy}{dt} = ky.}$$

This is a **differential equation** since it is an equation involving the function y and its derivative $\frac{dy}{dt}$.

We can solve this differential equation, i.e. find y as a function of t, by a trick called "separation of variables", which involves putting y's on one side of the equation and t's on the other.

Divide by y:

$$\frac{1}{y}\frac{dy}{dt} = k,$$

multiply by "dt":

$$\frac{1}{y}\,dy = k\,dt,$$

integrate:

$$\begin{aligned} \int \frac{1}{y}\,dy &= \int k\,dt \\ \ln y &= kt + c \\ e^{\ln y} &= e^{kt+c} \\ y &= e^{kt+c} = e^{kt}e^{c} = Ae^{kt}. \end{aligned}$$

Note that (1) the constant of integration in the second step above is needed on only one side of the equation (since the constant on the other side of could be absorbed into this one), and (2) the constant expression e^c in the final step can be relabelled as a constant A which is necessarily positive (since e^c is always positive).

Thus the solution of the differential equation is

$$\boxed{y = Ae^{kt}}$$

where $A = e^c$ is an unknown constant. If the population at time zero is given, say $y(0) = y_0$, then from the solution we find

$$y(0) = Ae^{k.0} = A$$

so that $A = y_0$ and

$$y = y_0 e^{kt}.$$

This is the equation of **simple exponential growth**.

Example

A culture of bacteria increases its population by one fifth daily. If the initial population is 10,000 how large is the population after 10 days? How long does it take the population to reach 1,000,000?

Solution

Here $k = \frac{1}{5}$ so the differential equation is

$$\frac{dy}{dt} = \frac{1}{5}y$$

with initial population 10,000 which has the solution

$$y = 10,000e^{\frac{1}{5}t}$$

When $t = 10$,

$$y = 10,000e^{(1/5)t} = 10,000e^2 \approx 73,891.$$

So the population after 10 days is about 73,900.

When $y = 1,000,000$,

$$\begin{aligned} 10,000e^{\frac{1}{5}t} &= 1,000,000 \\ e^{\frac{1}{5}t} &= 100 \\ \frac{1}{5}t &= \ln 100 \\ t &= 5 \ln 100 \approx 23. \end{aligned}$$

Thus the population reaches 1,000,000 after 23 days.

Exponential decay

Each kind of radio-active atom has a certain probability p of decaying in any given second. So in a sample of y atoms an expected number py decay per second.

$$\text{Rate of change of } y \text{ per sec.} = -py$$

$$\frac{dy}{dt} = -py$$

$$\boxed{y = Ae^{-pt}}$$

(Like the population model with birth rate $B = 0$.)
This is the equation of **simple exponential decay**.

The **half-life** of a radio-active element is the time it takes for half the atoms in a sample to decay (see Figure 8.1). Let the half-life be T.

$$\text{When } t = 0 \text{ no. of atoms } = Ae^0 = A.$$

$$\text{When } t = T \text{ no. of atoms } = Ae^{-pT} = \tfrac{1}{2}A.$$

So

$$\begin{aligned} e^{-pT} &= \tfrac{1}{2} \\ e^{pT} &= 2 \\ pT &= \ln 2 \\ T &= \frac{1}{p} \ln 2. \end{aligned}$$

The half-life *does not depend on the size of the sample.*

When does the sample disappear altogether? When $Ae^{-pt} = 0$, i.e., never!

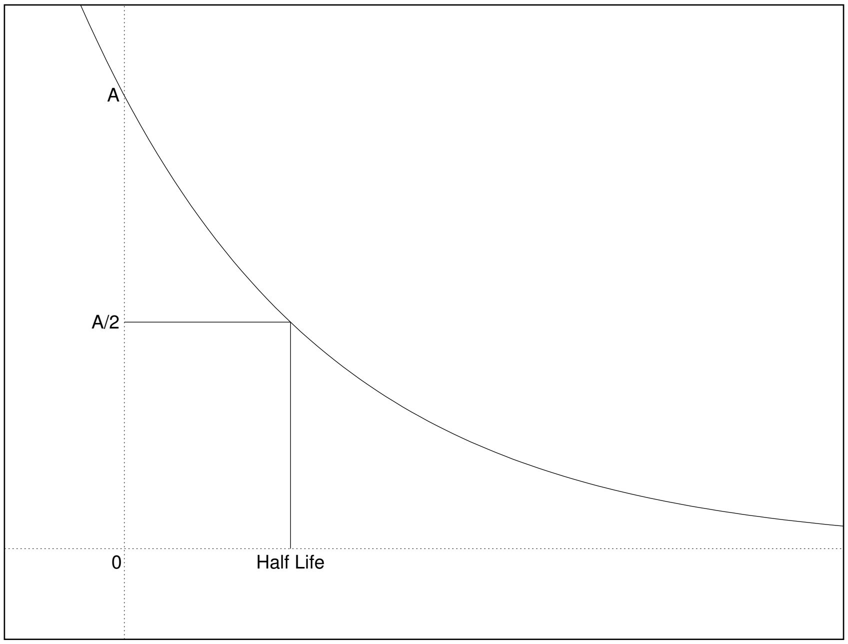

Figure 8.1: Half-life for exponential decay.

8.2 Restricted Exponential Decay

The equations of exponential growth and decay came from the situation where the rate of change of a variable y was proportional to y:

$$\frac{dy}{dx} = ky = k \times (\text{distance of } y \text{ from } 0).$$

This gave exponential growth when $k > 0$ and decay when $k < 0$. There are many situations where the rate of change of y is proportional to the distance of y from a steady value L:

$$\begin{aligned} \frac{dy}{dx} &= k \times (\text{distance of } y \text{ from } L) \\ &= k(L - y). \end{aligned}$$

where k and L are constants. Note that when $L = 0$ this reduces to equation for exponential decay

$$\frac{dy}{dx} = -ky.$$

The case when k is positive is the most important in practice and we will assume this is the case throughout this section.

Again the equation can be solved by "separation of variables":

$$\begin{aligned} dy &= k(L - y)dt \\ \frac{dy}{L - y} &= kdt \\ \int \frac{dy}{L - y} &= \int kdt \end{aligned}$$

