

GENERAL INFORMATION

Introduction

Welcome to MATH102. This booklet contains the important details of unit structure and administration for the unit MATH102 *Integral Calculus, Differential Equations and Introductory Statistics* as well as Assignment problems and Tutorial problems for Integral calculus and Differential equations with solutions.

Enquiries

Administrative and General Enquiries.

For all administrative matters and information in relation to your enrolment and candidature please write to, or see, the UNE Student Centre:

Student Centre

University of New England

Armidale NSW 2351

Phone: (02) 6773 4444

Fax: (02) 6773 4400

email: <mailto:studentcentre@une.edu.au>

Academic Enquiries.

For academic enquiries concerning the content and assessment requirements of this unit, contact the unit coordinator:

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School of Mathematics, Statistics and Computer Science

University of New England

Armidale NSW 2351

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email: <mailto:gerd@turing.une.edu.au>

Phone: (02) 6773 3182

I strongly prefer to be contacted by email. From my experience this is the most efficient way to solve any problems. You may email me your contact number and ask me to call back (rather than leaving voicemail messages on the phone).

Prerequisites

Topics

Integral Calculus:

Measuring and Integration, the definite integral, logarithms, exponentials and inverse trigonometric functions, techniques of integration, applications of the definite integral, curves defined parametrically, MacLaurin and Taylor expansions, power series, improper integrals, Lebesgue and Stieltjes integral.

Differential Equations:

separable equations, linear first order equations, linear second order differential equations with constant coefficients, applications of differential equations.

Introductory statistics:

Probability, Random variables, normal distribution, Poisson and Gamma distribution.

Reference Book

The recommended reference book for the Integral Calculus and Differential Equations component is *Calculus* by Howard Anton, Wiley. Please note that the 7th and following editions have Bivens and Davis as co-authors with Anton. The 8th and 9th editions exist in two versions: Early transcendentals and Late transcendentals (preferred version). It may

be purchased through the United Campus Bookshop: phone: (02) 6772 3468, fax: (02) 6772 3469. There are also second hand copies around as this book has been prescribed as a text in previous years and is also used in the follow up unit PMTH212. The text is extremely thorough and is designed for students taking calculus for the first time. If it is some time since you have done integral calculus, it is recommended that you spend some time studying the material on antiderivatives in 7.1, 7.2, 7.3, 7.4, 7.5 of Anton 6th edn. (6.1, 6.2, 6.3, 6.4, 6.5 of Anton 7th or 8th edn.) before beginning on the notes.

UNEonline

You can access an internet platform for **communication** with the unit coordinator, fellow students and for downloading unit material via the myStudy tab in the myUNE portal <https://my.une.edu.au/>. This platform will also be used for **announcements**. **Please check the Math102 website on myUNE regularly!**

You must be a currently enrolled UNE student and have registered your UNE username and password to access units through UNEonline. At the beginning of each semester, all students with a current UNE username and password are automatically loaded into the units in which they are enrolled.

To register

To access your myUNE and therefore your online units, you first need to register your username and set a password. You can register online by clicking on the 'Get access' link for new users on the UNEonline page at: <https://my.une.edu.au/>. You will need your student number. Take care to remember the password you create. Your login details take effect within three hours and you can access your online units from the first day of teaching.

List of usernames - disclaimer

Please note that your username within UNEonline appears as part of a list in some places. Only students enrolled in the unit have access to this list. Please contact your unit coordinator if you have any concerns.

Assessment

The assessment will be based on your performance in the examination (70%) and assignments (30%).

In order to pass the unit you must achieve a minimum of 50% on the examination and a final overall scaled mark of at least 50. In order to obtain a credit in the unit you must achieve a minimum of 60% on the examination and a final overall scaled mark of at least 65. In order to obtain a distinction in the unit you must obtain a final overall scaled mark of at least 75. In order to obtain a high distinction in the unit you must obtain a final overall scaled mark of at least 85.

Examination

There will be a three-hour examination in the November examination period. The examination will be worth 70% of the assessment marks. The whole semester's work is examinable. Copies of some past papers are available from the library.

The examination will consist, for the most part, of problems analogous to those in the assignments and lecture notes. It is essential that you work through all these exercises.

You will be allowed to take **five (5)** A4 sheets of *handwritten* notes. This means a maximum of ten (10) pages if both sides of each sheet are used. No printed material or photocopies.

Assignments

There are eleven assignments for the unit, and you are expected to hand in solutions to each assignment. Sample solutions will be returned with your assignment. It is important that you hand in the assignments at the nominated times, or at least make contact with us to let us know if there is some problem.

The assignments contribute 30% towards the assessment.

The standard procedure for submission is e-submission of a pdf scan of your work. This can be done through the e-submission webstite. Alternatively you may print the cover sheet from the e-submission web-site, sign it, attach it to your assignment and send it by post.

Assignment Schedule

Assignment	Latest date to be submitted/ posted by
1	30 July
2	6 August
3	13 August
4	20 August
5	27 August
6	3 September
7	24 September
8	1 October
9	8 October
10	15 October
11	22 October

REQUESTS FOR AN EXTENSION OF TIME FOR SUBMITTING AN ASSIGNMENT ARE TO BE ADDRESSED TO THE UNIT COORDINATOR.

Plagiarism

Students are warned to read the statements regarding the University's Policy on Plagiarism as set out in the following documents:

The University of New England Academic Board Policy on Plagiarism and Academic Misconduct: Coursework

<http://www.une.edu.au/policies/pdf/plagiarismcoursework.pdf>,

the document Avoiding Plagiarism and Academic Misconduct (Coursework)

<http://www.une.edu.au/policies/pdf/plagiarismstudentinfocw.pdf>,

the relevant sections on plagiarism provided in the UNE Referencing Guide

<http://www.une.edu.au/tlc/aso/students/publications/referencing.php>.

Library services

The UNE University Library has an extensive collection of books, journals and online resources. You can borrow books, obtain copies of articles and exam papers, and request advice from librarians on search strategies and information tools to use. Join the UNELibrary email list for updates on services and resources and for regular search tips. See <https://mail.une.edu.au/lists/cgi-bin/listinfo/unelibrary>. Additional information on the services and resources outlined below is available on the University Library page <http://www.une.edu.au/library>. Select either Library Services - Students - External or Library Services - Students - Local. New students are advised to consult Library Services - Students - New.

Online resources

There are many online resources available to staff and students from the University Library page at <http://www.une.edu.au/library>. Try the following resources:

- **Catalogue.** The catalogue tells you the books, journals, Reserve Collection items, audiovisual materials and other resources that are held in the Dixson and Law Library collections. You can renew your loans from within the catalogue. The catalogue includes online items accessible by clicking on the URL (web address) provided.
- **e-Reserve.** Here you will find electronic copies of articles and chapters of books recommended by lecturers as having high relevance to your unit/s of study.
- **Exam papers.** Use this to see officially released exam papers for UNE units.
- **Databases.** Databases and web search tools such as Google Scholar allow you to search for journal articles. A number of the University Library's databases include full-text, so you can read or print articles straightaway. E book databases are also found here. Choose an option from the pull-down menu.
- **Research guides.** These have been listed by discipline to make it easy to investigate what's most relevant to you. You can find highly recommended databases, websites, reference tools, assignment help and subject guides for your area of study.

- **Training and help.** Here you will find information about on-campus library tours and classes, online tutorials, library guides and a link to the Ask a Librarian service which includes help pages for frequently asked questions (FAQs). Training and help includes eSKILLS UNE, a series of lessons showing you how to find and evaluate information, write essays and correctly reference assignments. You can also try eSKILLS Plus to work on advanced information skills.

Access to online resources is restricted to UNE staff and students. Register online for a UNE username and password through the student portal, myUNE, at <https://my.une.edu.au>.

Borrowing from other university libraries

UNE students can apply for reciprocal (in-person) borrowing rights at most other Australian university libraries. There is usually a small fee for this service. For full details on how to apply in the various states of Australia, go to <http://www.une.edu.au/library/services/nbs.php>.

Your borrowing privileges are valid from January of the current academic year through to 28 February of the following academic year.

More information for off-campus students

(Students living outside the 2350, 2351 and 2358 postcode areas) There are a number of ways for off-campus students to ask the University Library to send material. Please use a separate request form for each request.

- Email the message 'OCS email forms please' (no subject) to <mailto:offcamp@une.edu.au>.
You will automatically receive email request forms. This is the best way to place requests and it is easy to set up a form with your details typed in, ready for re-use.
- Use the online request forms available from the External Students' page.
- Telephone the External Students' Library HelpLine on (02) 6773 3124.
- Fax request forms to (02) 6773 3273.

- Mail your requests to:

Document Services
 Dixson Library
 PO Box U246
 UNE NSW 2351

There is no charge for the loan of items, although you are responsible for the cost of return postage. Photocopy requests provided electronically or posted to you are free.

Support@UNE

The University is committed to providing all students with a high quality learning experience. Effective support services are available to assist you throughout the course of your studies. These services include information on academic, administrative, financial, IT, personal and resource needs. To access the information relating to these resources, go to <http://www.une.edu.au/for/current-students/#item2> and follow the links.

Timetable

There are three lectures per week on these topics throughout the semester.

The topics below correspond to the Chapter headings in the printed notes for Integral Calculus and Differential Equations. Relevant reading material from Anton is also shown.

Week	Dates	Topics as listed in printed notes	References to Anton (7,8th edn)	References to Anton (6th edn)
1	19 July to 23 July	1. Counting and Measuring 2.-3. Probability	6.1, 6.2, -	7.1, 7.2 -

Material for Assignment 1 has been completed

Week	Dates	Topics as listed in printed notes	References to Anton (7,8th edn)	References to Anton (6th edn)
2	26 July to 30 July	4. The definite Integral 5. Properties of the Integral 6. The Integral as a Function of the Upper Limit of Integration 7. Primitives	6.1, 6.2, 6.4 - 6.6	7.1, 7.2 7.4 - 7.6

Material for Assignment 2 has been completed

3	2 August to 6 August	8. The Substitution Rule 9. Applications of the Integral	6.3, 6.8 7.1-7.4, 7.6	7.3, 7.8, 8.1-8.4, 8.6
4	9 August to 13 August	10. The Logarithm 11. The Exponential and Power Functions 12. The Inverse Trigonometric Functions	6.9 4.4	7.9 4.5

Material for Assignment 3 has been completed

5	16 August to 20 August	13. Integration by Parts 14. Further Integration	8.2 8.1, 8.3-8.5	9.2 9.1, 9.3-9.5
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Material for Assignment 4,5 has been completed

6	23 August to 27 August	15. Curves Defined Parametrically 16. Polynomial Approximation	11.1- 11.4 10.8, 10.9	12.1- 12.4 11.5, 11.9
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Material for Assignment 6 has been completed

7	30 August to 3 September	16. Polynomial Approximation (cont.) 17. Improper Integrals	10.10 8.7, 8.8	10.11 9.7, 9.8
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Material for Assignment 7 has been completed

MID SEMESTER BREAK 4 September to 19 September

8	20 September to 24 September	18. Lebesgue and Stieltjes Integrals 19. The Gamma function 20. Random variables		
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Material for Assignment 8 has been completed

9	27 September to 1 October	21.-25. Distributions		
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Material for Assignment 9 has been completed

10	5 October to 8 October	26. First Order Differential Equations	9.1, 9.3	10.1, 10.3
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Material for Assignment 10 has been completed

11	11 October to 15 October	27. Second Order Differential Equations	9.4	10.3
12	18 October to 22 October	19. (continued) Second Order Differential Equations	9.4	

Material for Assignment 11 has been completed

13	25 October to 29 October	Revision		
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ASSIGNMENT PROBLEMS

ASSIGNMENT 1

Question 1.

