MATH110 FINAL 2004 – SOLUTIONS

Question 1

(a) $\mathbb{L} = \begin{pmatrix} 0 & 20 & 12 \\ 0.75 & 0 & 0 \\ 0 & 0.8 & 0 \end{pmatrix}$

(b) $\mathbb{A}^2 = \begin{pmatrix} 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{5}{4} \\ \frac{1}{12} & 0 & -\frac{25}{12} \end{pmatrix} \begin{pmatrix} 0 & \frac{4}{3} & 0 \\ 0 & 0 & \frac{5}{4} \\ \frac{1}{12} & 0 & -\frac{25}{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1.67 \\ 0.1 & 0 & -2.6 \\ -0.17 & 0.11 & 4.34 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1.33 & 0 \\ 0 & 0 & 1.25 \\ 0.08 & 0 & -2.08 \end{pmatrix} \begin{bmatrix} 400 \\ 30 \\ 15 \end{bmatrix} = \begin{bmatrix} 40 \\ 19 \\ 1 \end{bmatrix}$

(d) $\begin{pmatrix} 0 & 0 & 1.67 \\ 0.1 & 0 & -2.6 \\ -0.17 & 0.11 & 4.34 \end{pmatrix} \begin{bmatrix} 400 \\ 30 \\ 15 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 0 \end{bmatrix}$

Question 2

(a)

$$\begin{aligned} pH = -log_{10}(10x) &= -log_{10}(10) - log_{10}(x) \\ &= -1 + 6.2 = 5.2 \end{aligned}$$

(b)

$$\begin{aligned} \text{Slope} &= \frac{5-(-1)}{-2-1} = \frac{6}{-3} = -2 \\ \therefore\ y &= -2x + a \Longrightarrow -1 = -2 + a \Longrightarrow a = 1(y - \text{intercept}) \\ &\therefore y = -2x + 1. \end{aligned}$$

(c) $x_0 = \frac{-b}{2a} = \frac{6}{2} = 3 \quad y_0 = \frac{4ac-b^2}{4a} = \frac{20-36}{4} = -4$

Vertex a **minimum** since $a = 1 > 0$.

$x - intercepts : x = \frac{6\pm\sqrt{36-20}}{2} = \frac{6\pm 4}{2} = 5$ or 1

(d)

$$\begin{aligned}\text{Frequency} &= 2\\ \text{Amplitude} &= 1 \text{ (or } -1)\\ \text{Phase } shift &= \frac{\pi}{2}\end{aligned}$$

Question 3

(a) $x^{\frac{2}{3}} = 3 \Longrightarrow x = 3^{\frac{3}{2}} = (\pm\sqrt{3})^3 = \pm 3\sqrt{3}$

(b) $3ln(x+1) = 1 \Longrightarrow x+1 = e^{\frac{1}{3}} \Longrightarrow x = e^{\frac{1}{3}} - 1.$

(c) $2e^{3x} = 5 \Longrightarrow 3x = ln\left(\frac{5}{2}\right) \Longrightarrow x = \frac{ln\left(\frac{5}{2}\right)}{3}$

(d)

$$\begin{aligned}\mathbb{A} &= \begin{pmatrix} 2 & 3 \\ -1 & 4 \end{pmatrix} \ \mathbb{A}^{-1} = \frac{1}{11}\begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}\\ \therefore \ \mathbf{X} &= \frac{1}{11}\begin{pmatrix} 4 & -3 \\ 1 & 2 \end{pmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -11 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}\end{aligned}$$

(e)

$$\begin{aligned} y &= x - 1\\ y &= x^2 - 6x + 5\\ o &= x^2 - 7x + 6 \Longrightarrow \begin{array}{lcl} x &=& 1 \text{ or } 6\\ y &=& 0 \text{ or } 5\end{array}\end{aligned}$$

Question 4

(a) (i)

$$\begin{aligned} f'(x) &= 4x^3 + 21x^2 + 2x - 1 + \frac{1}{x^2}\\ f'(1) &= 4 + 21 + 2 - 1 + 1 = 27\end{aligned}$$

(ii)

$$\begin{aligned} f'(x) &= 2\pi x \cos(\pi x^2)\\ f'(1) &= -2\pi\end{aligned}$$

(iii)