Put $z = L - y$. then $dz = -dy$ so

$$\int \frac{dy}{L-y} = \int -\frac{dz}{z} = -\ln z + c_1 = -\ln(L-y) + c_1.$$

Hence

$$\begin{aligned} -\ln(L-y) + c_1 &= \int k dt = kt + c_2 \\ \ln(L-y) &= -kt + c \quad \text{(where } c = c_1 - c_2\text{)} \\ L - y &= e^{-kt}e^c \\ y - L &= -Ae^{-kt} \quad \text{(where } A = e^c\text{).} \end{aligned}$$

$$\boxed{y = L - Ae^{-kt}}$$

the equation of **restricted exponential decay**. The unknown constant A can be found from the starting value of y. If $y(0) = y_0$ then

$$y_0 = y(0) = L - Ae^{-0t} = L - A$$

thus

$$A = L - y_0$$

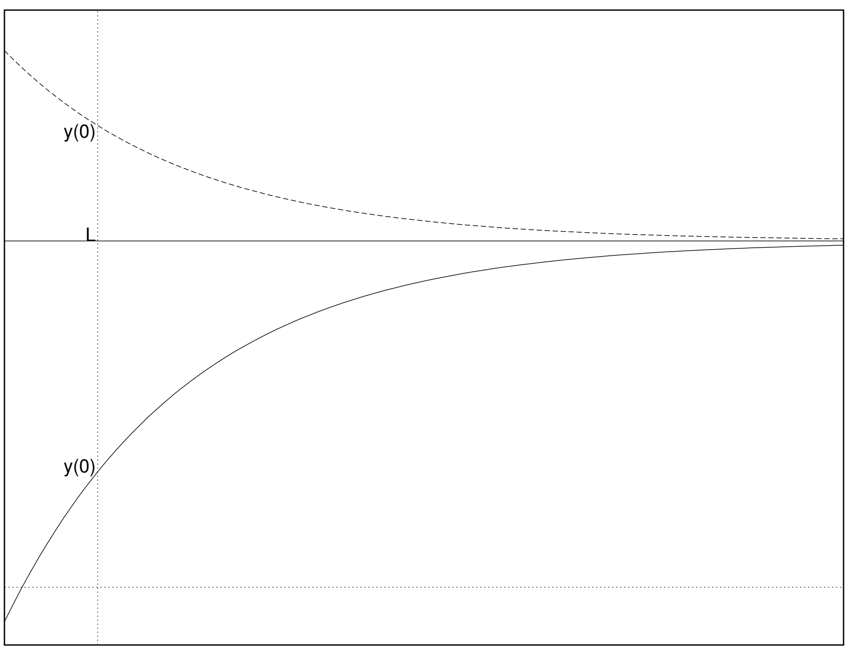

Figure 8.2: Solutions of the differential equation for restriced exponential growth.

Note that for large values of t, $y(t)$ approaches the value L.

Steady state solutions

It is much easier to find what values variables will have when a system reaches a **steady state** or **equilibrium** than to find exactly how they will change on the way to a steady state. In a steady state, when y is not changing, $dy/dt = 0$. For our equation this gives $k(L-y) = 0$ so $y = L$. Hence $y = L$ is the steady state solution of the equation. If y has the value L it stays at L.

Example

A freshly poured cup of tea has temperature 80°C. The air temperature is 20°C. Let the temperature of the tea at time t be $T(t)$. (So $T(0) = 80$.) Then $T(t)$ satisfies

$$\frac{dT}{dt} = k(20 - T)\,\text{deg/min},$$

where k is constant. Suppose $k = 0\cdot 05$. What is the temperature of the tea after 10 minutes? How long does it take to reach 40°?

Solution

The differential equation is

$$dT/dt = 0\cdot 05(20 - T),$$

so its solution is

$$T = 20 - Ae^{-0\cdot 05t}.$$

Find A:

$$A = L - T_0 = 20 - 80 = -60.$$

Therefore

$$T = 20 + 60e^{-0\cdot 05t}.$$

Find T when $t = 10$:

$$T = 20 + 60e^{-0\cdot 05\times 10} = 20 + 60e^{-0\cdot 5} = 56\cdot 4^\circ\text{C}.$$

Find t when $T = 40$:

$$\begin{aligned}
40 &= 20 + 60e^{-0\cdot 05t}\\
20 &= 60e^{-0\cdot 05t}\\
1 &= 3e^{-0\cdot 05t}\\
e^{0\cdot 05t} &= 3\\
0\cdot 05t &= \ln 3 \approx 1\cdot 1\\
t &\approx 1\cdot 1/0\cdot 05 = 22\,\text{mins}.
\end{aligned}$$

So the time taken to reach 40°C is about 22 mins.

Applications of restricted exponential growth equation

1. **Newton's Law of Cooling:** If an isolated object O has temperature T and its environment has temperature L then
$$\frac{dT}{dt} = k(L - T),$$
where k is a constant. This is illustrated in the example above.

2. **Restricted population growth.** Suppose a population y has a fixed food supply F and each individual consumes f per year. When the population is close to maximum capacity we would expect:
$$\begin{aligned} \text{Rate of increase of population} &\propto \text{excess food supply} \\ &= F - yf. \end{aligned}$$
So
$$\frac{dy}{dt} = c(F - yf) = k(L - y),$$
where $k = cf$ and $L = F/f$. For large values of t the population approaches $L = F/f$ the situation where each individual has just enough food. This equation also applies to plant populations growing in a restricted area, where it gives quite a good approximation.

8.3 The Logistic Equation

Models of population growth

1. *Simple exponential growth*
$$\frac{dy}{dt} = ky$$
This is a good approximation when y is small, but can't continue to be satisfied when y is large since the solution $y(t)$ grows rapidly forever which is physically impossible.

2. *Restricted exponential growth*
$$\frac{dy}{dt} = k(L - y)$$
This is a good approximation when y is near L, not good when y is small since when the population y is small the equation gives growth rate $\approx kL$, but the growth rate should be small when the population is small.

3. *Logistic equation* Assume the birth rate B per head is constant but the death rate D per head increases with population size y (due to reduced food ration, not enough space, more exposure to infection, for example). Let's assume $D = ky$. Then rate of change of population is
$$\frac{dy}{dt} = By - Dy = By - ky^2 = ky\left(\frac{B}{k} - y\right).$$
Put $B/k = L$. Then
$$\boxed{\frac{dy}{dt} = ky(L - y)}$$

the **logistic equation**.

When y is small $dy/dt \approx kLy \propto y$: the logistic equation behaves like simple exponential growth when y is small.

When y is near L, $dy/dt \approx kL(L-y)$: the logistic equation behaves like restricted exponential growth when y is near L.

Steady state solutions

The steady state solutions are the values of y for which $dy/dt = 0$. So $ky(L-y) = 0$ and $y = 0$ or L. If the population starts as 0 or L it stays there.

Solution of the logistic equation

Again we solve by separation of variables.

$$\frac{dy}{dt} = ky(L-y)$$

$$dy = ky(L-y)dt$$

$$\frac{dy}{y(L-y)} = kdt$$

$$\frac{L}{y(L-y)}dy = Lkdt$$

$$\left\{\frac{1}{y} + \frac{1}{L-y}\right\} dy = Lkdt \qquad (*)$$

(The last step was a "lucky guess" but you can check it's right by working backwards one step from $(*)$.)

$$\begin{aligned} \int \frac{dy}{y} &= \ln y + c_1, \\ \int \frac{dy}{(L-y)} &= -\ln(L-y) + c_2, \quad \text{(as we found in the previous section)} \\ \int Lkdt &= Lkt + c_3. \end{aligned}$$

So $(*)$ gives

$$\ln y - \ln(L-y) = Lkt + c \quad (\text{where } c = c_3 - c_1 - c_2)$$

$$\ln\frac{y}{L-y} = Lkt + c$$

$$\frac{y}{L-y} = e^{Lkt}e^c = Ae^{Lkt} \quad (\text{where } A = e^c)$$

$$y = (L-y)Ae^{Lkt}$$

$$e^{-Lkt}y = AL - Ay$$

$$(A + e^{-Lkt})y = AL$$

$$\boxed{y = \frac{LA}{A + e^{-Lkt}}}$$

The constant A can be found from the initial value of y. If $y(0) = y_0$ then

$$y_0 = y(0) = \frac{LA}{A+1}.$$

Solving this equation for A gives

$$A = \frac{y_0}{L - y_0}.$$

Note also that as t becomes large $e^{-Lkt} \to 0$ and so $y(t) \to L$.