- (a) Let A and B be sets with measures $m(A) = 1$, $m(B) = 2$ and $m(A \cap B) = \frac{1}{2}$. Find $m(A \cup B)$.
- (b) If it is known that $m(A \cup B) = m(A) + m(B)$, can we conclude $A \cap B = \emptyset$? Justify your answer.
- (c) Let m be the area of (measurable) sets in the plane. Give two different examples of non-empty sets with measure 0.

Question 2 A guided missile has five distinct sections through which a signal must pass if the missile is to operate properly. Each of the individual sections has two circuits through which the signal may pass, at least one of which must function if the signal is to traverse that section. The probability that any single circuit will fail is 0.02.

- (a) Calculate the probability that the signal passes through *any* section.
- (b) Calculate the probability that a signal passes through *all* sections, thus allowing the missile to function.

Question 3 Suppose a student who is about to take a multiple choice test has only learned 60% of the material covered by the exam. Thus, there is a 60% chance that she will know the answer to the question. However if she does not know the answer to a question, she still has a 20% chance of getting the right answer by guessing.

- (a) If we choose a question at random from the exam, what is the probability that she will get it right?
- (b) If we know that she correctly answered a question in the exam, what is the probability that she learned the material covered in the question?

Question 4. Evaluate $\sum_{n=1}^4 f(x_n)$ where $x_n = \frac{\pi n}{2}$ and $f(x) = \cos x$.

Assignment 2

Question 1. (a) Let $f(x) = x^2 + 1$, $0 \leq x \leq 3$. Find the smallest and the biggest Riemann sum for f on $[0, 3]$ with partition $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

(b) This Riemann sum is a (rough) approximation to the integral $\int_0^3 (x^2 + 1) dx$. What is the exact value of this integral?

Question 2. Find the following indefinite integrals

(a) $\int (x^2 7 - 3x^4 + x) dx$

(b) $\int \sqrt[5]{x^3} dx$

(c) $\int \frac{dx}{x \sqrt[3]{x}}$

(d) $\int (7 + x)(1 + 7x) dx$.

Question 3. Find the equation of the curve through $(2, 1)$ whose tangent (for $x > 0$) forms an angle with the x -axis, the tangent of which is equal to $\frac{1}{\sqrt[3]{x}}$. Sketch the curve for $0 < x \leq 4$.

Question 4. The velocity v of a body moving in a straight line is given by the equation $v = 3t^2 + 2t$. Find the distance s in terms of t given that $s = 10$ when $t = 2$. (Remember that $v = \frac{ds}{dt}$.)

ASSIGNMENT 3

Question 1. Evaluate

$$(a) \int x^2 \sqrt[3]{2x^3 + 3} dx \quad \text{by substituting } u = 2x^3 + 3,$$

and

$$(b) \int \frac{x dx}{a^2 + x^2} \quad \text{by substituting } u = a^2 + x^2.$$

Question 2. Evaluate the integrals by using a suitable substitution

$$(a) \int x^4 \sqrt{x^5 - 1} dx$$

and

$$(b) \int x \cos(3x^2 + 1) dx.$$

Question 3. Find the area of the region between the graph of $y = x^2 + 4$ and the x -axis over the interval $[-1, 3]$. Include a rough sketch of the region in your working.

Question 4. Find the length of the arc of the curve $y = x^{3/2}$ from the point $(1, 1)$ to the point $(4, 8)$.

Question 5. Evaluate

$$(a) \int x e^{-x^2} dx$$

$$(b) \int e^{2x} \sqrt{e^{2x} + 1} dx.$$

$$(c) \int \frac{3x^2 + 2x}{1 + x^3 + x^2} dx$$

$$(d) \int \tan x dx$$

ASSIGNMENT 4

Question 1. Evaluate

$$(a) \int_1^2 \frac{dx}{\sqrt{16-x^2}}$$

$$(b) \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{ds}{1+s^2}$$

Question 2. Evaluate

$$(a) \int \frac{x^2 dx}{1+x^6} \quad \text{by substituting } u = x^3$$

$$(b) \int \frac{x dx}{\sqrt{1-x^4}} \quad \text{by a suitable substitution}$$

$$(c) \int_0^{\pi/4} \frac{\sin \theta}{1+\cos^2 \theta} d\theta.$$

Question 3.

$$(a) \quad \text{Find } \int x \cos x dx \quad \text{by integration by parts.}$$

$$(b) \quad \text{Find } \int x^2 \sin x dx.$$

Hint: integrate by parts and use the result in (a).

Question 4. The region R is bounded by the curve $y = e^{-2x}$, the lines $x = 1, x = 2$, and the x -axis.

(a) Sketch the region R and find its area.

(b) Find the volume of the region obtained by rotating R around the x -axis.

(c) Find the volume of the region obtained by rotating R around the y -axis.

ASSIGNMENT 5

Question 1.

$$\int \frac{(2x + 3)dx}{(x - 2)(x + 3)}.$$

Question 2. Express in partial fractions and integrate with respect to x :

$$\frac{4x + 3}{(x + 2)^2}.$$

Question 3. Find

$$(a) \int \frac{5 + x^2}{1 + x^2} dx$$

Question 4. Sketch the graph of $y = |x^4 - 4|$ for $-4 \leq x \leq 4$ and find $\int_{-3}^3 |x^2 - 4| dx$.

Question 5. Classify the following functions as odd or even and in each case find $\int_{-1}^1 f(x) dx$.

$$(a) f(x) = \frac{x^4}{\sin x} \quad (b) f(x) = x^{2010} \quad (c) f(x) = \frac{\tan x}{1 + x^2 + x^6}.$$

Assignment 6

Question 1. Sketch roughly the following curves, and put names (ellipse, hyperbola, parabola) to them:

$$(a) x^2 + 4y^2 = 16 \quad (b) x^2 + 4y = 16 \quad (c) x^2 - 2y^2 = 16.$$

Question 2. Find the length of the curve $x = t - \sin t$, $y = 1 - \cos t$ for $0 \leq t \leq \pi$.

Question 3.

- (a) A curve is given in polar coordinates by $r = 2 + 2 \cos \theta$. Sketch its graph in the xy -plane.
- (b) Find the area enclosed by this curve.

Question 4. Use the MacLaurin polynomial of degree four for $\cos x$ to find $\cos(-0.01)$ approximately. Estimate the error.

Question 5. Use geometric series to

- (a) Express the repeating decimal $0.312312\dots$ as a rational number.
- (b) Express $\frac{1}{1+x^4}$ as a series in x valid for $|x| < 1$.
- (c) Express $\int_0^x \frac{dx}{1+x^4}$ as a series in x valid for $|x| < 1$.

ASSIGNMENT 7

Question 1. Write down the MacLaurin's series for e^x . Hence derive series expansions for

- (a) $x^2 e^x$
- (b) e^{-x^2}
- (c) $\int e^{-x^2} dx$

Question 2. Write down the binomial expansion for $(1+x)^k$. Hence derive series expansions for

(a) $\frac{1}{\sqrt{1-x^2}}$.

(b) $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.

Question 3. The Taylor expansion of a function f about the point x_0 is (see notes):

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$

Find the Taylor expansion of $f(x) = 1 + 3x + 5x^2 - 2x^3$ about $x_0 = -1$.

Question 4. Find, when they exist, the values of the following improper integrals:

$$(a) \int_1^{\infty} \frac{dx}{x^2} \quad (b) \int_0^1 \frac{dx}{\sqrt{x}} \quad (c) \int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$$

ASSIGNMENT 8

Question 1. (a) Find a distribution function F and its density ρ for the function $y = f(x) = \sqrt[3]{x}$ defined on $[0, 8]$.

(b) Compute $\int_0^2 y\rho(y)dy$ and $\int_0^8 f(x) dx$. Compare the results.

Question 2. Show that the Stieltjes integral $\int_{-1}^1 x^3 d|x|$ equals $\frac{1}{2}$.

Question 3. Show that

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} t^{-\frac{1}{2}} e^{-t} dt = \sqrt{\pi}.$$

Hint. Use the substitution $t = u^2$ and $\int_0^{\infty} e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$.

Question 4. A discrete random variable X has the following probability distribution :

$$P(X = x) = 1/3, \quad x = 0, 1, 2.$$

Calculate

- (a) $E(X)$
- (b) $V(X)$
- (c) $V(X^2)$

Show all working.

Question 5 The random variable X has density

$$f(x) = \frac{1}{2}e^{-x/2}, \quad x > 0$$

- (a) Find a number x_0 such that

$$P(X > x_0) = 1/2$$

- (b) Interpret your result.

ASSIGNMENT 9

Question 1. The random variable X follows the distribution given by the density

$$f(x) = cxe^{-x^2/2}, \quad x > 0$$

- (a) Determine c .
- (b) Find the expected value of X .

Question 2. The random variable X follows the Poisson distribution given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \dots$$

- (a) Verify that the given function is a probability distribution function, by showing that

$$\sum_x P(X = x) = 1$$

- (b) If $\lambda = 1$, verify that $E(X) = 1$.
- (c) When λ is very small, the term $P(X = 0)$ dominates all other terms, and the distribution is right skewed. Find the value of λ for which

$$P(X = 0) > P(X = 1) + P(X = 2) + \dots$$

Question 3.

The data in Table 1 shows the number of fires that occurred per day in 1979 in a town. The expected frequencies have been calculated on the basis of a Poisson distribution with mean 0.9, as calculated from the given data.

Number of fires	Number of days	Expected
0	151	148.4
1	118	133.6
2	77	60.1
3 or more	19	22.9
total (days)	365	365

Table 1: Observed and expected frequencies for fire data

- (a) Verify the value for the mean.
- (b) Verify the expected frequency for no fires per day.

Question 4. The following data give the number of man-hours (Y) required to complete a job as a function of units already completed (X).

