Sample Solutions for Tutorial 8

Question 1.

(i) Since \( \frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} \) for \( 0 < |x| < \pi \),

\[
\lim_{x \to 0} \left( \frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} = \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \quad \text{by the Bernoulli-de l'Hôpital Rule}
\]
\[
= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} \quad \text{by the Bernoulli-de l'Hôpital Rule}
\]
\[
= 0
\]

(ii) For \( u > 0, \ u = e^{\ln u} \).
Since both the exponential and natural logarithm functions are continuous,

\[
\lim_{x \to \infty} x^\frac{1}{x} = a \quad \text{if and only if} \quad \lim_{x \to \infty} \frac{\ln x}{x} = \ln a
\]

\[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0
\]
Thus, \( \lim_{x \to \infty} x^\frac{1}{x} = e^0 = 1 \).

(iii) Since \( \frac{x + \sin x}{x} = 1 + \frac{\sin x}{x} \) and \( \lim_{x \to \infty} \frac{\sin x}{x} = 0 \),

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = 1
\]

(iv) Since \( \frac{d}{dx} (x + \sin x) = 1 + \cos x \) and \( \frac{d}{dx} x = 1 \).

\[
\lim_{x \to \infty} \frac{\frac{d}{dx} (x + \sin x)}{\frac{d}{dx} x} = \lim_{x \to \infty} \frac{1 + \cos x}{1} = 1 + \lim_{x \to \infty} \cos x
\]
and there is no such limit, as \( \cos x \) is bounded, but does not converge as \( x \to \infty \).

Question 2.

If the internal radius of the cylindrical tank be \( r \) metres, then its volume is \( V = 10\pi r^2 \) cubic metres.

Let the error in the measure of the radius be \( \delta r \) and the error in the volume be \( \delta V \).

Then, for small errors,

\[
\delta V \approx \frac{dV}{dr} \delta r = 20\pi r \delta r,
\]
so that the relative error is

\[
\frac{\delta V}{V} \approx \frac{20\pi r \delta r}{10\pi r^2} = \frac{2\delta r}{r}
\]
Thus, to ensure that \( \frac{\delta V}{V} < 1\% = 0.01 \), we must have \( \frac{2\delta r}{r} < 0.01 \).
To achieve this, we must measure the internal radius within 0.5% of the correct value.
Question 3.

The domain of the function \( f: \mathbb{R} \rightarrow \mathbb{R}, \ x \mapsto e^{-x} \cos x \) contains no boundary points.

\[
f'(x) = -e^{-x} \cos x + e^{-x} (-\sin x) \\
= -e^{-x} (\cos x + \sin x) \\
= -e^{-x} \left( \frac{\cos x}{4} - x + \sin x \right) \\
= -e^{-x} 2 \sin \frac{x}{4} \cos \left( x - \frac{\pi}{4} \right)
\]

\[
\begin{align*}
< 0 & \quad \text{if } \frac{(8m-1)\pi}{4} < x < \frac{(8m+3)\pi}{4} \\
= 0 & \quad \text{if } x = \frac{(4n+1)\pi}{4} \\
> 0 & \quad \text{if } \frac{(8m+3)\pi}{4} < x < \frac{(8m+7)\pi}{4}
\end{align*}
\]

Thus \( f \) is differentiable everywhere.

Hence extrema can only occur at points where the derivative is 0.

The function \( f \) is

- monotonically decreasing on the intervals \( \left[ \frac{(8m-1)\pi}{4}, \frac{(8m+3)\pi}{4} \right] \) and
- monotonically increasing on the intervals \( \left[ \frac{(8m+3)\pi}{4}, \frac{(8m+7)\pi}{4} \right] \) \((m \in \mathbb{Z})\)

\[
f''(x) = d\left( e^{-x} (\cos x + \sin x) \right) \\
= e^{-x} (\cos x + \sin x) - e^{-x} (-\sin x + \cos x) \\
= 2 e^{-x} \sin x
\]

\[
\begin{align*}
> 0 & \quad \text{if } 2m\pi < x < (2m + 1)\pi \\
= 0 & \quad \text{if } x = n\pi \\
< 0 & \quad \text{if } (2m - 1)\pi < x < 2m\pi \\
\end{align*}
\]

Hence \( f \) is

- concave up on \( [2m\pi, (2m + 1)\pi] \) and
- concave down on \( [(2m - 1)\pi, 2m\pi] \) \((m \in \mathbb{Z})\).

Thus, \( f \) has

- a local maximum at \( \frac{(8n-1)\pi}{4} \) and
- a local minimum at \( \frac{(8n+3)\pi}{4} \) \((n \in \mathbb{Z})\).

The graph of \( f \) on \([0, 4\pi]\) is