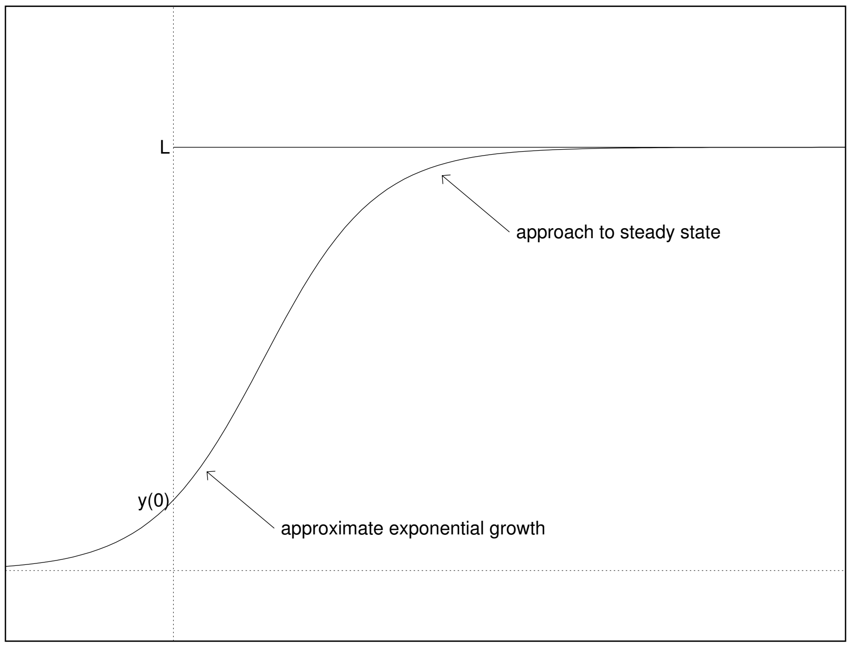

Figure 8.3: Solution of the logistic equation.

Example

When time is measured in years the world population of blue whales satisfies the logistic equation with $k = 6 \times 10^{-7}$ and steady state size 200,000. The whale population when a moratorium was introduced in 1972 was 5,000. What would we estimate the population to be in the year 2000? When can the population be expected to reach 100,000?

Solution

Let t be the time in years since 1972. The required logistic differential equation is

$$\frac{dy}{dt} = 6 \times 10^{-7} y(200,000 - y).$$

$Lk = 200,000 \times 6 \times 10^{-7} = 0 \cdot 12$, so its solution is

$$y = \frac{200,000A}{A + e^{-0 \cdot 12t}}.$$

Find A:

$$A = \frac{y_0}{L - y_0} = \frac{5,000}{200,000 - 5,000} = 0.026.$$

So

$$y = \frac{200,000A}{A + e^{-0 \cdot 12t}} = \frac{5200}{0 \cdot 026 + e^{-0 \cdot 12t}}.$$

Find y *when* $t = 28$:

$$y = \frac{5200}{0 \cdot 026 + e^{(-0 \cdot 12) \times 28}} = 85,000.$$

Find t *when* $y = 100,000$:

$$\begin{aligned}
100,000 &= \frac{5200}{0 \cdot 026 + e^{-0 \cdot 12t}} \\
1 &= \frac{0 \cdot 052}{0 \cdot 026 + e^{-0 \cdot 12t}} \\
0 \cdot 026 + e^{-0 \cdot 12t} &= 0 \cdot 052 \\
e^{-0 \cdot 12t} &= 0 \cdot 026 \\
-0 \cdot 12t &= \ln(0 \cdot 026) = -3 \cdot 66 \\
t &= \frac{3 \cdot 66}{0 \cdot 12} = 30 \cdot 5.
\end{aligned}$$

The population of blue whales will reach 100,000 by about the year 2003.

8.4 Gompertz Growth Equation

The final differential equation we will discuss provides a model which proves useful in describing the growth function connecting the weight W of an organism to time t. The basic is the differential equation

$$\frac{dW}{dt} = \mu W.$$

Now if μ were a constant, this equation would just be that of simple exponential growth (i.e. the rate of increase of W would be a constant fraction of the weight already present). However, for the Gompertz equation it is further assumed that the specific growth rate μ is not constant, but rather is itself governed by an exponential decay equation

$$\mu = \mu_0 \, e^{-Dt}.$$

Here D is a positive constant called the **decay constant** for the growth rate μ. Making this substitution for μ gives

$$\frac{dW}{dt} = \mu_0 \, e^{-Dt} W,$$

which is known as the **Gompertz differential equation**. Thus for the Gompertz equation the specific growth rate μ decays exponentially to zero.

To solve the Gompertz equation (i.e. to find a formula for W in terms of t), we again first separate the variables to obtain

$$\frac{1}{W}\,dW = \mu_0\,e^{-Dt}\,dt.$$

Integrating then gives

$$\ln(W) = \int \frac{1}{W}\,dW = \mu_0 \int e^{-Dt}\,dt = \frac{-\mu_0}{D}\,e^{-Dt} + C,$$

where C is constant. Next taking the exponential of both sides of this equation we obtain

$$W = e^{\ln(W)} = e^C\,e^{-\frac{\mu_0}{D}\,e^{-Dt}} = A\,e^{-\frac{\mu_0}{D}\,e^{-Dt}},$$

where $A = e^C$ is a constant. Now if W has the value W_0 at time $t = 0$, then

$$W_0 = A\,e^{-\frac{\mu_0}{D}},$$

and so

$$A = W_0 e^{\frac{\mu_0}{D}}.$$

Thus we have

$$W = W_0\,e^{\frac{\mu_0}{D}}\,e^{\frac{-\mu_0}{D}e^{-Dt}} = W_0\,e^{\frac{\mu_0}{D}(1-e^{-Dt})},$$

i.e. the solution to the Gompertz equation is

$$W = W_0\,e^{\frac{\mu_0}{D}(1-e^{-Dt})}.$$

We next look at what the graph of W against t looks like. First, at $t = 0$ we have $W = W_0$. Also, as $t \to \infty$, then $e^{-Dt} \to 0$, and so

$$W \to W_0\,e^{\frac{\mu_0}{D}}.$$

The quantity $W_0\,e^{\frac{\mu_0}{D}}$ is sometimes written as W_f for **final weight**. Thus, as t increases, W gets closer and closer to W_f (without actually reaching it). Next we recall that the derivative of the exponential function e^x at $x = 0$ is 1. Thus if x is near 0, then

$$\frac{e^x - 1}{x - 0} \simeq 1 \to e^x \simeq 1 + x$$

for x near 0. This means that if t is near 0, then

$$W \simeq W_0\,e^{\frac{\mu_0}{D}(1-(1-Dt))} = W_0\,e^{\mu_0 t}.$$

Therefore W starts out as approximately an exponential growth function.