X	0	1	2	3	4	5
Y	40	34	30	25	22	20

- (a) Plot the data.
- (b) Calculate the sample covariance and the sample correlation.
- (c) Are the two variables related? Give reasons!

Assignment 10

Question 1. Solve (that is, find all solutions of) the differential equation

$$\frac{dy}{dx} = \frac{x^2 y}{1 + x^3}.$$

Question 2. Solve the initial value problem

$$y^3 \frac{dy}{dx} = \sin 3x, \quad y(0) = 1.$$

Question 3. Solve

$$x \frac{dy}{dx} - 2y = x + 1 \text{ where } x > 0.$$

Question 4. Solve the initial value problem

$$\frac{dy}{dx} + y - x = 0, \quad y(0) = 1.$$

Question 5. Tests on a fossil show that 85% of its carbon-14 has decayed. Estimate the age of the fossil, assuming a half-life of 5750 years for carbon-14.

Question 6. The fish population in a lake is found to grow according to the logistic differential equation

$$\frac{dP}{dt} = 0.0001 P (10000 - P), \quad (P > 0)$$

where t is the time in years and $P = P(t)$ is the population at time t . Initially the population is 1000.

- Solve this differential equation, and express P in terms of t .
- What is the population after 5 years?
- How long does it take for the population to reach 7500?

ASSIGNMENT 11

Solve the following differential equations. Where applicable, find the solution which satisfies the given initial values.

Question 1.

$$2y'' + y' + y = 0.$$

Question 2.

$$y'' + 10y' + 25y = 0.$$

Question 3.

$$y'' + 2y' + 5y = 0, \text{ where } y(0) = 0, \quad y'(0) = 1.$$

Question 4.

$$y'' + 7y' - 8y = e^x.$$

Question 5.

$$y'' - y' - 2y = \cos x - 5 \sin x.$$

Question 6. When a mass M is set vibrating along the y -axis on the end of a spring, the distance y at time t of the mass from the equilibrium position satisfies the equation

$$M \frac{d^2y}{dt^2} + c \frac{dy}{dt} + ky = 0$$

where c is a constant depending on the frictional force, and k is a constant depending on the spring and the force of gravity.

- Solve this equation for the case when $M = 4$, $c = 4$, and $k = 101$. Assume the initial conditions $y(0) = 10$, and $y'(0) = 0$.
- Put the solution in cosine form.
- What is the period of the oscillation? (This is the time taken for the bobbing mass to complete one cycle.)
- What is the frequency of the oscillation? (This is the number of oscillations per second.)

Calculus Tutorials

Tutorial 1

1. Using axioms (S0)-(S2) on compatibility of the measure with the set operations from the Lecture Notes show that

(a) $m(A \cup B) = m(A) + m(B)$ if $A \cap B = \emptyset$.

(b) $m(A \setminus B) = m(A) - m(B)$ if $B \subset A$.

2. Evaluate

(a) $\sum_{k=1}^3 k^3$ (b) $\sum_{j=2}^6 (3j - 1)$ (c) $\sum_{i=-4}^1 (i^2 - i)$ (d) $\sum_{n=0}^5 1$.

3. Express in sigma notation

(a) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + 49 \cdot 50$ (b) $1 - 3 + 5 - 7 + 9 - 11$.

4. Evaluate

(a) $\sum_{k=1}^n n$ (b) $\sum_{i=0}^0 (-3)$ (c) $\sum_{k=1}^n kx$ (d) $\sum_{k=m}^n c$ ($n \geq m$).

5. Express $\sum_{k=4}^{18} k(k-3)$ in sigma notation with

(a) $k = 0$ as the lower limit of summation

(b) $k = 5$ as the lower limit of summation.

6. Evaluate the Riemann sum $\sum_{k=1}^n f(x_k^*)\Delta x_k$, where

$$f(x) = 4 - x^2; \quad a = -3, \quad b = 4; \quad n = 4;$$

$$\Delta x_1 = 1, \quad \Delta x_2 = 2, \quad \Delta x_3 = 1, \quad \Delta x_4 = 3;$$

$$x_1^* = -\frac{5}{2}, \quad x_2^* = -1, \quad x_3^* = \frac{1}{4}, \quad x_4^* = 3.$$

7. Find the smallest and largest values that the Riemann sum

$$\sum_{k=1}^3 f(x_k^*)\Delta x_k$$

can have on the interval $[0, 4]$ if $f(x) = x^2 - 3x + 4$, and

$$\Delta x_1 = 1, \quad \Delta x_2 = 2, \quad \Delta x_3 = 1.$$

Tutorial 2

1. Evaluate

$$(a) \int x^3\sqrt{x} \, dx \quad (b) \int \sec x(\sec x + \tan x) \, dx.$$

Check your answer by differentiating.

2. Find an equation for the curve passing through the point $(-3, 0)$ whose tangent at each point (x, y) has slope $2x + 1$.

3. Evaluate

$$(a) \int_4^9 2y\sqrt{y} \, dy \quad (b) \int_{\pi/6}^{\pi/2} \left(x + \frac{2}{\sin^2 x}\right) dx \quad (c) \int_0^2 |2x - 3| \, dx.$$

4. Find the total area between the curve $y = x^2 - 3x - 10$ and the interval $[-3, 8]$.

5. Find the average value of $\sin x$ over the interval $[0, \pi]$.

6. Let

$$F(x) = \int_1^x (t^3 + 1) dt.$$

(a) Use the Second Fundamental Theorem of Calculus to find $F'(x)$.

(b) Check your answer to part (a) by integrating and then differentiating.

7. Prove that the function

$$F(x) = \int_0^x \frac{1}{1+t^2} dt + \int_0^{1/x} \frac{1}{1+t^2} dt$$

is constant on the interval $(0, \infty)$.

Hint. Differentiate with respect to x .

Tutorial 3

1. Evaluate by making the indicated substitutions:

(a) $\int \cot x \csc^2 x dx, \quad u = \cot x;$

(b) $\int x^2 \sqrt{1+x} dx, \quad u = 1+x.$

2. Evaluate $\int t\sqrt{7t^2+12} dt$.

3. Evaluate

(a) $\int_0^1 \frac{du}{\sqrt{3u+1}}$ (b) $\int_0^{\pi/2} \sin^2 3x \cos 3x dx.$

4. For positive integers m and n , show that

$$\int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx.$$

[Hint: use a substitution.]

5. Sketch the region enclosed by the curves $y = x^2$, $y = \sqrt{x}$, $x = \frac{1}{4}$, $x = 1$ and find its area.
6. Find the volume of rotation about the x -axis of the region enclosed by the curves $y = x^2$, $x = 0$, $x = 2$, $y = 0$.
7. Find the volume of rotation about the y -axis of the region enclosed by the curves $y = x^2$, $x = 1$, $x = 2$, $y = 0$.
8. Find the arc length of the curve $y = \frac{x^4}{16} + \frac{1}{2x^2}$ between $x = 2$ and $x = 3$.

Tutorial 4

1. Simplify

$$(a) \log_2 16 \quad (b) \log_2 \frac{1}{32} \quad (c) \log_4 4 \quad (d) \log_9 3.$$

2. Solve for x :

$$(a) \ln \frac{1}{x} = -2 \quad (b) e^{-2x} - 3e^{-x} = -2.$$

3. Simplify and state the values of x for which the simplification is valid:

$$(a) e^{-\ln x} \quad (b) e^{\ln x^2} \quad (c) \ln e^{-x^2} \quad (d) \ln \frac{1}{e^x}.$$

4. Evaluate

$$(a) \int \cot x \, dx \quad (b) \int \frac{dx}{x \ln x} \quad (c) \int_{-1}^0 \frac{x \, dx}{x^2 + 5} \quad (d) \int_1^4 \frac{dx}{\sqrt{x}(1 + \sqrt{x})}.$$

5. For $y = x^{(e^x)}$, find $\frac{dy}{dx}$ by logarithmic differentiation.

6. Evaluate

$$(a) \int e^{\sin x} \cos x \, dx \quad (b) \int \pi^{\sin x} \cos x \, dx \quad (c) \int_{-\ln 3}^{\ln 3} \frac{e^x \, dx}{e^x + 4}.$$

Tutorial 5

1. Find the exact value of

$$(a) \sin^{-1} \frac{\sqrt{3}}{2} \quad (b) \cos^{-1} \frac{1}{2} \quad (c) \tan^{-1} 1.$$

2. Find $\frac{dy}{dx}$, where

$$(a) y = \sin^{-1} \frac{1}{x} \quad (b) y = \cos^{-1} \cos x \quad (c) y = \tan^{-1} \frac{1-x}{1+x}.$$

3. Evaluate

$$(a) \int_0^{1/\sqrt{2}} \frac{dx}{\sqrt{1-x^2}} \quad (b) \int \frac{dx}{1+16x^2} \quad (c) \int_1^3 \frac{dx}{\sqrt{x}(x+1)} \quad (d) \int \frac{dx}{x\sqrt{1-(\ln x)^2}}.$$

4. Use integration by parts to evaluate

$$(a) \int x e^{-x} \, dx \quad (b) \int x^2 \cos x \, dx \quad (c) \int_1^e x^2 \ln x \, dx.$$

5. Use a reduction formula to evaluate

$$(a) \int \sin^3 x \, dx \quad (b) \int_0^{\pi/4} \sin^4 x \, dx.$$

6. Suppose f is a function whose second derivative is continuous on $[-1, 1]$. Show that

$$\int_{-1}^1 x f''(x) \, dx = f'(1) + f'(-1) + f(-1) - f(1).$$

Tutorial 6

1. Complete the square and evaluate

$$(a) \int \frac{dx}{\sqrt{8 + 2x - x^2}}$$

$$(b)^* \int \frac{dx}{\sqrt{x^2 - 6x + 10}}.$$

2. Decompose into partial fractions and evaluate

$$(a) \int \frac{dx}{x^2 + 3x - 4}$$

$$(b) \int \frac{x^3 dx}{x^2 - 3x + 2}$$

$$(c) \int \frac{x^2 dx}{(x + 2)^3}$$

$$(d) \int \frac{dx}{x^4 - 16}$$

$$(e) \int \frac{x^4 + 6x^3 + 10x^2 + x}{x^2 + 6x + 10} dx.$$

Tutorial 7

1. Express the equation $x^2 + y^2 - 6y = 0$ in polar coordinates.

2. Sketch and name the following curves (given in polar coordinates):

$$(a) \theta = \frac{\pi}{6} \quad (b) r = 5 \quad (c) r = -6 \cos \theta \quad (d) r = 4 - 4 \cos \theta.$$

3. Find the area of

(a) the region in the first quadrant enclosed by the first loop of the spiral $r = \theta$, ($\theta \geq 0$), and the lines $\theta = \pi/6$ and $\theta = \pi/3$;

- (b) the region common to the circle $r = 3 \cos \theta$ and the cardioid $r = 1 + \cos \theta$.
4. Find the arc length of the curves (described parametrically)
- (a) $x = 4t + 3$, $y = 3t - 2$, $0 \leq t \leq 2$.
- (b) $x = (1 + t)^2$, $y = (1 + t)^3$, $0 \leq t \leq 1$.
5. Find the arc-length of the spiral described in polar coordinates by the equation

$$r = a\theta$$

for $\theta_0 \leq \theta \leq \theta_1$.

