TUTORIAL 4

Recall that $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

Question 1.

Determine whether the following sequences $(u_n)_{n \in \mathbb{N}^*}$ are monotonic. Are they increasing or decreasing? In each case, discuss the behaviour of the sequence as $n \to \infty$. Justify your answers.

(a) $u_n = \frac{n}{2n+1}$
(b) $u_n = \frac{2^n}{n^2}$
(c) $u_n = a + (n - 1)d \quad (a, d \in \mathbb{R})$
(d) $u_n = ar^n \quad (a, r \in \mathbb{R})$
(e) $u_n = n \sin \left( \frac{1}{n} \right)$
(f) $u_n = \sqrt{n+1} - \sqrt{n-1}$.

Question 2.

Given the sequence $(u_n)_{n \in \mathbb{N}^*}$, its $n$th partial sum, $S_n$, is $S_n = \sum_{j=1}^{n} u_j$.

Show that

(a) for $u_n = a + (n - 1)d$ with $a, d \in \mathbb{R}$, $S_n = na + \frac{n(n-1)}{2}d$;
(b) $u_n = ar^n$ with $a, r \in \mathbb{R}$, $S_n = a \frac{1 - r^n}{1 - r}$ if $r \neq 1$.

Discuss the convergence or otherwise of the corresponding series.

Question 3.

Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Question 4 is on the next page.
Question 4.

Use the comparison test to determine whether the series below converge or diverge.

(a) \( \sum_{n=0}^{\infty} \frac{n^3}{n^4 + 1} \)

(b) \( \sum_{n=2}^{\infty} \frac{|\sin n|}{n^2 - 1} \)

Question 5.

Use the ratio test to discuss the behaviour of the following series.

(a) \( \sum_{n=1}^{\infty} \frac{6^n}{n^5} \)

(b) \( \sum_{n=0}^{\infty} \frac{4^n}{n^n} \)