Putting these observations together, the graph of W is as follows:

The logistic and Gompertz equations have similar characteristics, both have early exponential growth and approach a steady state for large t. For the same initial rate of growth and same steady state, the Gompertz equation approaches the steady state more slowly.

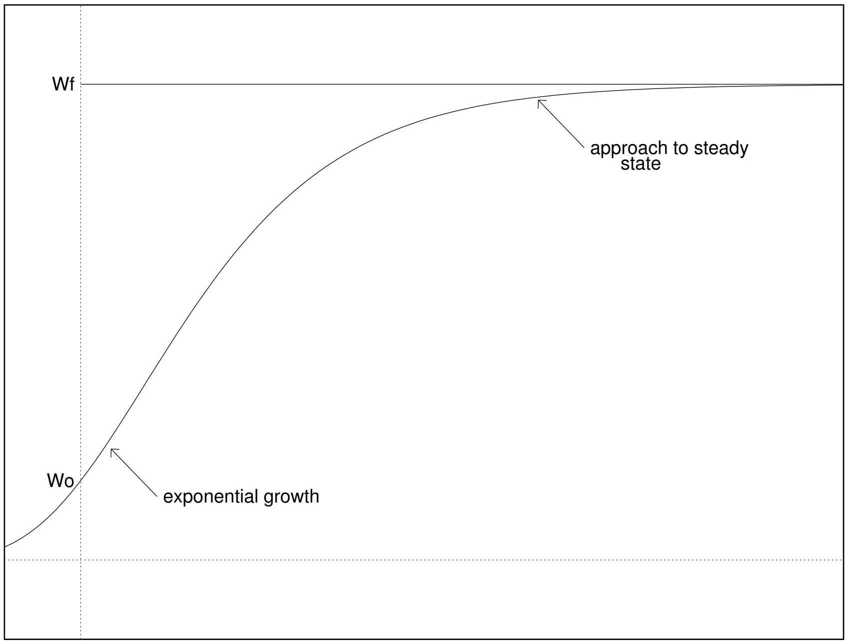

Figure 8.4: Solution of Gompertz's equation.

Example

Solve the Gompertz differential equation

$$\frac{dW}{dt} = 0.25\, e^{-0.05t} W,$$

with initial value $W(0) = 10$.

Solution

Here $\mu_0 = 0.25$, $D = 0.05$ and $W_0 = 10$ so

$$\frac{\mu_0}{D} = 5$$

and the solution is

$$W = 10\, e^{5(1-e^{0.05t})}.$$

Note that as $t \to \infty$, then $W \to 10\, e^5 \simeq 1484$, i.e. $W_f \simeq 1484$.

8.5 Exercises

1. Yeast is growing in a sugar solution in such a way that the rate of increase in the weight of yeast is equal to one-third of the weight already formed (when t is measured in hours). Describe the change in weight by a differential equation, and find the weight at time t if at $t = 0$ the weight is 1 g.

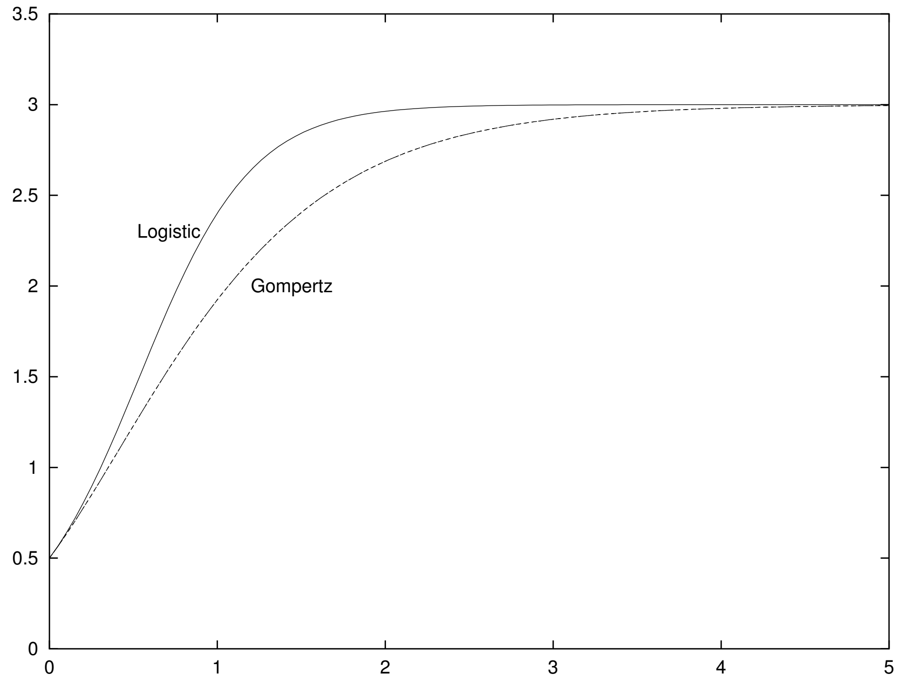

Figure 8.5: Comparison of logistic and Gompertz growth.

2. The specific growth rate of a population is 0.05/yr. If at $t = 0$ the population size is 1 million, what will be its size 20 years later?
3. Assume that the proportional growth rate $\frac{y'(t)}{y(t)}$ of the human population of the earth is constant. The population in 1930 was 2 billion and in 1975 was 4 billion. Taking 1930 to be $t = 0$, determine the population $y(t)$ of the earth at time t. According to this model, what should the population have been in 1960? When will the population reach 6 billion?
4. A bacteria population grows at a rate (cells per hour) equal to 9% of the population size. There were 1000 cells at time $t = 0$. How long will it take for the population to reach 12,000 cells?
5. Thorium is used to date coral and shells. Its decay satisfies the differential equation

$$\frac{dy}{dt} = (-9.2)10^{-6}y,$$

when t is measured in years. What is the half-life of radioactive thorium?
6. The number of worker bees of the species Bombus humilis still alive after d days from a hive initially containing 100 workers is $N = 100\,e^{-0.04d}$. How many days elapse before the number is halved?
7. Chemical tests on charred bison bones in North America showed that the bones had lost 70% of their carbon-14. Using the known decay constant for carbon-14 (i.e. 0.00012 per year), estimate the age of the bones.
8. Suppose a given environment can support a maximum population 1000 of a certain species. The differential equation

$$\frac{dy}{dt} = 0.1(1000 - y),$$

is proposed as a simple model of the population y at time t. Write down the solution of the differential equation given an initial population of 200. When will the population reach 500? Sketch the graph of the population against time.