Tutorial 8

1. Find the fourth degree Maclaurin polynomials for the functions

$$(a) \quad e^{-2x} \quad (b) \quad \tan x \quad (c) \quad xe^x.$$

2. Find the third degree Taylor polynomial for e^x about $x = 1$.
3. Find the Maclaurin series for $\frac{1}{1+x}$ and express your answer with sigma notation.
4. Find the Taylor series for $\frac{1}{x}$ about $x = -1$ and express your answer with sigma notation.
5. Use the Maclaurin series for e^x to approximate \sqrt{e} to four decimal places.
6. Derive the Maclaurin series for $\frac{1}{(1+x)^2}$ by differentiating an appropriate Maclaurin series term by term.
7. Use any method to find the first four nonzero terms in the Maclaurin series for $e^{-x^2} \cos x$.

Tutorial 9

1. Evaluate the integrals that converge:

(a) $\int_0^{\infty} e^{-x} dx$

(b) $\int_1^{\infty} \frac{dx}{\sqrt{x}}$

(c) $\int_0^{\pi/2} \tan x dx$

(d) $\int_0^{\infty} \frac{dx}{x^2}$.

2. Use the Weierstrass M-test to show that the improper integral

$$\int_1^{\infty} \frac{\sin x}{x^2} dx$$

converges.

3. (a) Find a distribution function F and its density ρ for the function $y = f(x) = \sin x$ defined on $[0, \frac{\pi}{2}]$.

(b) Compute $\int_0^1 y\rho(y)dy$ and $\int_0^{\pi/2} f(x) dx$. Compare the results.

4. Show that the Stieltjes integral $\int_{-1}^1 x d|x|$ equals 1.

Tutorial 10

Solve the given separable differential equation. Where convenient, express the solution explicitly as a function of x .

1. $\frac{dy}{dx} = \frac{y}{x}$

2. $\sqrt{1+x^2} y' + x(1+y) = 0$

3. $e^{-y} \sin x - y' \cos^2 x = 0$

4. Solve the initial-value problem

$$y^2 t \frac{dy}{dt} - t + 1 = 0, \quad y(1) = 3 \quad (t > 0).$$

5. Polonium-210 is a radioactive element with a half-life of 140 days. Assume that a sample weighs 10 mg initially.

(a) Find a formula for the amount that will remain after t days.

(b) How much will remain after 10 weeks?

Solve the given first-order linear differential equation. Where convenient, express the solution explicitly as a function of x .

6.

$$\frac{dy}{dx} + 3y = e^{-2x}$$

7.

$$x^2 y' + 3xy + 2x^6 = 0 \quad (x > 0)$$

8. Solve the initial-value problem

$$\frac{dy}{dx} - xy = x, \quad y(0) = 3.$$

9. Find an equation of the curve in the xy -plane that passes through the point $(1, 1)$ and has slope $= \frac{y^2}{2\sqrt{x}}$.

Tutorial 11

1. Verify that $c_1 e^{2x} + c_2 e^{-x}$ is a solution of $y'' - y' - 2y = 0$ by substituting the function into the equation.

2. Find the general solution of

$$y'' + 3y' - 4y = 0.$$

3. Find the general solution of

$$y'' - 2y' + y = 0.$$

4. Find the general solution of

$$y'' + 5y = 0.$$

5. Find the general solution of

$$y'' - 4y' + 13y = 0.$$

6. Solve the initial value problem

$$y'' + 2y' - 3y = 0, \quad y(0) = 1, \quad y'(0) = 5.$$

7. Solve the initial value problem

$$y'' + 4y' + 5y = 0, \quad y(0) = -3, \quad y'(0) = 0.$$

Tutorial 12

1. Use the method of undetermined coefficients to find the general solution of the differential equation

$$y'' + 6y' + 5y = 2e^{3x}.$$

2. Use the method of undetermined coefficients to find the general solution of the differential equation

$$y'' + 3y' - 4y = 5e^{7x}.$$

3. Use the method of undetermined coefficients to find the general solution of the differential equation

$$y'' - 9y' + 20y = -3e^{5x}.$$

4. Use the method of undetermined coefficients to find the general solution of the differential equation

$$y'' + y' - 12y = 4x^2.$$

Tutorial Solutions

Tutorial 1

1. (a) From (S3) $m(A \cup B) = m(A) + m(B) - m(A \cap B)$. But $A \cap B = \emptyset$ and due to (S0) $m(A \cap B) = m(\emptyset) = 0$.

(b) If $B \subset A$ then $A = (A \setminus B) \cup B$ and $B \cap (A \setminus B) = \emptyset$. From (a) then $m(A) = m(B) + m(A \setminus B)$. Thus, $m(A \setminus B) = m(A) - m(B)$.

2. (a) $1^3 + 2^3 + 3^3 = 36$

(b) $3(2) - 1 + 3(3) - 1 + \dots + 3(6) - 1 = 5 + 8 + 11 + 14 + 17 = 55$

(c) $(16 + 4) + (9 + 3) + (4 + 2) + (1 + 1) + (0 + 0) + (1 - 1) = 40$

(d) $1 + 1 + 1 + 1 + 1 + 1 = 6$

3. (a) $\sum_{k=1}^{49} k(k+1)$

(b) $1 - 3 + 5 - 7 + 9 - 11$

$$\begin{aligned} &= (2(0) + 1) - (2(1) + 1) + \dots - (2(5) + 1) \\ &= (-1)^0(2(0) + 1) + (-1)^1(2(1) + 1) + \dots + (-1)^5(2(5) + 1) \\ &= \sum_{k=0}^5 (-1)^k(2k + 1) \\ &= \sum_{k=1}^6 (-1)^{k+1}(2k - 1) \quad (\text{alternatively}) \end{aligned}$$

4. (a) $\underbrace{n + n + \dots + n}_n = n \times n = n^2$

(b) -3

(c) $1x + 2x + \dots + nx = (1 + 2 + \dots + n)x = \frac{1}{2}n(n+1)x$ (for $n \geq 1$)

(d) $\underbrace{c + c + \dots + c}_{n-m+1} = c(n - m + 1)$

5. (a) Put $k' = k - 4$, so $k = k' + 4$. When $k = 4$, $k' = 0$; when $k = 18$, $k' = 14$. Hence,

$$\sum_{k=4}^{18} k(k-3) = \sum_{k'=0}^{14} (k'+4)(k'+4-3) = \sum_{k=0}^{14} (k+4)(k+1),$$

(since k and k' are dummy variables).

(b) As above, putting $k' = k + 1$.

6.

$$\begin{aligned} \sum_{k=1}^n f(x_k^*) \Delta x_k &= \sum_{k=1}^4 (4 - (x_k^*)^2) \Delta x_k \\ &= (4 - (-\frac{5}{2})^2)(1) + (4 - (-1)^2)(2) + (4 - (\frac{1}{4})^2)(1) \\ &\quad + (4 - (3)^2)(3) \\ &= -\frac{9}{4} + 6 + \frac{63}{16} - 15 \\ &= -\frac{117}{16} \end{aligned}$$

6. $f(x) = x^2 - 3x + 4$ has a stationary point when $f'(x) = 2x - 3 = 0$, i.e. when $x = \frac{3}{2}$. Since f and f' are continuous, maximum and minimum values of f on the intervals $[0, 1]$, $[1, 3]$ and $[3, 4]$ will occur at the endpoints of these intervals or at the stationary point. Observing that

$$f(0) = 4, \quad f(1) = 2, \quad f(\frac{3}{2}) = \frac{7}{4}, \quad f(3) = 4, \quad f(4) = 8 :$$

the smallest value for the Riemann sum occurs when we take $x_1^* = 1$, $x_2^* = \frac{3}{2}$, $x_3^* = 3$, giving

$$\sum_{k=1}^3 f(x_k^*) \Delta x_k = (2)(1) + (\frac{7}{4})(2) + (4)(1) = \frac{19}{2};$$

the largest value for the Riemann sum occurs when we take $x_1^* = 0$, $x_2^* = 3$, $x_3^* = 4$, giving

$$\sum_{k=1}^3 f(x_k^*) \Delta x_k = (4)(1) + (4)(2) + (8)(1) = 20.$$

Tutorial 2

$$1. \text{ (a) } \int x^3 \sqrt{x} \, dx = \int x^3 x^{\frac{1}{2}} \, dx = \int x^{\frac{7}{2}} \, dx = \frac{x^{\frac{9}{2}}}{\frac{9}{2}} + C = \frac{2x^{\frac{9}{2}}}{9} + C$$

$$\text{(b) } \int \sec x (\sec x + \tan x) \, dx = \int \sec^2 x + \sec x \tan x \, dx = \tan x + \sec x + C$$

2. $\frac{dy}{dx} = 2x + 1$, so $y = \int 2x + 1 \, dx = x^2 + x + C$. When $x = -3$, $y = 0$, so $0 = (-3)^2 + (-3) + C$, which gives $C = -6$. Hence the equation of the curve is $y = x^2 + x - 6$.

$$3. \text{ (a) } \int_4^9 2y\sqrt{y} \, dy = \int_4^9 2y^{\frac{3}{2}} \, dy = \left[\frac{4}{5}y^{\frac{5}{2}} \right]_4^9 = \frac{4}{5}(9^{\frac{5}{2}} - 4^{\frac{5}{2}}) = \frac{844}{5}$$

$$\begin{aligned} \text{(b) } \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(x + \frac{2}{\sin^2 x} \right) dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (x + 2 \csc^2 x) dx = \left[\frac{x^2}{2} - 2 \cot x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \left(\frac{\pi^2}{8} - 2 \cot \frac{\pi}{2} \right) - \left(\frac{\pi^2}{72} - 2 \cot \frac{\pi}{6} \right) = \frac{\pi^2}{9} + 2\sqrt{3} \end{aligned}$$

(c) Note that

$$|2x - 3| = \begin{cases} 3 - 2x, & x \leq \frac{3}{2}, \\ 2x - 3, & x \geq \frac{3}{2}. \end{cases}$$

$$\begin{aligned} \text{Thus } \int_0^2 |2x - 3| \, dx &= \int_0^{\frac{3}{2}} (3 - 2x) \, dx + \int_{\frac{3}{2}}^2 (2x - 3) \, dx \\ &= [3x - x^2]_0^{\frac{3}{2}} + [x^2 - 3x]_{\frac{3}{2}}^2 = \left(\frac{9}{2} - \frac{9}{4} \right) + \left((4 - 6) - \left(\frac{9}{4} - \frac{9}{2} \right) \right) = \frac{5}{2}. \end{aligned}$$

