MATH110 — 2003 Exam Solutions

Question 1

(a) 30 (b) 0.7

(c) $L^2 = \begin{pmatrix} 0 & 30 \\ 0.7 & 0 \end{pmatrix} \begin{pmatrix} 0 & 30 \\ 0.7 & 0 \end{pmatrix} = \begin{pmatrix} 21 & 0 \\ 0 & 21 \end{pmatrix}$

(d) $L^2 \begin{bmatrix} 1800 \\ 749 \end{bmatrix} = \begin{bmatrix} 37800 \\ 15729 \end{bmatrix}$

(e) $L^{-1} = \frac{-1}{21} \begin{pmatrix} 0 & -30 \\ -0.7 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{10}{7} \\ \frac{1}{30} & 0 \end{pmatrix}$

(f) $L^{-1} \begin{bmatrix} 1800 \\ 749 \end{bmatrix} = \begin{bmatrix} 1070 \\ 60 \end{bmatrix}$

Question 2

(a)
$$\begin{aligned} pH = -\log_{10}(8 \times 10^{-7}) &= -(\log_{10}(2^3) + \log_{10}(10^{-7})) \\ &= -(3\log_{10}(2) - 7\log_{10}(10)) \\ &\doteq -3 \times 0.3 + 7 = 6.1 \end{aligned}$$

(b) Slope $= \dfrac{\Delta y}{\Delta x} = \dfrac{7+2}{-3-1} = \dfrac{-9}{4} \Rightarrow -2 = -\dfrac{9}{4} \times 1 + b$
$\Rightarrow$ y-intercept $b = \dfrac{9}{4} - 2 = \dfrac{1}{4}$
Equation: $y = -\dfrac{9}{4}x + \dfrac{1}{4}$

(c) $x_0 = -\dfrac{b}{2a} = \dfrac{3}{2}$; $y_0 = c - \dfrac{b^2}{4a} = 4 + \dfrac{9}{4} = \dfrac{25}{4}$
$a = -1 \Rightarrow$ vertex a **maximum**

$$\textbf{Roots}: \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 \pm 5}{2} = 4 \text{ or } -1$$

sketch on page 2

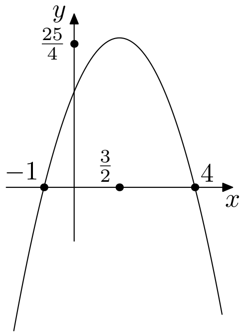

Figure 1: Graph of $y = -x^2 + 3x + 4$

(d) Frequency = 3; amplitude = 2; phase = $-\frac{1}{3}$

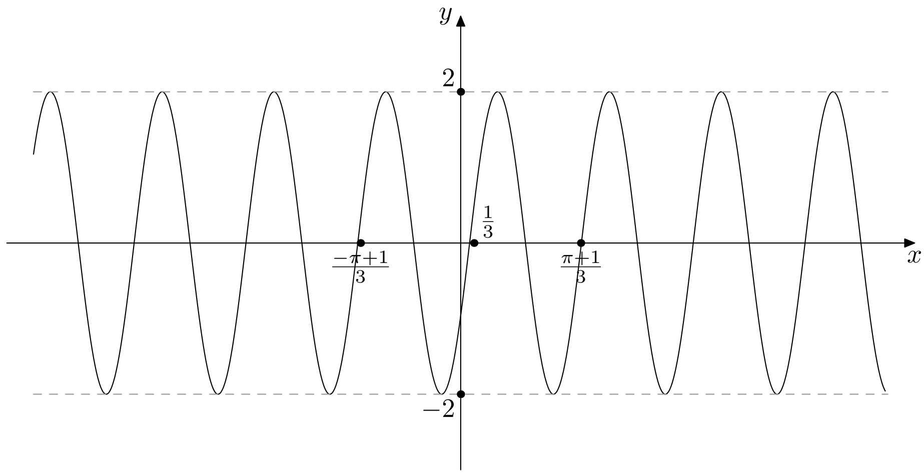

Figure 2: Graph of $y = 2\sin(3x - 1)$

Question 3

(a) $\log_2(x) = 3 \Leftrightarrow 2^3 = x \Rightarrow x = 8$

(b) $3e^x = 2 \Leftrightarrow e^x = \frac{2}{3} \Rightarrow x = \ln\left(\frac{2}{3}\right)$

(c)
$$\begin{aligned} A^{-1} &= \frac{1}{14}\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{14} & \frac{1}{14} \\ \frac{-1}{7} & \frac{2}{7} \end{pmatrix} \\ \Rightarrow \underline{X} &= \begin{pmatrix} \frac{3}{14} & \frac{1}{14} \\ \frac{-1}{7} & \frac{2}{7} \end{pmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{-1}{14} \\ \frac{5}{7} \end{bmatrix} \end{aligned}$$

