Question 1

Suppose that a species of earthworm has a maximum life span of three months. During its life a typical worm manages to reproduce twice, the first time at an average of twenty offspring per adult in the age-bracket one to two months, and then once again at an average of twelve offspring per adult in the two-to-three-months age-bracket. Suppose that the infant worms (aged between zero and one month) have a survival rate of 0.75, while adults aged one to two moths have a survival rate of 0.8.

(a) Write down the Leslie matrix $\mathbf{L}$ for this species with respect to the three age-groups above. *[4 marks]*

Consider the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1.33 & 0 \\ 0 & 0 & 1.25 \\ 0.08 & 0 & -2.08 \end{pmatrix}$.

(b) Compute the matrix $\mathbf{A}^2$, rounding the entries in your answer to two decimal places. *[6 marks]*

Suppose that a particular sample of worms of this species is isolated and left to breed for two months. At the end of this period the sample is found to consist of 400 infants, 30 aged between one and two months, and 15 aged between two and three months.

(c) If the matrix $\mathbf{A}$ above is taken to be equal to $\mathbf{L}^{-1}$, the inverse of the Leslie matrix of our species, what number of worms would there have been in each age-group one month earlier? Give your answer in vector form, with entries rounded to the nearest whole number. *[5 marks]*

(d) What would have been the population vector for this sample two months before, i.e., when the sample was first isolated? Again give your answer in vector form, rounded to nearest whole numbers. *[5 marks]*

Question 2 is on page 3

Question 2

(a) The acidity, or pH, of a soil sample is defined by the equation

$$\mathrm{pH} = -\log_{10}(x),$$

where x measures the hydrogen ion concentration of the sample. If a particular sample has $\mathrm{pH} = 6.2$, what is the pH for another sample that has a ten-times higher concentration (i.e., a concentration equal to $10x$)? Use properties of logarithms to determine your answer. *[4 marks]*

(b) Find the equation of a straight line passing through $(1, -1)$ and $(-2, 5)$. Beside your answer indicate the value of the slope and y-intercept of this line. *[4 marks]*

(c) Find the coordinates of the vertex of the parabola described by

$$y = x^2 - 6x + 5$$

and indicate whether the vertex corresponds to a maximum or a minimum of the function. Sketch the graph of this parabola, showing the location of the x-intercepts, or "*roots*", of the quadratic function. *[6 marks]*

(d) Consider the generalised trigonometric function

$$y = -\sin(2x + \pi).$$

Indicate the frequency, amplitude, and phase shift, then sketch the graph. Be sure to display at least three x-intercepts, and the maximum and minimum y-values on your graph. *[6 marks]*

Question 3 is on page 4.

Question 3

Solve the following equations:

(a) $x^{2/3} = 3$ *[3 marks]*

(b) $3 \ln(x + 1) = 1$ *[3 marks]*

(c) $2e^{3x} = 5$ *[3 marks]*

Consider the matrix equation $\mathbf{A}X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ determined by the intersection of lines $2x + 3y = -2$ and $-x + 4y = 1$.

(d) Compute $\mathbf{A}^{-1}$ and use this to solve for the vector $X = \begin{bmatrix} x \\ y \end{bmatrix}$. *[6 marks]*

(e) Find the points of intersection between the straight line $y = x - 1$ and the parabola $y = x^2 - 6x + 5$. Be sure to determine the x and y coordinates of both points. *[5 marks]*

Question 4

(a) For each of the following functions find $f'(x)$ and determine the slope of the tangent to $y = f(x)$ at $x = 1$.

(i) $f(x) = x^4 + 7x^3 + x^2 - x + 2 - \frac{1}{x}$ *[3 marks]*

(ii) $f(x) = \sin(\pi x^2)$ *[3 marks]*

(iii) $f(x) = (\sqrt{x}) \cdot (\ln(x))$ *[3 marks]*

(b) Suppose the sugar level, measured in the blood stream of an adult human after a normal meal, changes over time t according to the formula

$$S(t) = 0.01[1 + (\sqrt{t})(e^{-t})]$$

Question 4(b)(i) is on page 5

Question 4(b) continued

(i) What is the blood sugar level at time $t = 0$ (i.e., when the meal is first eaten)? Is this also the *steady state* value of the function $S(t)$? Explain your answer. *[3 marks]*

(ii) Find the stationary point of $S(t)$. Assuming that

$$\frac{d^2S}{dt^2} = (0.01)e^{-t}\left[\sqrt{t} - \frac{1}{\sqrt{t}} - \frac{1}{4\sqrt{t^3}}\right],$$

use the second derivative test to determine whether the stationary point is a local maximum or minimum. *[6 marks]*

(iii) What is the blood-sugar level at the stationary point? *[2 marks]*

(c) Compute the following integrals:

(i) $\int 3x^7 - 4x + 1\; dx$ *[2 marks]*

(ii) $\int_2^3 \frac{1}{x} dx$ *[2 marks]*

(iii) $\int_0^{\pi/6} \cos(6x)dx$ *[3 marks]*

(d) Calculate the average value of $f(x) = e^{-x}$ over the interval $[0, 2]$. *[3 marks]*

Question 5

(a) Show that the function $A(t) = A_0e^{-kt}$ satisfies the general differential equation

$$\frac{dA}{dt} = -kA(t).$$

[4 marks]

(b) Radioactive decay of a specific nuclear isotope is governed by the differential equation

$$\frac{dA}{dt} = -0.2A(t),$$

where $A(t)$ represents the amount of isotope present after t minutes. If the initial amount of the isotope is $A_0 = 23$ grams, how long will it take until the amount remaining is 11.5 grams? *[5 marks]*

Question 5 continued

(c) Show that the function $P(t) = L - Ae^{-kt}$ satisfies the general differential equation

$$\frac{dP}{dt} = k(L - P(t)).$$

[5 marks]

(d) Population growth in a particular environment is governed by the restricted decay equation

$$\frac{dP}{dt} = 0.1(2000 - P(t)),$$

where $P(t)$ represents the population after t weeks. If the initial population is 500, how long will it take for the population to reach 1500? What is the *steady state* population approached over an extended time? *[6 marks]*

Question 6

Recall that (x_0, y_0) is a local *maximum* of $z = f(x, y)$ if

(i) $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$ and

(ii) $\det(H(x_0, y_0)) > 0$, $f_{xx}(x_0, y_0) < 0$, $f_{yy}(x_0, y_0) < 0$, where

$$H(x, y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}.$$

Verify that $f(x, y) = -x^2 - y^2 + 4x^2y$ has a local maximum at $(0, 0)$. *[10 marks]*

Useful Formulae is on page

Useful Formulae

1.

$$\left.\begin{array}{ll}\textbf{Vertex of Parabola:} & x_0 = -\dfrac{b}{2a} \quad y_0 = c - \dfrac{b^2}{4a} \\[2ex] \textbf{Quadratic Formula:} & x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\end{array}\right\} y = ax^2 + bx + c$$

2. **Matrices:**

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det(A) = ad - bc$$

$$A^{-1} = \frac{1}{\det(A)} \times \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

3. **Average Value:** $\bar{y} = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)dx$