4. The required area is $\int_{-3}^8 |x^2 - 3x - 10| \, dx$. Now

$$x^2 - 3x - 10 = (x - 5)(x + 2) \begin{cases} \geq 0, & x \leq -2, \\ \leq 0, & -2 \leq x \leq 5, \\ \geq 0, & x \geq 5. \end{cases}$$

So the above integral becomes

$$\begin{aligned} &\int_{-3}^{-2} (x^2 - 3x - 10) \, dx - \int_{-2}^5 (x^2 - 3x - 10) \, dx + \int_5^8 (x^2 - 3x - 10) \, dx \\ &= \left[\frac{x^3}{3} - \frac{3x^2}{2} - 10x \right]_{-3}^{-2} - \left[\frac{x^3}{3} - \frac{3x^2}{2} - 10x \right]_{-2}^5 + \left[\frac{x^3}{3} - \frac{3x^2}{2} - 10x \right]_5^8 \\ &= \frac{23}{6} - \left(-\frac{343}{6} \right) + \frac{243}{6} \\ &= \frac{203}{2}. \end{aligned}$$

$$5. \frac{1}{\pi - 0} \int_0^\pi \sin x \, dx = \frac{1}{\pi} [-\cos x]_0^\pi = \frac{2}{\pi}$$

$$6. \text{ (a) } F'(x) = x^3 + 1$$

$$\text{(b) } F(x) = \left[\frac{1}{4}t^4 + t \right]_1^x = \frac{1}{4}x^4 + x - \frac{5}{4}; F'(x) = x^3 + 1.$$

7. To prove that $F(x)$ is constant, differentiate to get

$$\begin{aligned} F'(x) &= \frac{d}{dx} \int_0^x \frac{1}{1+t^2} dt + \frac{d}{dx} \int_0^{\frac{1}{x}} \frac{1}{1+t^2} dt \\ &= \frac{1}{1+x^2} + \frac{1}{1+(\frac{1}{x})^2} \frac{d}{dx} \left(\frac{1}{x} \right) \\ &= \frac{1}{1+x^2} + \frac{x^2}{1+x^2} \cdot \frac{-1}{x^2} \\ &= 0. \end{aligned}$$

So $F(x)$ is constant on the interval $(0, \infty)$ (noting that the functions $\frac{1}{x}$ and $\frac{1}{1+x^2}$ are continuous on the same interval).

Tutorial 3

1. (a) $u = \cot x$, $\frac{du}{dx} = -\csc^2 x$, i.e. $du = -\csc^2 x dx$. Hence

$$\int \cot x \csc^2 x dx = -\int u du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cot^2 x + C.$$

(b) $u = 1 + x$ (so $x = u - 1$), $du = dx$, hence $\int x^2 \sqrt{1+x} dx =$
 $\int (u-1)^2 \sqrt{u} du = \int (u^2 - 2u + 1)u^{\frac{1}{2}} du = \int (u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}) du =$
 $\frac{2}{7}u^{\frac{7}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{7}(1+x)^{\frac{7}{2}} - \frac{4}{5}(1+x)^{\frac{5}{2}} + \frac{2}{3}(1+x)^{\frac{3}{2}} + C.$

2. Put $u = 7t^2 + 12$, $du = 14t dt$,

$$\int t\sqrt{7t^2+12} dt = \frac{1}{14} \int \sqrt{u} du = \frac{1}{14} \frac{2}{3} u^{\frac{3}{2}} + C = \frac{1}{21} (7t^2 + 12)^{\frac{3}{2}} + C.$$

3. (a) Put $v = 3u + 1$, $dv = 3du$. When $u = 0$, $v = 1$. When $u = 1$, $v = 4$. Hence

$$\int_0^1 \frac{du}{\sqrt{3u+1}} = \frac{1}{3} \int_1^4 \frac{dv}{\sqrt{v}} = \frac{1}{3} \left[2v^{\frac{1}{2}} \right]_1^4 = \frac{2}{3} (2 - 1) = \frac{2}{3}.$$

(b) Put $u = \sin 3x$, $du = 3 \cos 3x dx$. $x = 0 \rightarrow u = 0$, $x = \frac{\pi}{2} \rightarrow u = -1$. So

$$\int_0^{\frac{\pi}{2}} \sin^2 3x \cos 3x dx = \frac{1}{3} \int_0^{-1} u^2 du = \frac{1}{3} \left[\frac{1}{3} u^3 \right]_0^{-1} = \frac{1}{9} (-1 - 0) = -\frac{1}{9}.$$

4. Put $u = 1 - x$ (so $x = 1 - u$). $du = -dx$, $x = 0 \rightarrow u = 1$, $x = 1 \rightarrow u = 0$, so

$$\int_0^1 x^m(1-x)^n dx = -\int_1^0 (1-u)^m u^n du = \int_0^1 (1-u)^m u^n du = \int_0^1 x^n(1-x)^m dx,$$

since u and x are dummy variables.

$$5. \int_{\frac{1}{4}}^1 \sqrt{x} - x^2 dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \right]_{\frac{1}{4}}^1 = \left(\frac{2}{3} - \frac{1}{3} \right) - \left(\frac{2}{3} \cdot \frac{1}{8} - \frac{1}{3} \cdot \frac{1}{64} \right) = \frac{49}{192}$$

$$6. \pi \int_0^2 y^2 dx = \pi \int_0^2 x^4 dx = \pi \left[\frac{1}{5}x^5 \right]_0^2 = \frac{32\pi}{5}$$

$$7. 2\pi \int_1^2 xy dx = 2\pi \int_1^2 x \cdot x^2 dx = 2\pi \left[\frac{1}{4}x^4 \right]_1^2 = \frac{\pi}{2}(16 - 1) = \frac{15\pi}{2}$$

$$8. \frac{dy}{dx} = \frac{x^3}{4} - \frac{1}{x^3}.$$

$$\begin{aligned} L &= \int_2^3 \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx = \int_2^3 \sqrt{1 + \left(\frac{x^3}{4} - \frac{1}{x^3} \right)^2} dx = \int_2^3 \sqrt{\frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6}} dx \\ &= \int_2^3 \frac{x^3}{4} + \frac{1}{x^3} dx = \left[\frac{x^4}{16} - \frac{1}{2x^2} \right]_2^3 = \frac{81}{16} - \frac{1}{18} - 1 + \frac{1}{8} \end{aligned}$$

Tutorial 4

1. (a) $\log_2 16 = \log_2 2^4 = 4 \log_2 2 = 4(1) = 4$

(b) $\log_2 \frac{1}{32} = \log_2 2^{-5} = -5$

(c) $\log_4 4 = 1$

(d) $\log_9 3 = \log_9 9^{\frac{1}{2}} = \frac{1}{2}$

2. (a) $\ln \frac{1}{x} = -2 \Rightarrow -\ln x = -2 \Rightarrow \ln x = 2 \Rightarrow e^{\ln x} = e^2 \Rightarrow x = e^2$

(b) $e^{-2x} - 3e^{-x} + 2 = 0$ is a quadratic equation in $e^{-x} \Rightarrow (e^{-x} - 2)(e^{-x} - 1) = 0 \Rightarrow e^{-x} = 2$

or $e^{-x} = 1 \Rightarrow x = \ln \frac{1}{2}$ or $x = 0$.

3. (a) $e^{-\ln x} = e^{\ln \frac{1}{x}} = \frac{1}{x}$, for $x > 0$.

(b) $e^{\ln x^2} = x^2$, for $x^2 > 0 \Leftrightarrow x \neq 0$.

(c) $\ln(e^{-x^2}) = -x^2$, for all real x .

(d) $\ln\left(\frac{1}{e^x}\right) = \ln e^{-x} = -x$, for all real x .

4. (a) Put $u = \sin x$, $du = \cos x dx$, $\int \cot x dx = \int \frac{\cos x dx}{\sin x} = \int \frac{du}{u} = \ln |u| + C = \ln |\sin x| + C$.

(b) Put $u = \ln x$, $du = \frac{1}{x} dx$, $\int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln |u| + C = \ln |\ln x| + C$.

(c) Let $u = x^2 + 5$, $du = 2x dx$, $x = -1 \rightarrow u = 6$, $x = 0 \rightarrow u = 5$.

$$\int_{-1}^0 \frac{x dx}{x^2 + 5} = \frac{1}{2} \int_6^5 \frac{du}{u} = \frac{1}{2} [\ln u]_6^5 = \frac{1}{2} (\ln 5 - \ln 6) = \ln \sqrt{\frac{5}{6}}.$$

(d) Let $u = 1 + \sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$, $x = 1 \rightarrow u = 2$, $x = 4 \rightarrow u = 3$.

$$\int_1^4 \frac{dx}{\sqrt{x}(1 + \sqrt{x})} = 2 \int_2^3 \frac{du}{u} = 2 [\ln u]_2^3 = 2(\ln 3 - \ln 2) = \ln \frac{9}{4}.$$

5.

$$\begin{aligned} y &= x^{(e^x)} \\ \ln y &= e^x \ln x \\ \frac{1}{y} \frac{dy}{dx} &= \frac{d}{dx} e^x \ln x \\ \frac{dy}{dx} &= y e^x \left(\ln x + \frac{1}{x} \right) \\ &= x^{(e^x)} e^x \left(\ln x + \frac{1}{x} \right) \end{aligned}$$

6. (a) Put $u = \sin x$, $du = \cos x dx$, $\int e^{\sin x} \cos x dx = \int e^u du = e^u + C = e^{\sin x} + C$.

(b) Put $u = \sin x$, $du = \cos x dx$, $\int \pi^{\sin x} \cos x dx = \int \pi^u du = \int e^{u \ln \pi} du = \frac{e^{u \ln \pi}}{\ln \pi} + C = \frac{\pi^u}{\ln \pi} + C = \frac{\pi^{\sin x}}{\ln \pi} + C$.

