Sample Solutions for Tutorial 7

Question 1.
The function
\[
f: \mathbb{R}_0^+ \rightarrow \mathbb{R}, \quad x \mapsto \ln(1 + x)
\]
is continuous everywhere and differentiable on \(\mathbb{R}_0^+\), with
\[
f'(x) = \frac{1}{1 + x}
\]
Take \(x > 0\).
By the Mean Value Theorem, there is a \(c\) with \(0 < c < x\) and
\[
\frac{1}{1 + c} = f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\ln(1 + x)}{x}
\]
Since \(0 < c < x\), we have, \(1 < 1 + c < 1 + x\), whence
\[
\frac{1}{1 + x} < \frac{1}{1 + c} < 1
\]
Since \(\frac{1}{1 + c} = \frac{\ln(1 + x)}{x}\) and \(x > 0\), this is equivalent to
\[
\frac{x}{1 + x} < \ln(1 + x) < x
\]

Question 2.
Take \(f: \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \frac{(x + 1)^2}{x^2 + 1}\)
\[
f'(x) = \frac{2(x + 1)(x^2 + 1) - (x + 1)^2 2x}{(x^2 + 1)^2} \quad \text{by the quotient rule}
\]
\[
= 2 \left( 1 - \frac{x^2}{(x^2 + 1)^2} \right)
\]
\[
\begin{cases}
< 0 & \text{for } x < -1 \text{ and for } x > 1 \\
= 0 & \text{for } x = \pm 1 \\
> 0 & \text{for } -1 < x < 1
\end{cases}
\]
It follows that
(i) \(f\) is monotonically decreasing on \([-\infty, -1]\) and on \([1, \infty]\);
(ii) \(f\) is monotonically increasing on \([-1, 1]\);
(iii) \(f\) has critical points at \(\pm 1\).
\[
f''(x) = 2 \frac{2x(x^2 + 1)^4 - (1 - x^2)(4x(x^2 + 1))}{(x^2 + 1)^4} \quad \text{by the quotient rule}
\]
\[
= 4 \frac{x(x^2 - 3)}{(x^2 + 1)^3}
\]
\[
\begin{cases}
< 0 & \text{for } x < -\sqrt{3} \text{ and for } 0 < x < \sqrt{3} \\
= 0 & \text{for } x = \pm \sqrt{3} \\
> 0 & \text{for } -\sqrt{3} < x < 0 \text{ and for } x > \sqrt{3}
\end{cases}
\]
It follows that
(iv) \(f\) is concave down on \([-\infty, -\sqrt{3}]\) and on \([0, \sqrt{3}]\);
(v) \(f\) is concave up on \([-\sqrt{3}, 0]\) and on \([\sqrt{3}, \infty]\);
(vi) \(f\) has points of inflexion when \(x = \pm \sqrt{3}\).
Consequently,
(a) a local minimum of 0 at −1 and
(b) a local maximum of 2 at 1.

Since the domain of $f$ has no boundary points, and since $f$ is differentiable everywhere, the only extrema occur when $x = \pm 1$.
Since $f(x) \geq 0$ for all $x$, $f(-1) = 0$ is the absolute minimum value of $f$.
Since $f(x) \to 1$ as $x \to \pm \infty$, $f(1) = 2$ is the absolute maximum value of $f$.
Finally, the graph of $f$ is

![Graph of f](image)

**Question 3.**

Since $2x < 30$, $0 < x < 15$.
The area of the base of the container is $(30 - 2x)^2 = 4(15 - x)^2$ sq. cm.
Since its height is $x$ cm., the volume, $V$ is given by function
$$V: [0, 15] \to \mathbb{R}, \quad x \mapsto 4x(15 - x)^2$$

Plainly, $V(x) > 0$ for all $x$, and $V(x) \to 0$ both as $x \to 0^+$ and as $x \to 15^-$.
The domain of $V$ contains no boundary points.
Being a polynomial functional function, $V$ is differentiable everywhere.
Thus extrema occur only where the derivative of $V$ is 0.
$$\frac{dV}{dx} = 4 \left( \frac{d}{dx} (x - 15)^2 + x \frac{d}{dx} (x - 15)^2 \right)$$
$$= 4 \left( (x - 15)^2 + x \cdot 2(x - 15) \right)$$
$$= 12(x - 15)(x - 5)$$

\begin{align*}
&> 0 \quad \text{for } 0 < x < 5 \\
&= 0 \quad \text{for } x = 5 \\
&< 0 \quad \text{for } 5 < x < 15
\end{align*}

Hence, the only extremum occurs when $x = 5$, and $V(5) = 2,000$ is a maximum.
Thus, the maximum capacity of the container is 2 litres, achieved when the small squares have sides of length 5 centimetres.