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} \cdot \ln(x) + \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}\left(\frac{1}{2}\ln(x) + 1\right) \\ f'(1) &= 1 \end{aligned}$$

(b) (i) $S(0) = 0.01$

$\lim_{t \longrightarrow \infty} S(t) = 0.01$ since $\lim_{t \longrightarrow \infty} \sqrt{t}\ e^{-t} = 0$

(ii)

$$\begin{aligned} S'(t) &= 0.01\left[\frac{1}{2\sqrt{t}}e^{-t} - \sqrt{t}e^{-t}\right] \\ &= 0 \Longrightarrow \frac{1}{2\sqrt{t}} - \sqrt{t} = 0 \Longrightarrow t = \frac{1}{2} \\ S''\left(\frac{1}{2}\right) &= 0.01e^{-\frac{1}{2}}\left[\frac{1}{\sqrt{2}} - \sqrt{2} - \frac{2\sqrt{2}}{4}\right] \\ &= 0.01e^{-\frac{1}{2}}\left[\frac{1}{\sqrt{2}} - \frac{3\sqrt{2}}{2}\right] \\ &= 0.01e^{-\frac{1}{2}}[-\sqrt{2}] < 0 \therefore t = \frac{1}{2}\ (a\ \textbf{max}.) \end{aligned}$$

(iii) $S\left(\frac{1}{2}\right) = 0.01\left[1 + \frac{e^{-\frac{1}{2}}}{\sqrt{2}}\right]$

(c) (i) $\int 3x^7 - 4x + 1\ dx = \frac{3}{8}\ x^8 - 2x^2 + x + c$

(ii) $\int_2^3 \frac{1}{x}\ dx = \ln(3) - \ln(2)$

(iii)

$$\begin{aligned} \int_0^{\frac{\pi}{6}} \cos(6x)\ dx &= \left[\frac{1}{6}\sin(6x)\right]_0^{\frac{\pi}{6}} &= \frac{1}{6}\sin(\pi) - \frac{1}{6}\sin(0) \\ && = 0 \end{aligned}$$

(d) $\bar{f} = \frac{1}{2}\int_0^2 e^{-x}\ dx = \frac{1}{2}\left[-e^{-x}\right]_0^2 = \frac{1}{2}\left[-\frac{1}{e^2} + 1\right]$

Question 5

(a)

$$\begin{aligned} A(t) &= A_0e^{-kt} \\ A'(t) &= -kA_0e^{-kt} = -kA(t) \end{aligned}$$

(b)

$$\begin{aligned} A(t) &= 23e^{-0.2t} \\ 11.5 &= 23e^{-0.2t} \Longrightarrow t = \frac{-ln(2)}{-0.2} = 5\ln(2) \text{ minutes} \end{aligned}$$

(c)

$$\begin{aligned} P(t) &= L - Ae^{-kt} \\ P'(t) &= kAe^{-kt} \therefore k(L - P(t)) = k(Ae^{-kt}) \\ & \qquad\qquad\qquad\qquad = P'(t) \end{aligned}$$

(d)

$$\begin{aligned} P(t) &= 2000 - Ae^{-0.1t} \\ P(o) &= 500 = 2000 - A \Longrightarrow A = 1500 \end{aligned}$$

$\therefore 1500 = 2000 - 1500e^{-0.1t} \Longrightarrow$

$e^{-0.1t} = \frac{500}{1500} = \frac{1}{3} \therefore t = \frac{-\ln(3)}{-0.1} = 10\ \ \ln(3)$ weeks.

Steady State $L = 2000$.

Question 6

$\dfrac{\partial f}{\partial x} = -2x + 8xy \quad \dfrac{\partial f}{\partial y} = -2y + 4x^2$

$f_{xx} = -2 + 8y \quad f_{xy} = f_{yx} = 8x \quad f_{yy} = -2$

Now $\dfrac{\partial f}{\partial x}(0,0) = \dfrac{\partial f}{\partial y}(0,0) = 0$ and

$f_{xx}(0) = -2 < 0 \quad f_{yy}(0) = -2 < 0$

$H(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \therefore detH = 4 > 0 \therefore (0,0)$ (a **max**.)