(c) Let $u = e^x + 4$, $du = e^x dx$, $x = -\ln 3 \rightarrow u = \frac{13}{3}$, $x = \ln 3 \rightarrow u = 7$.

$$\int_{-\ln 3}^{\ln 3} \frac{e^x dx}{e^x + 4} = \int_{\frac{13}{3}}^7 \frac{du}{u} = [\ln u]_{\frac{13}{3}}^7 = \ln 7 - \ln \frac{13}{3} = \ln \frac{21}{13}.$$

Tutorial 5

1. (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$

2. (a) $\frac{dy}{dx} = \frac{1}{\sqrt{1 - (\frac{1}{x})^2}} \cdot \frac{-1}{x^2} = \frac{-1}{\frac{x^2}{|x|} \sqrt{x^2 - 1}} = \frac{-1}{|x| \sqrt{x^2 - 1}}$

(b) $\frac{dy}{dx} = \frac{-1}{\sqrt{1 - \cos^2 x}} (-\sin x) = \frac{\sin x}{|\sin x|}$

$$(c) \frac{dy}{dx} = \frac{1}{1 + \left(\frac{1-x}{1+x}\right)^2} \cdot \frac{-(1+x) - (1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2 + (1-x)^2} = \frac{-1}{1+x^2}$$

$$3. (a) \int_0^{\frac{1}{\sqrt{2}}} \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1} x]_0^{\frac{1}{\sqrt{2}}} = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$(b) \int \frac{dx}{1+16x^2} \stackrel{u=4x}{=} \frac{1}{4} \int \frac{du}{1+u^2} = \frac{1}{4} \tan^{-1} u + C = \frac{1}{4} \tan^{-1}(4x) + C$$

$$(c) \int_1^3 \frac{dx}{\sqrt{x}(x+1)} \stackrel{u=\sqrt{x}}{=} 2 \int_1^{\sqrt{3}} \frac{du}{u^2+1} = 2 [\tan^{-1} u]_1^{\sqrt{3}} = 2\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\pi}{6}$$

$$(d) \int \frac{dx}{x\sqrt{1-(\ln x)^2}} \stackrel{u=\ln x}{=} \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(\ln x) + C$$

$$4. (a) \text{ Put } u = x, u' = 1; v' = e^{-x}, v = -e^{-x}.$$

$$\int x e^{-x} dx = -x e^{-x} - \int -e^{-x} = -x e^{-x} - e^{-x} + C.$$

$$(b) \text{ Put } u = x^2, u' = 2x; v' = \cos x, v = \sin x.$$

$$\int x^2 \cos x dx = x^2 \sin x - \int 2x \sin x dx.$$

Now put $u = x, u' = 1; v' = \sin x, v = -\cos x$ and the integral becomes

$$x^2 \sin x - 2 \left(-x \cos x - \int -\cos x dx \right) = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

$$(c) \text{ Put } u = \ln x, u' = \frac{1}{x}; v' = x^2, v = \frac{x^3}{3}. \int x^2 \ln x dx =$$

$$\left[\frac{x^3 \ln x}{3} \right]_1^e - \int_1^e \frac{1}{x} \frac{x^3}{3} dx = \left[\frac{x^3 \ln x}{3} \right]_1^e - \int_1^e \frac{x^2}{3} dx = \left[\frac{x^3 \ln x}{3} - \frac{x^3}{9} \right]_1^e = \frac{2e^3 + 1}{9}.$$

$$5. (a) \int \sin^3 x dx = -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} \int \sin x dx = -\frac{1}{3} \sin^2 x \cos x - \frac{2}{3} \cos x + C$$

$$(b) \int_0^{\frac{\pi}{4}} \sin^4 x dx = \left[-\frac{1}{4} \sin^3 x \cos x \right]_0^{\frac{\pi}{4}} + \frac{3}{4} \int_0^{\frac{\pi}{4}} \sin^2 x dx =$$

$$\left[-\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} \cdot \frac{-1}{2} \sin x \cos x \right]_0^{\frac{\pi}{4}} + \frac{3}{4} \cdot \frac{1}{2} \int_0^{\frac{\pi}{4}} dx =$$

$$\left[-\frac{1}{4} \sin^3 x \cos x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x \right]_0^{\frac{\pi}{4}} = \frac{3\pi}{32} - \frac{1}{4}$$

$$6. \text{ Put } u = x, u' = 1; v' = f'', v = f'.$$

$$\int_{-1}^1 x f''(x) dx = [x f'(x)]_{-1}^1 + \int_{-1}^1 f'(x) dx = f'(1) - (-1)f'(-1) + [f(x)]_{-1}^1$$

$$= f'(1) + f'(-1) + f(1) - f(-1).$$

Tutorial 6

1. (a) $\int \frac{dx}{\sqrt{8+2x-x^2}} = \int \frac{dx}{\sqrt{9-(x-1)^2}} = \sin^{-1} \left(\frac{x-1}{3} \right) + C.$

(b) $I = \int \frac{dx}{\sqrt{x^2-6x+10}} = \int \frac{dx}{\sqrt{(x-3)^2+1}} \stackrel{u=x-3}{=} \int \frac{du}{\sqrt{u^2+1}}.$ Now put $u = \tan \theta$,
 $du = \sec^2 \theta d\theta$, (noting that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, so $\sec \theta > 0$) to get

$$I = \int \frac{\sec^2 \theta d\theta}{\sqrt{\tan^2 \theta + 1}} = \int \frac{\sec^2 \theta d\theta}{\sqrt{\sec^2 \theta}} = \int \frac{\sec^2 \theta d\theta}{|\sec \theta|} = \int \sec \theta d\theta = \int \frac{d\theta}{\cos \theta} = \int \frac{\cos \theta d\theta}{\cos^2 \theta}$$

$$= \int \frac{\cos \theta d\theta}{1 - \sin^2 \theta}.$$
 Now put $v = \sin \theta$, $dv = \cos \theta d\theta$, and we have

$$I = \int \frac{dv}{1-v^2} = \frac{1}{2} \ln \left| \frac{1+v}{1-v} \right| + C$$

after integrating by partial fractions. Back-substituting and simplifying, we get

$$I = \frac{1}{2} \ln \left| \frac{1 + \sin \theta}{1 - \sin \theta} \right| + C = \frac{1}{2} \ln \left| \frac{1 + \sin \tan^{-1} u}{1 - \sin \tan^{-1} u} \right| + C = \frac{1}{2} \ln \left| \frac{1 + \frac{u}{\sqrt{1+u^2}}}{1 - \frac{u}{\sqrt{1+u^2}}} \right| + C =$$

$$\frac{1}{2} \ln \left| \frac{\sqrt{1+u^2} + u}{\sqrt{1+u^2} - u} \right| + C = \frac{1}{2} \ln \left| \frac{(\sqrt{1+u^2} + u)^2}{1+u^2 - u^2} \right| + C = \ln \left| \sqrt{1+u^2} + u \right| + C$$

$$= \ln \left| \sqrt{1+(x-3)^2} + x - 3 \right| + C.$$

Of course, this answer could have been achieved more easily with the use of integrating formulae, but it is worthwhile seeing how such formulae are unnecessary if one perseveres with appropriate substitutions.

2. (a) $\frac{1}{x^2+3x-4} = \frac{1}{(x-1)(x+4)} = \frac{A}{x-1} + \frac{B}{x+4}$, where $1 = A(x+4) + B(x-1)$.
 Put $x = 1$: $1 = 5A \Rightarrow A = \frac{1}{5}$; put $x = -4$: $1 = -5B \Rightarrow B = -\frac{1}{5}$.

$$\int \frac{dx}{x^2+3x-4} = \frac{1}{5} \int \left(\frac{1}{x-1} - \frac{1}{x+4} \right) dx = \frac{1}{5} \ln \left| \frac{x-1}{x+4} \right| + C.$$

(b) $\frac{x^3}{x^2-3x+2} = x+3 + \frac{7x-6}{(x-1)(x-2)} = x+3 + \frac{A}{x-1} + \frac{B}{x-2}$,

where $7x-6 = A(x-2) + B(x-1)$. Put $x = 1 \Rightarrow A = -1$, put $x = 2 \Rightarrow B = 8$.

$$\int \frac{x^3 dx}{x^2-3x+2} = \int \left(x+3 + \frac{8}{x-2} - \frac{1}{x-1} \right) dx = \frac{1}{2}x^2 + 3x + 8 \ln |x-2| - \ln |x-1| + C.$$

(c) $\frac{x^2}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$ where $x^2 = A(x+2)^2 + B(x+2) + C$. Equating coefficients of $x^2 \Rightarrow A = 1$. Put $x = 2 \Rightarrow C = 4$. Put $x = 0 \Rightarrow 0 = 4A + 2B + C \Rightarrow B = -4$.

$$\int \frac{x^2 dx}{(x+2)^3} = \int \left(\frac{1}{x+2} - \frac{4}{(x+2)^2} + \frac{4}{(x+2)^3} \right) dx = \ln|x+2| + \frac{4}{x+2} - \frac{2}{(x+2)^2} + C.$$

(d) $\frac{1}{x^4 - 16} = \frac{1}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$, where $1 = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2)$. Put $x = 2 \Rightarrow A = \frac{1}{32}$. Put $x = -2 \Rightarrow B = -\frac{1}{32}$. Put $x = 0 \Rightarrow 1 = 8A - 8B - 4D \Rightarrow D = -\frac{1}{8}$. Equating coefficients of $x^3 \Rightarrow 0 = A + B + C \Rightarrow C = 0$.

$$\int \frac{dx}{x^4 - 16} = \frac{1}{32} \int \left(\frac{1}{x-2} - \frac{1}{x+2} \right) dx - \frac{1}{8} \int \frac{dx}{x^2+4} = \frac{1}{32} \ln \left| \frac{x-2}{x+2} \right| - \frac{1}{16} \tan^{-1} \frac{x}{2} + C.$$

(e)

$$\begin{aligned} \int \frac{x^4 + 6x^3 + 10x^2 + x}{x^2 + 6x + 10} dx &= \int \left(x^2 + \frac{x}{x^2 + 6x + 10} \right) dx = \frac{x^3}{3} + \int \frac{x}{(x+3)^2 + 1} dx \\ &\stackrel{u=x+3}{=} \frac{x^3}{3} + \int \frac{u-3}{u^2+1} du = \frac{x^3}{3} + \frac{1}{2} \int \frac{2u}{u^2+1} du - 3 \int \frac{du}{u^2+1} \\ &= \frac{x^3}{3} + \frac{1}{2} \ln(u^2+1) - 3 \tan^{-1} u + C = \frac{x^3}{3} + \frac{1}{2} \ln((x+3)^2+1) - 3 \tan^{-1}(x+3) + C. \end{aligned}$$

Tutorial 7

1.

$$\begin{aligned} x^2 + y^2 - 6y &= 0 \\ r^2(\cos^2 \theta + \sin^2 \theta) - 6r \sin \theta &= 0 \\ r^2 &= 6r \sin \theta \\ r &= 6 \sin \theta \end{aligned}$$

2. (a) Line through the origin.

(b) Circle centre origin radius 5.

(c) $r = -6 \cos \theta \Rightarrow r^2 = -6r \cos \theta \Rightarrow x^2 + y^2 = -6x \Rightarrow (x + 3)^2 + y^2 = 9$, upon completing the square, which is a circle centre $(-3, 0)$, radius 3.