9. On a day when the air temperature is 36° a can of coke at a temperature of 4° is taken from the fridge at 2:30pm. Thereafter the temperature of the can satisfies the differential equation

$$\frac{dT}{dt} = 0.1(36 - T),$$

where t is the number of minutes after 2:30pm. What is the temperature of the can at 2:35pm? At what time does the temperature reach 30°?

10. A city, planned to have a population 3 million, had a population of 1.0 million in the 1985 census and a population of 1.1 million in the 1990 census. We think the population y satisfies the differential equation

$$\frac{dy}{dt} = k(3 \times 10^6 - y)$$

where t is the number of years since 1985, but we don't know what k is. Determine k from the solution of the differential equation and the given data. What is the predicted population for 1990? When can we expect the population to reach 2 million? When can we expect the population to reach 3 million?

11. The population P of maggots in a rubbish tin is given by the equation

$$\frac{dP}{dt} = \frac{P^2}{100},$$

where time t is measured in hours.

(a) By separating the variables, then integrating, find a formula for P in terms of t.

(b) Your answer to (a) should involve an unknown constant. Find this constant by using the information that the initial population is 10.

(c) Hence find how long it takes for the population of maggots to become infinite.

12. The growth of a certain rabbit population follows the logistic equation

$$\frac{dy}{dt} = ky(M - y),$$

with the equilibrium size M equal to 1000, and k equal to 0.00025 when t is measured in months. The population is suddenly reduced from its equilibrium size M to a size equal to 1% of M by an epidemic of myxamatosis. How many months does it take for the population to return to 50% of its equilibrium size?

13. In a town whose population is 2,000, the spread of an influenza epidemic follows the differential equation

$$\frac{dy}{dt} = 0.002y(2000 - y),$$

where y is the number of people who have been infected by time t (measured in weeks). If there are initially 2 people infected, find y as a function of t. How long is it before three-quarters of the population have become infected?

14. Solve the Gompertz equation

$$\frac{dW}{dt} = 0.2\, e^{-0.08t} W$$

if $W = 5$ when $t = 0$. What is the equilibrium value of W?

15. In data presented by B.J. Wilson in 1980, the growth rate of male broiler chickens with initial weight 0.25 kg seemed to be very well described by the Gompertz equation

$$\frac{dW}{dt} = 0.068\, e^{-0.022t} W,$$

where t is measured in days.

(a) Solve the equation to find W in terms of t.

(b) How many days elapse before the bird reaches a weight of 4 kg?

(c) What is the asymptotic weight of the birds as time increases?

(d) For the commercial market, some of the birds are killed at the point when their rate of putting on weight is a maximum. What is the weight of the birds at this time, and after how many days do the birds reach this weight?

Chapter 9

Functions Of Several Variables

9.1 Partial Derivatives

In the solution

$$y = y_0 e^{kt}$$

of the differential equation for exponential growth it is usual to think of y as function of the variable t, ie. $y = y(t)$. However there are two other parameters y_0 and k appearing in the solution, and although we have so far considered these as constants, they can take different values and may therefore be considered as variables. That is we can think of y as a function of three variables t, y_0 and k, i.e $y = y(t, y_0, k)$.

A function of two (or more) variables is a mathematical formula f that assigns a single number z to a pair (or more) of numbers (x, y). We write

$$z = f(x, y) \ .$$

For most of this course we have referred to x as an *independent* variable, and y as a *dependent* one, meaning that x can be chosen freely from a domain of possible values, whereas y depends on the output obtained from the function. For functions of, say, two variables, the domain of free choices, or input, is made up of pairs (x, y), and the dependent output is now denoted by z. Examples of such functions could be

$$z = 3x^2 - 2x + y^3 + 7e^{xy} \ ,$$

$$z = \sin(x^2 + y^2) \ ,$$

$$z = \ln(\sqrt{4x - 3y}) \ .$$

Whereas the graph of a function $y = f(x)$ is just a curve in the two-dimensional coordinate plane, the graphs of functions $z = g(x, y)$ require a space of **three** dimensions, to display the variation of x, y and z in the form of a **surface**. Such graphs can be difficult to draw in practice, but two of the most basic examples have been represented for you in the next section. In the same way that points on the graph of $y = f(x)$ are written as a pair $(x, f(x))$, we can also denote points on the graph of $z = g(x, y)$ as **triples** $(x, y, g(x, y))$ in three-dimensional space.

Let $z = f(x, y)$ be a function of two variables. If y is kept fixed and only x varied then z becomes a function of x. The derivative of this function holding y constant is called the "**partial derivative** of z with respect to x" and is formally defined in the same way as the ordinary derivative, but with a new notation

$$\frac{\partial z}{\partial x} := \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \ .$$

Similarly, we can define the partial derivative of $f(x,y)$ in the "y-direction" (i.e., holding x fixed) by the formula

$$\frac{\partial z}{\partial y} := \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \ .$$

Just as for ordinary derivatives, $\frac{\partial z}{\partial x}$ has a different value for each point (x,y), and so also does $\frac{\partial z}{\partial y}$, hence

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}$$

is a function of x and y. The same again holds for $\frac{\partial f}{\partial y}$. These formulae may appear complicated, but partial differentiation is no more difficult in practice than ordinary differentiation. The trick is to "close one eye", in a sense, and always treat one variable (either the x or the y, depending on the direction of differentiation) as if it is a constant.

Example

Let

$$z = x^2 + y^3 + \sin(xy).$$

In order to calculate $\frac{\partial z}{\partial x}$, it may help the reader to replace y^3 by the letter C in this formula, and y by the letter D, since they are both held constant, i.e.,

$$z = x^2 + C + \sin(Dx).$$

Notice that $\frac{d(\sin(Dx))}{dx} = D\cos(Dx)$ if we apply the chain rule with $v = Dx$, while $\frac{d(x^2+C)}{dx} = 2x$. The important thing to understand is that once y, and any powers of y, have been re-assigned as constants, then partial differentiation just becomes ordinary differentiation with respect to x, so we get

$$\frac{\partial z}{\partial x} = 2x + D\cos(Dx) \ .$$

But D was only introduced as a device to help us "close one eye" to the y-variable, so we should re-instate y in the final answer

$$\frac{\partial z}{\partial x} = 2x + y\cos(xy) \ .$$

A similar game can be played with x when we calculate $\frac{\partial z}{\partial y}$, hence we may write

$$z = C + y^3 + \sin(Dy) \ ,$$

and find

$$\frac{d(C + y^3 + \sin(Dy))}{dy} = 3y^2 + D\cos(Dy) \ .$$

Here D "closes the eye" to the x-variable, so we conclude

$$\frac{\partial z}{\partial y} = 3y^2 + x\cos(xy) \ .$$

The usual rules for differentiation, for example the product and chain rules, also work for partial derivatives.

