Sample Solutions for Tutorial 1

Question 1.
(a) If we let $P(x)$ be \textit{x is a positive integer divisible by 4 and 6} and $Q(x)$ be \textit{x is a positive integer divisible by 24}, then the proposition is of the form

$$(\forall x)(P(x) \Rightarrow Q(x)).$$

The negation is then

$$(\exists x)\neg(P(x) \Rightarrow Q(x)), \quad \text{which is} \quad (\exists x)(P(x) \land \neg Q(x)),$$

or, in English,

\textbf{There is a positive integer is divisible by 4 and by 6, which is not divisible by 24.}

The converse is

$$(\forall x)(Q(x) \Rightarrow P(x)),$$

or, in English,

\textbf{If a positive integer is divisible by 24, then it is divisible by 4 and by 6.}

(b) If we write $P(x)$ for \textit{x is a prime number}, $Q(x)$ for \textit{x is an odd number}, then the first proposition is

$$(\forall x)(P(x) \Rightarrow Q(x)).$$

Its negation is thus

$$(\exists x)(P(x) \land \neg Q(x)),$$

or, in English,

\textbf{There is a prime number which is not odd.}

If we write $C(x)$ for \textit{x is a clever person}, $D(x)$ for \textit{x does dumb things}, then the second proposition is

$$(\exists x)(C(x) \land D(x)).$$

Its negation is thus

$$\neg(\exists x)(C(x) \land D(x)), \quad \text{which is} \quad (\forall x)\neg(C(x) \land D(x)), \quad \text{or} \quad (\forall x)\ ((\neg C(x)) \lor (\neg D(x)))$$

or, in English,

\textbf{No clever person does dumb things.}

Question 2.
Let $a$ be a real number and suppose that for every real number, $b$,

$$(a + b)^2 = a^2 + b^2.$$

Then, since $\frac{1}{2}$ is a real number, we must have

$$a^2 + \left(\frac{1}{2}\right)^2 = (a + \frac{1}{2})^2$$

$$= a^2 + a + \left(\frac{1}{2}\right)^2$$

Subtracting $a^2 + \left(\frac{1}{2}\right)^2$ from both sides of the equation, we see that $a = 0$. 
Question 3.

Let $P(n)$ be the proposition $\sum_{j=1}^{n} j^3 = \left(\frac{1}{2}1(1 + 1)\right)^2$

$n = 1$:  
\[
\sum_{j=1}^{1} j^3 = 1^3 = 1 = \left(\frac{1}{2} \times 2\right)^2 = \left(\frac{1}{2}1(1 + 1)\right)^2.
\]

Hence $P(1)$ is true.

$n \geq 1$: We make the Inductive hypothesis that $P(n)$ is true, that is,
\[
\sum_{j=1}^{n} j^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{1}{2}n(n + 1)\right)^2.
\]

Then
\[
\sum_{j=1}^{n+1} j^3 = \left(\sum_{j=1}^{n} j^3\right) + (n + 1)^3
\]
\[= \left(\frac{1}{2}n(n + 1)\right)^2 + (n + 1)^3 \quad \text{by the Inductive Hypothesis}
\]
\[= (n + 1)^2 \left(\frac{1}{2}n + (n + 1)\right)
\]
\[= (n + 1)^2 \frac{n^2 + 4n + 4}{4}
\]
\[= \frac{(n + 1)^2(n + 2)^2}{4}
\]
\[= \left(\frac{1}{2}(n + 1)((n + 1) + 1)\right)^2
\]

Hence $P(n + 1)$ is true whenever $P(n)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for every counting number, $n$. That is to say, $\sum_{j=1}^{n} j^3 = \left(\frac{1}{2}1(1 + 1)\right)^2$ for every counting number, $n$.

Question 4.

Let $a$ be a positive real number, and $P(n)$ the proposition $(1 + a)^n \geq 1 + na$.

$n = 1$: Then
\[ (1 + a)^1 = 1 + a = 1 + 1.a, \]
showing that $P(1)$ is true.

$n \geq 1$: We make the Inductive hypothesis that $P(n)$ is true, that is,
\[ (1 + a)^n \geq 1 + na. \]

Then
\[ (1 + a)^{n+1} = (1 + a)(1 + a)^n \geq (1 + a)(1 + na) \quad \text{by the Inductive Hypothesis and the fact that } 1 + a > 0
\]
\[ = 1 + (n + 1)a + na^2
\]
\[ > 1 + (n + 1)a \quad \text{as } na^2 > 0
\]
Hence \( P(n + 1) \) is true whenever \( P(n) \) is true.

By the Principle of Mathematical Induction, \( P(n) \) is true for every counting number, \( n \). That is to say, \((1 + a)^n \geq 1 + na\) for every counting number.

**Question 5.**

Let \( P(n) \) the proposition that if \( n \) is a counting number, then \( 3^{2n} - 1 \) is divisible by 8.

\( n = 1 \): Then

\[ 3^{2 \times 1} - 1 = 9 - 1 = 8, \]

so that \( P(1) \) is true.

\( n \geq 1 \): We make the Inductive hypothesis that \( P(n) \) is true, that is, there is a counting number \( k \), such that

\[ 3^{2n} - 1 = 8k. \]

Then

\[
3^{2(n+1)} - 1 = 3^{2n+2} - 1
= 9 \times 3^{2n} - 1
= 9 \times 3^{2n} - 9 + 8
= 9(3^{2n} - 1) + 8
= 9 \times 8k + 8 \quad \text{by the Inductive Hypothesis}
= 8 \times (9k + 1),
\]

which is plainly divisible by 8.

Hence \( P(n + 1) \) is true whenever \( P(n) \) is true.

By the Principle of Mathematical Induction, \( P(n) \) is true for every counting number, \( n \). That is to say, if \( n \) is a counting number, \( 3^{2n} - 1 \) is divisible by 8.

**Question 6.**

We illustrate, successively \( A \setminus B, A, B' \) and \( A \cap B' \)

\[ A \setminus B \]
Since the shaded areas in the first and last diagrams agree, the sets they depict coincide.