(d) Cardioid.

$$3. \text{ (a) } \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} r^2 d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} \theta^2 d\theta = \left[\frac{\theta^3}{6} \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{7\pi^3}{1296}$$

(b) The two curves intersect when $3 \cos \theta = 1 + \cos \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pm \frac{\pi}{3}$. By symmetry about the x -axis, the area is

$$\begin{aligned} A &= 2 \left[\int_0^{\frac{\pi}{3}} \frac{1}{2} (1 + \cos \theta)^2 d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2} (3 \cos \theta)^2 d\theta \right] \\ &= \int_0^{\frac{\pi}{3}} (1 + 2 \cos \theta + \cos^2 \theta) d\theta + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 9 \cos^2 \theta d\theta \\ &= \int_0^{\frac{\pi}{3}} \left(\frac{3}{2} + 2 \cos \theta + \frac{1}{2} \cos 2\theta \right) d\theta + \frac{9}{2} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\cos 2\theta + 1) d\theta \\ &= \left[\frac{3\theta}{2} + 2 \sin \theta + \frac{\sin 2\theta}{4} \right]_0^{\frac{\pi}{3}} + \frac{9}{2} \left[\frac{\sin 2\theta}{2} + \theta \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \\ &= \left[\frac{\pi}{2} + \sqrt{3} + \frac{\sqrt{3}}{8} - 0 - 0 - 0 \right] + \frac{9}{2} \left[0 + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{3} \right] \\ &= \frac{5\pi}{4}. \end{aligned}$$

$$4. \text{ (a) } \int_0^2 \sqrt{4^2 + 3^2} dt = \int_0^2 5 dt = 10$$

(b)

$$\begin{aligned} \int_0^1 \sqrt{(2(1+t))^2 + (3(1+t)^2)^2} dt &= \int_0^1 (1+t) \sqrt{4 + 9(1+t)^2} dt \\ &= \left[\frac{(4 + 9(1+t)^2)^{\frac{3}{2}}}{18} \cdot \frac{2}{3} \right]_0^1 \\ &= \frac{1}{27} (40\sqrt{40} - 13\sqrt{13}) \end{aligned}$$

5. We have $r = a\theta$ and $r' = a$. Hence

$$\ell = \int_{\theta_1}^{\theta_2} \sqrt{a^2 \theta^2 + a^2} d\theta = |a| \int_{\theta_1}^{\theta_2} \sqrt{\theta^2 + 1} d\theta.$$

Substitute $\theta = \sinh t$, $d\theta = \cosh t dt$. Then

$$\ell = \int_{\sinh^{-1} \theta_1}^{\sinh^{-1} \theta_2} \cosh^2 t dt.$$

Use $\cosh^2 t = \frac{\cosh 2t + 1}{2}$.

$$\begin{aligned} \ell &= \int_{\sinh^{-1} \theta_1}^{\sinh^{-1} \theta_2} \frac{\cosh 2t + 1}{2} dt = \left[\frac{\sinh 2t}{4} + \frac{t}{2} \right]_{\sinh^{-1} \theta_1}^{\sinh^{-1} \theta_2} \\ &= \frac{\sinh(2 \sinh^{-1} \theta_2) - \sinh(2 \sinh^{-1} \theta_1)}{4} + \frac{\sinh^{-1} \theta_2 - \sinh^{-1} \theta_1}{2}. \end{aligned}$$

Tutorial 8

1.

(a)	$f(x) = e^{-2x}$	$f(0) = 1$
	$f'(x) = -2e^{-2x}$	$f'(0) = -2$
	\vdots	\vdots
	$f^{(n)}(x) = (-2)^n e^{-2x}$	$f^{(n)}(0) = (-2)^n$

$$S_4(x) = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4$$

(b)	$f(x) = \tan x$	$f(0) = 0$
	$f'(x) = \sec^2 x$	$f'(0) = 1$
	$f''(x) = 2 \sec^2 x \tan x$	$f''(0) = 0$
	$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x$	$f'''(0) = 2$
	$f^{(4)}(x) = \dots$	$f^{(4)}(0) = 0$

$$S_4(x) = x + \frac{1}{3}x^3$$

(c)

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

$$xe^x = x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \cdots$$

$$S_4(x) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{6}$$

2. $f^{(n)}(x) = e^x$, so $f^{(n)}(1) = e$ for all n .

$$S_3(x) = e + e(x-1) + \frac{e(x-1)^2}{2} + \frac{e(x-1)^3}{6}$$

3.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1, \text{ so}$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \quad \text{for } |x| < 1.$$

4.

$$\frac{1}{x} = \frac{-1}{1-(x+1)} = -\sum_{n=0}^{\infty} (x+1)^n$$

5. For e^x we have the Maclaurin polynomial $S_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$ with remainder term

$$R_n(x) = \frac{f^{(n+1)}(\epsilon)x^{n+1}}{(n+1)!}$$

for some ϵ between 0 and x . Now $f^{(n+1)}(\epsilon) = e^\epsilon \leq e^{\frac{1}{2}} \leq 2$ for $0 \leq \epsilon \leq \frac{1}{2}$. So

$$\left| R_n\left(\frac{1}{2}\right) \right| \leq \frac{2\left(\frac{1}{2}\right)^{n+1}}{(n+1)!} = \frac{1}{2^n(n+1)!} \leq 0.00005 \quad \text{for } n = 5, \text{ hence}$$

$$e^{\frac{1}{2}} \simeq 1 + \frac{1}{2} + \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{6} + \frac{\left(\frac{1}{2}\right)^4}{24} + \frac{\left(\frac{1}{2}\right)^5}{120}$$

$$\simeq 1.6487 \quad \text{to 4 decimal places.}$$

6.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Differentiate both sides of this equation to get

$$\frac{-1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n n x^{n-1},$$

hence

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n x^{n-1} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n.$$

7.

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$$

$$\begin{aligned} e^{-x^2} \cos x &= \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \dots\right) \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots\right) \\ &= 1 + \left(-1 - \frac{1}{2}\right)x^2 + \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{24}\right)x^4 + \left(-\frac{1}{6} - \frac{1}{4} - \frac{1}{24} - \frac{1}{720}\right)x^6 + \dots \\ &= 1 - \frac{3x^2}{2} + \frac{25x^4}{24} - \frac{331x^6}{720} + \dots \end{aligned}$$

(keeping only terms of degree 6 or less.)

Tutorial 9

1. (a) $\int_0^{\infty} e^{-x} dx = \lim_{R \rightarrow \infty} \left([-e^{-x}]_0^R = e^0 - e^{-R} \right) = 1 - 0 = 1$

(b) $\lim_{R \rightarrow \infty} \int_1^R \frac{dx}{\sqrt{x}} = \lim_{R \rightarrow \infty} \left[2x^{\frac{1}{2}} \right]_1^R = \lim_{R \rightarrow \infty} 2\sqrt{R} - 2$ which diverges to ∞ .

(c) $\lim_{r \rightarrow \frac{\pi}{2}^-} \int_0^r \tan x dx = \lim_{r \rightarrow \frac{\pi}{2}^-} [-\ln |\cos x|]_0^r = 0 - \lim_{r \rightarrow \frac{\pi}{2}^-} \ln |\cos r| = -\lim_{s \rightarrow 0} \ln |s|$ which diverges to ∞ .

(d) $\lim_{\substack{r \rightarrow 0^+ \\ R \rightarrow \infty}} \int_r^R \frac{dx}{x^2} = \lim_{\substack{r \rightarrow 0^+ \\ R \rightarrow \infty}} \left[-\frac{1}{x} \right]_r^R = \lim_{r \rightarrow 0^+} \frac{1}{r} - \lim_{R \rightarrow \infty} \frac{1}{R} = \lim_{r \rightarrow 0^+} \frac{1}{r} - 0$ which diverges to ∞ .

2. Since $|\sin x| \leq 1$ we have $\left| \frac{\sin x}{x^2} \right| \leq \frac{1}{x^2}$. Since $\int_1^{\infty} \frac{dx}{x^2}$ converges, the integral $\int_1^{\infty} \frac{\sin x dx}{x^2}$ must converge as well.

3. (a) The distribution function $F(y) = \text{mes}\{0 \leq \sin x < y\} = \text{mes}\{0 \leq x < \sin^{-1} y\} = \sin^{-1} y$ for $0 \leq y \leq \frac{\pi}{2}$, it vanishes for $y < 0$ and equals 1 for $y > \frac{\pi}{2}$. Therefore the density $\rho(y) = F'(y) = \frac{1}{\sqrt{1-y^2}}$.

$$(b) \quad \int_0^1 \frac{y dy}{\sqrt{1-y^2}} = -\frac{1}{2} \int_0^1 \frac{-2y}{\sqrt{1-y^2}} dy$$

We substitute $1 - y^2 = t$. Then the lower bound becomes 1 and the upper bound becomes 0. Hence

$$\int_0^1 \frac{y dy}{\sqrt{1-y^2}} = -\frac{1}{2} \int_1^0 \frac{du}{\sqrt{u}} dy = -\frac{1}{2} [2\sqrt{u}]_1^0 = 1.$$

On the other hand $\int_0^{\pi/2} \sin x dx = [-\cos x]_0^{\pi/2} = 1$. The results coincide.

4. Let $x_0 = -1 < x_1 < \dots < x_N = 1$ be a partition of $[-1, 1]$ such that non of the $x_i = 0$. Then there is exactly one subinterval $[x_{i-1}, x_i]$ that contains 0.