9.2 Maxima and Minima of Functions of Two Variables

Maxima and minima

Suppose a maximum of a function $f(x, y)$ of two variables occurs at the point (x_0, y_0). If we keep y fixed at y_0 then the function $f(x, y_0)$ of the one variable x will have a maximum at the point x_0. For this to be so the derivative must vanish at x_0, i.e

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0.$$

Similarly, if we keep x fixed at x_0 the function $f(x_0, y)$ will have a maximum at y_0 and

$$\frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

Thus at a maximum (or minimum) both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are zero. Points where both partial derivatives are zero are called **stationary points** of the function.

Recall from chapter 6 that our intuition about maxima and minima of functions $y = f(x)$ comes from thinking about the example of the parabola $y = ax^2 + bx + c$, where $a > 0$ means that the vertex is a minimum, and $a < 0$ means that the vertex is a maximum. On the other hand, $y = x^3$ provides the basic example of a function with a stationary point at $x = 0$, which is neither a maximum nor a minimum, but a *horizontal inflection.* The first and second examples below provide the corresponding models of behaviour at stationary points for functions $z = f(x, y)$.

Examples

1. $z = x^2 + y^2$.

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y.$$

If both partial derivatives are zero then $x = 0$ and $y = 0$ (and hence $z = 0$). So there is only one stationary point on the graph at $(0, 0, 0)$). We have sketched the graph of this function below. Notice that $z = x^2 + y^2$ is greater than zero for all pairs (x, y) different from $(0, 0)$, so the stationary point must be a minimum, as it is with the case of a concave-up parabola. If we were to take instead the example

$$z = -x^2 - y^2 \ ,$$

we would find that the point $(0, 0, 0)$ is now a maximum, as in the case of a concave-down parabola (see fig. 9.2.1).

2. $z = x^2 - y^2$.

$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = -2y.$$

If both partial derivatives are zero then $x = 0$ and $y = 0$. So again there is only one stationary point on the graph at $(0, 0, 0)$. Now $(x, y) = (0, 0)$ implies $z = 0$, but $y = 0$ implies $z = x^2 > 0$ when $x \neq 0$. So there are points close to $(0, 0)$ with larger z values and $(0, 0, 0)$ can't be a maximum. Also $x = 0$ implies $z = -y^2 > 0$ when $y \neq 0$. So there are points close to $(0, 0)$ with smaller z values and $(0, 0, 0)$ can't be a minimum. In fact $(0, 0, 0)$ is a kind of stationary point called a **saddle point** (see fig. 9.2.2).

(Fig. 9.2.1) (Fig. 9.2.2)

3. $z = x^2 + y^3 + x - 3y$.

$$\frac{\partial z}{\partial x} = 2x + 1 \text{ , and } \frac{\partial z}{\partial y} = 3y^2 - 3 \text{ .}$$

In this example we see that the general problem of finding stationary points of a function $z = f(x, y)$ boils down to solving a **system of two simultaneous equations**. Here we have to solve

$$2x + 1 = 0 \text{ and}$$

$$3y^2 - 3 = 0 \text{ .}$$

The solution to the first equation is $x = -\frac{1}{2}$, while the second equation has $y^2 = 1$, or $y = \pm 1$. The fact that we have obtained one x-value and two y-values means that we have **two** stationary points : when $x = -\frac{1}{2}$, $y = +1$, then

$$z = (-\frac{1}{2})^2 + (+1)^3 + (-\frac{1}{2}) - 3(+1) = -2\frac{1}{4} \text{ .}$$

On the other hand, $x = -\frac{1}{2}$, $y = -1$, then

$$z = (-\frac{1}{2})^2 + (-1)^3 + (-\frac{1}{2}) - 3(-1) = 1\frac{3}{4} \text{ .}$$

So the stationary points are at $(-\frac{1}{2}, 1, -2\frac{1}{4})$ and $(-\frac{1}{2}, -1, 1\frac{3}{4})$.

There are three main kinds of stationary point a function of two variables can have:

1. local maximum
2. local minimum
3. saddle point

In the next section we will apply our intuition about stationary points to devise a "second derivative test" which will help us to identify which type of stationary point we may have for a given function of two variables.

Second order partial derivatives

If $z = f(x, y)$ is a function of x and y then $\partial z/\partial x$ is also a function of x and y. We can differentiate it again with respect to either x or y to get

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}.$$

We can also differentiate $\partial z/\partial y$ to get

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}.$$

Example

$$z = x^4 + 2x^3y + 3x^2y^2 + 4xy^3 + 5y^4.$$

$$\frac{\partial z}{\partial x} = 4x^3 + 6x^2y + 6xy^2 + 4y^3, \qquad \frac{\partial z}{\partial y} = 2x^3 + 6x^2y + 12xy^2 + 20y^3.$$

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 + 12xy + 6y^2, \qquad \frac{\partial^2 z}{\partial y \partial x} = 6x^2 + 12xy + 12y^2,$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6x^2 + 12xy + 12y^2, \qquad \frac{\partial^2 z}{\partial y^2} = 6x^2 + 24xy + 60y^2.$$

We notice that $\partial^2 z/\partial y\partial x$ is the same as $\partial^2 z/\partial x\partial y$, though we obtained it in quite different ways.

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}}$$

for all (reasonable) functions z.

The matrix of second partial derivatives

$$H = \begin{bmatrix} \frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial y \partial x} \\ \frac{\partial^2 z}{\partial x \partial y} & \frac{\partial^2 z}{\partial y^2} \end{bmatrix}$$

is called the **Hessian Matrix** of the function z. In the example above the Hessian matrix is

$$H = \begin{bmatrix} 12x^2 + 12xy + 6y^2 & 6x^2 + 12xy + 12y^2 \\ 6x^2 + 12xy + 12y^2 & 6x^2 + 24xy + 60y^2 \end{bmatrix}.$$

A very convenient shorthand notation for partial derivatives of first and second order will be the following

$$\frac{\partial f}{\partial x} := f_x \ ; \ \frac{\partial f}{\partial y} := f_y \ ,$$
$$\frac{\partial^2 f}{\partial x^2} := f_{xx} \ ; \ \frac{\partial^2 f}{\partial y^2} := f_{yy} \ ,$$
$$\frac{\partial^2 f}{\partial x \partial y} := f_{xy} \ ; \ \frac{\partial^2 f}{\partial y \partial x} := f_{yx} \ .$$

Classifying stationary points

The following argument is far from being a rigorous mathematical demonstration. It is designed to get across the essential idea about classifying stationary points, as was done in chapter 6. In a sense which can be made precise, the graph of every function $z = f(x, y)$ looks like that of $z = x^2+y^2$ in a small neighbourood of any minimum (assuming that we have first translated the coordinate axes so that the minimum is situated at the origin $(0, 0, 0)$). Notice in this case that $f_{xx}(0, 0) = 2 > 0$ and also $f_{yy}(0, 0) = 2 > 0$. Similarly it looks like $z = -x^2 - y^2$ near any maximum ($f_{xx}(0, 0) = -2 < 0$ and $f_{yy}(0, 0) = -2 < 0$), and like $z = x^2 - y^2$ near any saddle point ($f_{xx}(0, 0) = 2 > 0$ and $f_{yy}(0, 0) = -2 < 0$), if these exist in the graph of $z = f(x, y)$. On the strength of these models, we might reasonably guess in general that if $(x_0, y_0, f(x_0, y_0))$ is a stationary point with $f_{xx}(x_0, y_0) > 0$ and $f_{yy}(x_0, y_0) > 0$ then it is a minimum. Similarly we might guess that a stationary point where f_{xx} and f_{yy} are both negative is a maximum, or that f_{xx} positive and f_{yy} negative (or vice versa) implies a saddle point.

