

# MATH110 — 2003 Exam

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## Question 1.

A certain species of butterfly spends one month in the “juvenile” (egg-catapillar-larva) stage, then one further month in adult-hood. The entire life cycle of these butterflies is therefore completed in two months.

The Lesley matrix of a particular population of the species is

$$\mathbf{L} = \begin{pmatrix} 0 & 30 \\ 0.7 & 0 \end{pmatrix}.$$

(a) What is the average number of eggs laid by each adult?

**2 marks**

(b) What is the estimated survival rate of juveniles through the first month?

**2 marks**

(c) Compute the matrix corresponding to  $\mathbf{L}^2$ .

**4 marks**

(d) If there are 1800 juveniles and 749 adult butterflies at the beginning of the experiment, what are the predicted numbers of juveniles and adults in the population after two months? Give your answer in **vector** form.

**4 marks**

(e) Compute the inverse matrix  $\mathbf{L}^{-1}$ .

**4 marks**

(f) What is the supposed population vector one month before the experiment began?

**4 marks**

## Question 2.

(a) Let  $x$  be the concentration of hydrogen ions (moles/litre) dissolved in a soil sample. The acidity, or “ $pH$ ”, of the sample is defined by the equation

$$pH = -\log_{10}(x).$$

Use the estimate

$$\log_{10}(2) \doteq 0.3$$

to find the  $pH$  of a sample when

$$x = 8 \times 10^{-7} (= 0.0000008).$$

**4 marks**

(b) Find the equation of the straight line passing through  $(1, -2)$  and  $(-3, 7)$ . With your answer indicate the value of the slope and  $y$ -intercept of this line.

**4 marks**

**Question 2 continued**

(c) Find the coordinates of the vertex of the parabola described by

$$y = -x^2 + 3x + 4$$

and indicate whether the vertex corresponds to a maximum or minimum value of the function. Sketch the graph of this parabola, showing the  $x$ -intercepts, or “roots” of the function.

**6 marks**

(d) Consider the generalised trigonometric function

$$y = 2 \sin(3x - 1).$$

Indicate the frequency, amplitude, and phase shift, then sketch the graph. Be sure to display at least two  $x$ -intercepts, and the maximum and minimum  $y$ -values, on your sketch.

**6 marks**

**Question 3.**

Solve for  $x$  the equations

(a)  $\log_2(x) = 3$

**2 marks**

(b)  $3e^x = 2$

**2 marks**

Consider the system of equations  $\mathbf{A}\underline{X} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ , where

$$\underline{X} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}.$$

(c) Find  $\mathbf{A}^{-1}$  and use this to solve the system for  $\underline{X}$ .

**4 marks**

(d) Write down the equations of each of the two straight lines which intersect at the point  $\underline{X}$ , as determined by the system above.

**4 marks**

(e) Use Least Squares to determine the straight line of best fit through the data points

$$(1.5, 2.7), (3.2, 7.1), (4.1, 9.2).$$

**8 marks**

**Question 4.**

(a) For each of the following functions find  $f'(x)$  and evaluate  $f'(1)$ .

(i)  $f(x) = \sin\left(\frac{\pi}{x}\right)$

**3 marks**

(ii)  $f(x) = \ln(x^3 + 1)$

**3 marks**

(iii)  $f(x) = \frac{x^2 - 3x + 2}{(x^2 + 1)}$

**3 marks**

(b) A colony of bacteria is cultured in a controlled environment so that the population is assumed to change over time according to the formula

$$P(t) = 10,000(1 + te^{-t}), \quad t \geq 0.$$

(i) What is the population at the beginning of the experiment? What happens to  $P(t)$  as  $t$  approaches infinity?

**3 marks**

(ii) At what time  $t_0$  does  $P(t_0)$  reach a stationary point? Use the Second Derivative Test to determine whether  $P(t)$  achieves a maximum or minimum value at this point.

**6 marks**

(iii) What is the approximate population of bacteria at time  $t_0$ ?

**2 marks**

**Question 5.**

(a) Compute the following integrals

(i)  $\int -x^2 + 3x + 4 dx$

**2 marks**

(ii)  $\int e^{-2x} dx$

**2 marks**

(iii)  $\int_0^{\pi/3} \sin(3x) dx$

**4 marks**

(b) Find the average value of  $f(x) = \frac{1}{x}$  over the interval  $[1, 2]$ .

**4 marks**

(c) Find the intersections of the line  $y = 4x + 1$  with the parabola  $y = x^2 + x + 3$ , and compute the area of the closed region which is bounded completely by these two graphs.

**8 marks**

**Question 6.**

(a) Show that the function

$$P(t) = 10,000 - 2,000e^{-0.5t}$$

satisfies the differential equation

$$\frac{dP}{dt} = 0.5(10,000 - P(t)) \quad (*).$$

**6 marks**

(b) The growth of a population under specific environmental constraints is described by the restricted exponential decay equation (\*) above, with solution  $P(t)$  as described in (a).

(i) What is  $P(0)$ ?

**2 marks**

(ii) At what time  $t$  is  $P(t) = 9000$ ?

**4 marks**

(c) Recall that  $(x_0, y_0)$  is a “saddle point” of a function  $f(x, y)$  if

(i)  $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$ ,  
and

(ii)  $\det(H(x_0, y_0)) < 0$ , where

$$H(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

Verify that  $f(x, y) = xy$  has a saddle point at  $(0, 0)$ .

**8 marks**

## USEFUL FORMULAE

$$\left. \begin{array}{l} \text{1. Vertex of Parabola: } x_0 = -\frac{b}{2a} \quad y_0 = c - \frac{b^2}{4a} \\ \text{Quadratic Formula: } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{array} \right\} y = ax^2 + bx + c$$

### 2. Matrices:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det(\mathbf{A}) = ad - bc$$
$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \times \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

### 3. Least Squares:

$$y = bx + a$$
$$b = \frac{N \times \sum(xy) - (\sum x)(\sum y)}{N \times \sum x^2 - (\sum x)^2}$$
$$a = \frac{1}{N}(\sum y - b \times \sum x)$$

### 4. Average Value: $\bar{y} = \frac{1}{b-a} \int_a^b f(x) dx$