(d) $4x - y = -1$ i.e. $y = 4x + 1$
$2x + 3y = 2$ i.e. $y = -\frac{2}{3}x + \frac{2}{3}$

(e)

$$\begin{aligned}
\sum(xy) &= (1.5\times 2.7)+(3.2\times 7.1)+(4.1\times 9.2)\\
&= 64.49\\
\sum x &= 8.8; \qquad \sum y = 19.0; \qquad \left(\sum x\right)^2 = 77.44\\
\sum x^2 &= 2.25+10.24+16.81 = 29.30\\
\Rightarrow b &= \frac{3\times(64.49)-8.8\times 19.0}{3(29.3)-77.44} \doteq 2.49\\
a &= \frac{1}{3}(19.0-2.49\times 8.8) \doteq -0.97\\
\Rightarrow y &= bx+a = 2.49x-0.97
\end{aligned}$$

Question 4

(a)(i) $f'(x) = -\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right)$; $f'(1) = \pi$

(ii) $f'(x) = \frac{1}{1+x^3}3x^2$; $f'(1) = \frac{3}{2}$

(iii)

$$\begin{aligned}
f'(x) &= \frac{(2x-3)(x^2+1)-(x^2-3x+2)2x}{(x^2+1)^2}\\
&= \frac{3x^2-2x-3}{(x^2+1)^2};\ f'(1) = -2
\end{aligned}$$

(b)(i) $P(0) = 10,000$
$t\to\infty \Rightarrow P(t)\to 10,000$

(ii)

$$\begin{aligned}
P'(t) = 10,000(e^{-t}-te^{-t}) &= 10,000e^{-t}(1-t)\\
&= 0 \text{ when } t = 1 = t_0
\end{aligned}$$

$$P''(t) = 10,000e^{-t}(t-2) \Rightarrow P''(t_0) = 10,000\left(-\frac{1}{e}\right) < 0$$

$\Rightarrow P(1)$ a **maximum**.

(iii) $P(t_0) = 10,000(1+\frac{1}{e}) \doteq 13700$

Question 5

(a)(i) $\int -x^2+3x+4dx = -\frac{1}{3}x^3+\frac{3}{2}x^2+4x+C$

(ii) $\int e^{-2x}dx = -\frac{1}{2}e^{-2x}+C$

(iii)

$$\begin{aligned}
\int_0^{\frac{\pi}{3}}\sin(3x)dx &= \left[-\frac{1}{3}\cos(3x)\right]_0^{\frac{\pi}{3}}\\
&= -\frac{1}{3}\cos(\pi)+\frac{1}{3}\cos(0)\\
&= \frac{1}{3}+\frac{1}{3} = \frac{2}{3}
\end{aligned}$$

(b)

$$\begin{aligned}\bar{y} &= \frac{1}{b-a}\int_a^b f(x)dx \\ &= \frac{1}{2-1}\int_1^2 \frac{1}{x}dx = \ln(2) - \ln(1) = \ln(2)\end{aligned}$$

(c)

$$\begin{aligned}4x+1 = x^2+x+3 \Rightarrow x^2-3x+2 &= 0 \\ (x-1)(x-2) &= 0\end{aligned}$$

$\Rightarrow$ Intersection of graphs at $(1,5)$ and $(2,9)$.

$$\text{Area} = \int_1^2 (4x+1) - (x^2+x+3)dx$$

$$\begin{aligned}= \int_1^2 -x^2+3x-2dx &= \left[-\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x\right]_1^2 \\ &= \left(-\frac{8}{3} + \frac{12}{2} - 4\right) - \left(-\frac{1}{3} + \frac{3}{2} - 2\right) \\ &= -\left(\frac{7}{3} + \frac{3}{2}\right) + 4 = \frac{1}{6}\end{aligned}$$

Question 6

(a) $P'(t) = 1000e^{-0.5t}$

$$\begin{aligned}0.5(10000 - P(t)) &= 0.5(2000e^{-0.5t}) \\ &= 1000e^{-0.5t}\end{aligned}$$

$\Rightarrow \dfrac{dP}{dt} = 0.5(10000 - P(t))$

(b)(i) $P(0) = 8000$

(ii)

$$\begin{aligned}10000 - 2000e^{-0.5t} &= 9000 \Rightarrow \\ e^{-0.5t} &= \frac{-1000}{-2000} = 0.5 \\ \Rightarrow -0.5t = \ln\left(\frac{1}{2}\right) &\Rightarrow t = 2\ln(2)\end{aligned}$$

(c) $f(x,y) = xy$

$\dfrac{\partial f}{\partial x} = y;\ \dfrac{\partial f}{\partial y} = x \ \Rightarrow\ \dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0.$

$H(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ \Rightarrow\ \det(H) = -1 < 0$ for **all** (x,y)

$(f_{xx} = f_{yy} = 0;\ f_{xy} = f_{yx} = 1)$