We have

$$\int_1^1 x d|x| \approx \sum_{k=1}^{i-1} x_{k-1}(|x_k| - |x_{k-1}|) + x_{i-1}(|x_i| - |x_{i-1}|) + \sum_{k=i+1}^N x_{k-1}(|x_k| - |x_{k-1}|)$$

Now, $|x_k| = -x_k$ for $k \leq i-1$ and $|x_k| = x_k$ for $k \geq i$. Hence

$$\int_1^1 x d|x| \approx -\sum_{k=1}^{i-1} x_{k-1}(x_k - x_{k-1}) + x_{i-1}(-x_i - x_{i-1}) + \sum_{k=i+1}^N x_{k-1}(x_k - x_{k-1})$$

As the partition becomes finer the first sum tends to $-\int_{-1}^0 x dx = \frac{1}{2}$, the term $x_{i-1}(-x_i - x_{i-1})$ tends to 0 and the last sum tends to $\int_0^1 x dx = \frac{1}{2}$. It follows

$$\int_1^1 x d|x| = 1.$$

Tutorial 10

- 1.

$$\frac{dy}{y} = \frac{dx}{x} \text{ so } \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln |y| = \ln |x| + c$$

$$|y| = e^{\ln|x|+c} = |x| e^c$$

$$y = \pm e^c x = Kx, \text{ where } K \text{ is a constant.}$$

2.

$$\sqrt{1+x^2} \frac{dy}{dx} = -x(1+y)$$

$$\int \frac{dy}{1+y} = - \int \frac{x dx}{\sqrt{1+x^2}}$$

$$\ln |1+y| = -\sqrt{1+x^2} + c$$

$$|1+y| = e^c e^{-\sqrt{1+x^2}}$$

$$1+y = K e^{-\sqrt{1+x^2}}$$

$$y = -1 + K e^{-\sqrt{1+x^2}}, \text{ where } K \text{ is a constant.}$$

3.

$$\frac{dy}{dx} \cos^2 x = e^{-y} \sin x$$

$$\int e^y dy = \int \frac{\sin x}{\cos^2 x} dx$$

$$e^y = \frac{1}{\cos x} + c = \sec x + c$$

$$y = \ln(\sec x + c)$$

4.

$$y^2 \frac{dy}{dt} = 1 - \frac{1}{t}$$

$$\int y^2 dy = \int \left(1 - \frac{1}{t}\right)$$

$$\frac{y^3}{3} = t - \ln t + c$$

When $x = 1$, $y = 3$ so $9 = 1 + c$. Hence $c = 8$ and

$$y^3 = 3t - 3 \ln t + 24$$

$$y = \sqrt[3]{3t - 3 \ln t + 24}.$$

5. Let y be the number of grams left after t days. Then

$$y' = ky, \quad y(0) = 10.$$

which has solution $y = 10e^{kt}$. After 140 days, $y = 5$, so $5 = 10e^{140k}$, whence $k = -\ln 2/140$. Hence, $y = 10e^{-(\ln 2)t/140}$. When $t = 70$, $y = 10e^{-(\ln 2)70/140} = 10e^{-(\ln 2)/2} \approx 7.07$ (mg).

6. Here $p(x) = 3$, so $\rho = e^{\int p(x)dx} = e^{3x}$. Thus $\frac{dy}{dx}e^{3x} + 3ye^{3x} = e^{3x}e^{-2x} = e^x$, $(ye^{3x})' = e^x$, whence $ye^{3x} = e^x + c$. $y = e^{-2x} + ce^{-3x}$.

7. Here $p(x) = 3/x$, so $\rho = e^{\int p(x)dx} = e^{3\ln x} = x^3$. On multiplying by ρ we get $x^3y' + 3x^2y = -2x^7$, $(x^3y)' = -2x^7$, whence $x^3y = -\frac{x^8}{4} + c$. $y = -x^5/4 + c/x^3$.

8. $\rho = e^{-\int x dx} = e^{-x^2/2}$. So

$$\begin{aligned} e^{-x^2/2} \frac{dy}{dx} - x e^{-x^2/2} y &= x e^{-x^2/2} \\ (e^{-x^2/2} y)' &= x e^{-x^2/2} \\ e^{-x^2/2} y &= -e^{-x^2/2} + c \\ y &= -1 + c e^{x^2/2}. \end{aligned}$$

When $x = 0$, $y = 3$, so $3 = -1 + ce^0 = -1 + c$. Hence $c = 4$ and $y = -1 + 4e^{x^2/2}$.

9.

$$\begin{aligned} \frac{dy}{dx} &= \frac{y^2}{2\sqrt{x}} \\ \int \frac{dy}{y^2} &= \int \frac{dx}{2\sqrt{x}} \\ -\frac{1}{y} &= \sqrt{x} + c \end{aligned}$$

When $x = 1$, $y = 1$, so $-1 = 1 + c$ and $c = -2$. Hence

$$-\frac{1}{y} = \sqrt{x} - 2 \quad \text{and} \quad y = \frac{1}{2 - \sqrt{x}}.$$

Tutorial 11

1. $y = c_1 e^{2x} + c_2 e^{-x}$, $y' = 2c_1 e^{2x} - c_2 e^{-x}$, $y'' = 4c_1 e^{2x} + c_2 e^{-x}$. Hence

$$\begin{aligned} y'' - y' - 2y &= (4c_1 e^{2x} + c_2 e^{-x}) - (2c_1 e^{2x} - c_2 e^{-x}) - 2(c_1 e^{2x} + c_2 e^{-x}) \\ &= c_1 e^{2x}(4 - 2 - 2) + c_2 e^{-x}(1 + 1 - 2) \end{aligned}$$

Hence y is a solution of the given d.e.

2. The auxiliary equation is $m^2 + 3m - 4 = 0$. Hence $(m - 4)(m + 1) = 0$ and $m_1 = 4$, $m_2 = -1$.

$$y = c_1 e^{-4x} + c_2 e^x.$$

3. The auxiliary equation is $m^2 - 2m + 1 = 0$. Hence $(m - 1)^2 = 0$ and $m_1 = m_2 = 1$.

$$y = c_1 e^x + c_2 x e^x.$$

4. The auxiliary equation is $m^2 + 5 = 0$. Hence $m = \pm i\sqrt{5}$. $u = 0$, $v = 5$.

$$y = c_1 \cos(\sqrt{5}x) + c_2 \sin(\sqrt{5}x).$$

5. The auxiliary equation is $m^2 - 4m + 13 = 0$. Hence $m = 2 \pm 3i$. $u = 3$, $v = 3$.

$$y = e^{2x}(c_1 \cos(3x) + c_2 \sin(3x)).$$

6. The auxiliary equation is $m^2 + 2m - 3 = 0$. Hence $(m + 3)(m - 1) = 0$ and $m_1 = -3$, $m_2 = 1$.

$$y = c_1 e^{-3x} + c_2 e^x.$$

Now, $y' = -3c_1 e^{-3x} + c_2 e^x$. The conditions $y(0) = 1$, $y'(0) = 5$ give

$$c_1 + c_2 = 1$$

$$-3c_1 + c_2 = 5.$$

Subtracting the first of these equations from the second gives $-4c_1 = 4$, whence $c_1 = -1$ and $c_2 = 2$.

$$y = -e^{-3x} + 2e^x.$$

7. The auxiliary equation is $m^2 + 4m + 5 = 0$. Hence $m = -2 \pm i$. $u = -2$, $v = 1$.

$$y = e^{-2x}(c_1 \cos x + c_2 \sin x).$$

Now, $y' = -2e^{-2x}(c_1 \cos x + c_2 \sin x) + e^{-2x}(-c_1 \sin x + c_2 \cos x)$. The conditions $y(0) = -3$, $y'(0) = 0$ give $c_1 = -3$ and $-2c_1 + c_2 = 0$, whence $c_2 = -6$.

$$y = -3e^{-2x}(\cos x + 2 \sin x).$$

Tutorial 12

1. To get y_1 and y_2 we solve $y'' + 6y' + 5y = 0$. The auxiliary equation is $m^2 + 6m + 5 = 0$. Hence $(m + 5)(m + 1) = 0$ and $m_1 = -5$ and $m_2 = -1$. $y_1 = e^{-5x}$, $y_2 = e^{-x}$. Since the RHS is e^{3x} multiplied by constant, we look for a solution which is also e^{3x} multiplied by constant, i.e. of the form $y = Ae^{3x}$. Substituting into $y'' + 6y' + 5y = 2e^{3x}$ gives

$$(9Ae^{3x}) + 6(3Ae^{3x}) + 5(Ae^{3x}) = 2e^{3x}.$$

That is $32Ae^{3x} = 2e^{3x}$. Hence $A = 1/16$. Our particular solution is $y_p = e^{3x}/16$. The general solution is $y = c_1 e^{-5x} + c_2 e^{-x} + e^{3x}/16$.

2. To get y_1 and y_2 we solve $y'' + 3y' - 4y = 0$. The auxiliary equation is $m^2 + 3m - 4 = 0$. Hence $(m + 4)(m - 1) = 0$ and $m_1 = 4$ and $m_2 = 1$. $y_1 = e^x$, $y_2 = e^{-4x}$. Since the RHS is e^{7x} multiplied by constant, we look for a solution of the form $y = Ae^{7x}$. Substituting into $y'' + 3y' - 4y = 5e^{7x}$ gives

$$(49Ae^{7x}) + 3(7Ae^{7x}) - 4(Ae^{7x}) = 5e^{7x}.$$

That is $66Ae^{7x} = 5e^{7x}$. Hence $A = 5/66$. Our particular solution is $y_p = 5e^{7x}/66$. The general solution is $y = c_1 e^x + c_2 e^{-4x} + 5e^{7x}/66$.

3. To get y_1 and y_2 we solve $y'' - 9y' + 20y = 0$. The auxiliary equation is $m^2 - 9m + 20 = 0$. Hence $(m - 4)(m - 5) = 0$ and $m_1 = 4$ and $m_2 = 5$. $y_1 = e^{4x}$, $y_2 = e^{5x}$. Since the RHS is e^{5x} multiplied by constant and this is a solution of the homogeneous equation, we need to use the modification rule. Thus we look for a solution of the form $y = Ax e^{5x}$. Then

$y' = Ae^{5x} + 5Ax e^{5x}$, $y'' = 10Ae^{5x} + 25Ax e^{5x}$. Substituting into $y'' - 9y' + 20y = -3e^{5x}$ gives

$$(10Ae^{5x} + 25Ax e^{5x}) - 9(Ae^{5x} + 5Ax e^{5x}) + 20(Ax e^{5x}) = -3e^{5x}.$$

That is $Ae^{5x} = -3e^{5x}$. Hence $A = -3$. Our particular solution is $y_p = -3x e^{5x}$. The general solution is $y = c_1 e^{4x} + c_2 e^{5x} - 3x e^{5x}$.

4. To get y_1 and y_2 we solve $y'' + y' - 12y = 0$. The auxiliary equation is $m^2 + m - 12 = 0$. Hence $(m+4)(m-3) = 0$ and $m_1 = 3$ and $m_2 = -4$. $y_1 = e^{3x}$, $y_2 = e^{-4x}$. Since the RHS is a polynomial of degree two, we look for a solution of the same form $y = A_0 + A_1x + A_2x^2$. Then $y' = A_1 + 2A_2x$ and $y'' = 2A_2$. Substituting into $y'' + y' - 12y = 4x^2$ gives

$$2A_2 + (A_1 + 2A_2x) - 12(A_0 + A_1x + A_2x^2) = 4x^2.$$

That is $(2A_2 + A_1 - 12A_0) + (2A_2 - 12A_1)x - 12A_2x^2 = 4x^2$. Now we equate coefficients of like powers: $-12A_2 = 4$, so $A_2 = -1/3$; $2A_2 - 12A_1 = 0$, so $A_1 = -1/18$; $2A_2 + A_1 - 12A_0 = 0$, so $A_0 = -13/216$. Our particular solution is $y_p = -13/216 - x/18 - x^2/3$. The general solution is $y = c_1 e^{3x} + c_2 e^{-4x} - 13/216 - x/18 - x^2/3$.