While this is not far from the truth, it doesn't give sufficient information in general. Further information about the mixed derivatives f_{xy} is needed. The extra condtions involve the determinant of the Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} ,$$

$$\text{i.e., } \det(H) = f_{xx} \cdot f_{yy} - f_{yx} \cdot f_{xy} = f_{xx} \cdot f_{yy} - f_{xy}^2 \ .$$

The complete classification is: let (x_0, y_0) be a stationary point of $f(x, y)$, i.e. $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ then

1. If
$$\det(H(x_0, y_0)) < 0$$
(x_0, y_0) is saddle point.

2. If
$$\det(H(x_0, y_0)) > 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2} > 0 \quad \frac{\partial^2 f}{\partial y^2} > 0$$
(x_0, y_0) is a minimum.

3. If
$$\det(H(x_0, y_0)) > 0 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x^2} < 0 \quad \frac{\partial^2 f}{\partial y^2} < 0$$
(x_0, y_0) is a maximum.

Just as in the case of one variable, these criteria tell us nothing when $\det(H) = 0$.

Example

Find and classify the stationary points of $x^3 + y^3 - 3xy$.

Solution

$$\frac{\partial z}{\partial x} = 3x^2 - 3y, \qquad \frac{\partial z}{\partial y} = 3y^2 - 3x.$$

The stationary points are found by solving

$$\begin{aligned} 3x^2 - 3y &= 0 \\ 3y^2 - 3x &= 0. \end{aligned}$$

From the first equation we get $y = x^2$, and substituting this value of y in the second equation gives

$$3x^4 - 3x = 0,$$

$$3x(x^3 - 1) = 0,$$

$$3x(x-1)(x^2 + x + 1) = 0,$$

$$x = 0 \text{ or } x - 1 = 0.$$

(Because $x^2 + x + 1 = (x + 1/2)^2 + 3/4 > 0$ can never be 0.) So $x = 0$ or 1. From the first equation we have $y = x^2$, so $y = 0$ when $x = 0$ (hence $z = 0$) and $y = 1$ when $x = 1$ (hence $z = 1 + 1 - 3 = -1$). The stationary points are therefore $(0, 0, 0)$ and $(1, 1, -1)$. Now

$$\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial x \partial y} = -3, \quad \frac{\partial^2 z}{\partial y^2} = 6y.$$

So we can summarise as follows

Point	$(0,0,0)$	$(1,1,-1)$
f_{xx}	0	6
f_{xy}	-3	-3
f_{yy}	0	6
$\det(H)$	-9	27
Type	Saddle	Minimum

9.3 Lagrange Multipliers and constrained maxima and minima

Level curves of a function

Let's return once more to our basic examples $z = x^2 + y^2$ and $z = x^2 - y^2$. We define the *level curves* of the graph of a function $z = f(x, y)$ to be the locus of points (x, y) such that $f(x, y) = c$ for some constant c. When $f(x, y) = x^2 + y^2$ we see that $x^2 + y^2 = c$ describes the locus of a circle of radius $\sqrt{c}$, provided $c > 0$. When $z = x^2 - y^2$, the equation $x^2 - y^2 = c$ describes the locus of a hyperbola, provided $c \neq 0$ (otherwise, $x^2 - y^2 = 0$ just describes a pair of straight lines - see figure 9.3.1 below). If we now consider the graph of a second function, $z = g(x, y)$ we may want to "lift" the locus of $f(x, y) = c$ onto the graph of g, i.e., we can consider the locus of points $(x, y, g(x, y))$ in three-dimensional space, such that x and y are *constrained* by the extra condition $f(x, y) = c$. In effect, the level curve defined by this equation in the (x, y)-plane gets lifted to a "mountain pass" on the graph $z = g(x, y)$, as you can see below (fig. 9.3.2).

(Fig. 9.3.1) (Fig. 9.3.2)

Lagrange multipliers

It is a somewhat different problem to look for maxima and minima of a function $g(x, y)$ when it is constrained to one of these mountain passes. Recall that we have a stationary point (x_0, y_0) of g whenever the vector

$$\nabla g(x_0, y_0) := (g_x(x_0, y_0), g_y(x_0, y_0)) = (0, 0) \ .$$

We can think of this vector, called the **gradient** of g, as an arrow which points in the direction of increasing or decreasing values of g. In this rough sense, the gradient disappears at a stationary point because g has reached its maximum or minimum value (at least locally) on the graph. But such a point need not be the maximum or minimum value of g along a given mountain pass. For this problem, it turns out that if $(x_0, y_0, g(x_0, y_0))$ is a "constrained" maximum or minimum of g thenit must satisfy two conditions:

(1) $f(x_0, y_0) = c$ (i.e., the point lies on the given mountain pass), and

(2) the gradients of g and f *lie along the same straight line* (i.e., the arrows point in the same direction or opposite directions) through the point (x_0, y_0). In other words, we can find a number t_0 such that

$$\nabla g(x_0, y_0) = t_0 \nabla f(x_0, y_0) \ .$$

The essential idea behind these two conditions is not hard to understand. Think of all the level curves of the function $z = g(x, y)$ as "contours" in the (x, y)–plane. These are the curves defined by $g(x, y) = a$ for a fixed number "a". When we vary the value of a we produce a family of curves, just like the contours on a map, which make up a plane "projection" of the three–dimensional "landscape" of the graph $z = g(x, y)$. At each point (x_0, y_0), the gradient vector $\nabla g(x_0, y_0)$ points in the direction perpendicular to the particular level curve of g that passes through that point (ie., in the direction of increasing or decreasing values of a). In figure 9.3.3 we illustrate a family of three such curves, $g(x, y) = a$ $(a = 1, 2, 3)$, intersecting the one level curve $f(x, y) = c$. As we pass along this one level curve of f, we see that the values of g, at the places where $g(x, y) = a$ intersects $f(x, y) = c$ for $a = 1, 2, 3$, increase to the maximum value $a = 3$ at the point where the level curves $g(x, y) = 3$ and $f(x, y) = c$ are "tangent" to each other. In other words, at this point the gradients of f and g are parallel.

(Fig. 9.3.3)

In practice, to locate the constrained maximum or minimum, we have to find (x_0, y_0, t_0) which solves three equations:

(i) $f(x_0, y_0) = c$
(ii) $g_x(x_0, y_0) - t_0 f_x(x_0, y_0) = 0$
(iii) $g_y(x_0, y_0) - t_0 f_y(x_0, y_0) = 0$.

We remark that any equation of the form $f(x, y) = c$ can be rewritten $f(x, y) - c = 0$, or $\tilde{f}(x, y) = 0$, where $\tilde{f}(x, y) := f(x, y) - c$. On the other hand, $\nabla \tilde{f} = \nabla f$, hence we may rewrite these equations in the form

(i) $\tilde{f}(x_0, y_0) = 0$
(ii) $g_x(x_0, y_0) - t_0 \tilde{f}_x(x_0, y_0) = 0$
(iii) $g_y(x_0, y_0) - t_0 \tilde{f}_y(x_0, y_0) = 0$.

The number t_0 that we determine as part of the solution is called the "Lagrange multiplier" for the constrained maximum or minimum, after the French mathematician J.-L. Lagrange who first discovered this method.

Example Find the minimum value of the function $z = \ln(x^2 + y^2)$, subject to the constraint $x^2 - y^2 = 1$.(This example is quite hard, but it is a very good one to understand before you go on to try some easier exercises yourself.) First, let's find the constrained stationary points in this problem, and then we will discuss how to classify them as maxima or minima (or neither).

The equation that describes the mountain pass is $f(x, y) = x^2 - y^2 = 1$. The locus of this equation in the (x, y)-plane is a hyperbola, which we've sketched opposite. If you don't see why the locus looks like this, contact your lecturer for further discussion. On the other hand, the graph of $z = g(x, y) = \ln(x^2 + y^2)$ falls into a bottomless hole around the $z - axis$ in three-dimensional space. We've provided a sketch of this below (cf. fig. 9.3.4).

(Fig. 9.4.4)

Now

$$\frac{\partial f}{\partial x} = 2x \ ; \qquad \frac{\partial f}{\partial y} = -2y \ ,$$

and

$$\frac{\partial g}{\partial x} = \frac{2x}{x^2+y^2} \ ; \qquad \frac{\partial g}{\partial y} = \frac{2y}{x^2+y^2} \ ,$$

So the three equations we need to solve are

$$x^2 - y^2 = 1$$
$$\frac{2x}{x^2+y^2} - 2tx = 0$$
$$\frac{2y}{x^2+y^2} + 2ty = 0 \ .$$

Whatever values x and y may have, we notice right away that they cannot satisfy $x^2+y^2 = 0$ - for at least two reasons! First, this equation can only be satisfied if $x = 0$ and $y = 0$ - but the point $(0,0)$ does not lie on the mountain pass $x^2 - y^2 = 1$. On the other hand, $x^2+y^2 = 0$ means that $\ln(x^2+y^2)$ is not defined (remember that the logarithm function isn't defined at zero). This means in turn that we can multiply the second and third equations through by x^2+y^2 and also divide through by 2, to get

$$x + tx(x^2+y^2) = x(1 + t(x^2+y^2)) = 0 \qquad (*)$$
$$y - ty(x^2+y^2) = y(1 - t(x^2+y^2)) = 0 \qquad (\dagger) \ .$$

We now have two possibilities. If $x \neq 0$ then we can divide the first of these equations through by x to get

$$1 + t(x^2+y^2) = 0 \ , \qquad \text{i.e.,} \qquad x^2+y^2 = -\frac{1}{t} \ .$$

But if $y \neq 0$ we get

$$1 - t(x^2+y^2) = 0 \ , \qquad \text{i.e.,} \qquad x^2+y^2 = +\frac{1}{t} \ .$$

Hence we get a contradiction : $\frac{1}{t} = -\frac{1}{t}$ when we assume that x and y are both different from zero, and we are therefore forced to assume either (i) $x = 0$, or (ii) $y = 0$ (but not both at the same time, for the reasons already given) . In the first case, the equation (*) above is automatically solved by $x = 0$, while the equation (†) becomes $y(1 - ty^2) = 0$ or more simply $1 - ty^2 = 0$, because $y \neq 0$. Hence $y = \pm\frac{1}{\sqrt{t}}$. But now recall that when $x = 0$, the mountain pass equation becomes $-y^2 = 1$, or $y^2 = -1$, which has no solutions in y, so we are forced to discard the case $x = 0$. For the second case, however, we find that the equation (†) is solved automatically by $y = 0$, and the equation (*) now becomes $1 + tx^2 = 0$, because $x \neq 0$. Hence $x = \pm\frac{1}{\sqrt{-t}}$, where we see that t will have to be a negative number for $\sqrt{-t}$ to make sense. But now the mountain pass equation for $y = 0$ becomes $x^2 = 1$, or $x = \pm 1$, and we see that our solution for x can be consistent only if $t = -1$.

So now we have our constrained stationary points at $(x, y, z) = (1, 0, 0)$ and $(-1, 0, 0)$ (note that in both cases $z = \ln(x^2 + y^2) = \ln(1) = 0$). As a matter of fact, both of these points are minima of $\ln(x^2 + y^2)$ along the mountain pass, but how do we know this? The second derivative test only helps us to classify maxima and minima on the graph of $z = g(x, y)$ when the stationary points are unconstrained, so how do we identify constrained maxima and minima? There are various ways to address this issue, though we will not go into them in this course. Fortunately in our example we can resolve the matter quite simply.

Notice that $\ln(x^2 + y^2)$ takes constant values whenever it is restricted to circles of the form $x^2 + y^2 = c$ $(c > 0)$. The values of $\ln(x^2 + y^2)$ increase as c increases, because the graph of the natural logarithm function was seen to behave this way in chapter 3. You can see from the sketches that the hyperbola corresponding to the mountain pass $x^2 - y^2 = 1$ touches the circle $x^2 + y^2 = 1$ at the points $(-1, 0)$ and $(1, 0)$. All other points on the hyperbola must intersect with a circle $x^2 + y^2 = c$ for some $c > 1$, hence $\ln(x^2 + y^2)$ must take values $\ln(c) > 0$. For this reason, $\ln(x^2 + y^2)$ takes its *minimum* value, namely zero, at the two points $(-1, 0, 0)$ and $(1, 0, 0)$ which lie on our mountain pass.

9.4 Exercises

1. Calculate the the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ for each of the following functions:

1. x^2y^4.
2. $\sqrt{x^2 - y^2}$.
3. e^{x+3y}.
4. $\sin(xy)$.
5. $\ln(x^2 + y^2)$.
6. $\cos(x^2)/y$.

2. For each of the following functions of x and y calculate $\partial^2 f/\partial x\partial y$ and $\partial^2 f/\partial y\partial x$ separately and verify that they are the same:

1. $3x^4y^2 + 7x^2y$.
2. $\sin(xy) - \cos(xy)$.

3. Find and classify all of the stationary points of the following functions:

1. $2x^2 + xy + y^2$.
2. $2x^2 + xy - y^2$.
3. $2x^2 - 2y^2 + x^4 + y^4$.

4. (From Goldstein et al., p.396) Use the method of Lagrange multipliers to maximize the following functions under the given constraints.

1. $g(x, y) = x^2 - y^2$ under the constraint $2x + y - 3 = 0$.
2. $g(x, y) = x^2 + xy - 3y^2$ under the constraint $2 - x - 2y = 0$.