Sample Solutions for Tutorial 6

Question 1.
(a) Take $f: \mathbb{R} \rightarrow \mathbb{R}, \ x \mapsto x^2e^x$
Then
\[ f'(x) = \frac{d}{dx} (x^2)e^x + x^2 \frac{d}{dx} (e^x) \]
\[ = 2xe^x + x^2e^x \]
\[ = x(x+2)e^x \]

(b) Take $f: \mathbb{R} \rightarrow \mathbb{R}, \ x \mapsto e^x - e^{-x} \]
Then $f(x) = \frac{e^{2x} - 1}{e^{2x} + 1} = 1 - \frac{2}{e^{2x} + 1}$. Thus
\[ f'(x) = \frac{d}{dx} (1) - 2 \frac{d}{dx} \left( \frac{1}{e^{2x} + 1} \right) \]
\[ = 0 - 2 \frac{-1}{(e^{2x} + 1)^2} \left( \frac{d}{dx} (e^{2x} + 1) \right) \]
\[ = \frac{4e^{2x}}{(e^{2x} + 1)^2} \]
\[ = \left( \frac{2}{e^x + e^{-x}} \right)^2 \]

(c) $f: \mathbb{R} \rightarrow \mathbb{R}, \ y \mapsto \ln(y + \sqrt{y^2 + 1})$
Then
\[ f'(y) = \frac{d}{dy} \ln(y + \sqrt{y^2 + 1}) \]
\[ = \frac{1}{y + \sqrt{y^2 + 1}} \frac{d}{dy} (y + \sqrt{y^2 + 1}) \]
\[ = \frac{1}{y + \sqrt{y^2 + 1}} \left( 1 + \frac{2y}{2\sqrt{y^2 + 1}} \right) \]
\[ = \frac{1}{\sqrt{y^2 + 1}} \]

Question 2.
(a) Given that $x^2 + y^2 = 1$ we must have $-1 \leq x \leq 1$.
Hence we take $X = [-1, 1]$.
Differentiating both sides of the above equation with respect to $x$, we obtain
\[ 2x + 2y \frac{dy}{dx} = 0, \]
so that $\frac{dy}{dx} = \frac{-x}{y}$ as long as $y \neq 0$, that is, $x \neq \pm 1$.

(b) Differentiating both sides of the equation $xy^2 + y\sin(xy) + e^x = 0$ with respect to $x$, we obtain
\[ y^2 + 2xy \frac{dy}{dx} + \frac{dy}{dx} \sin(xy) + y \cos(xy)(y + x \frac{dy}{dx}) + e^x = 0, \]
or 
\[
\frac{dy}{dx} = -\frac{e^x + y^2(1 + \cos(xy))}{\sin(xy) + xy(2 + \cos(xy))}
\]
as long as \(\sin(xy) + xy(2 + \cos(xy)) \neq 0\)

(c) Differentiating both sides of the equation \(e^{2x} - 2ye^x - 1 = 0\) with respect to \(x\), we obtain
\[
2e^{2x} - 2\frac{dy}{dx}e^x - 2ye^x = 0,
\]
or
\[
\frac{dy}{dx} = e^x - y
\]
as \(e^x \neq 0\) for all \(x \in \mathbb{R}\).

Question 3.
Take \(f: \mathbb{R} \rightarrow \mathbb{R}, \ x \mapsto x^3 - 6x^2 + 3x - 7\).
Being a polynomial function, \(f\) is differentiable everywhere, thereby satisfying the hypotheses of the Mean Value Theorem on \([0,6]\). Now
\[
\frac{f(6) - f(0)}{6 - 0} = \frac{6^3 - 6.6^2 + 3.6}{6} = 3
\]
As \(f'(x) = 3x^2 - 12x + 3\), \(f'(c) = \frac{f(6) - f(0)}{6 - 0} = 3\) if and only if \(3c^2 - 12c + 3 = 3\), or, equivalently, \(c(c - 4) = 0\).
Thus \(f'(c) = \frac{f(6) - f(0)}{6 - 0}\) for \(c \in [0,6]\) if and only if \(c = 0, 4\).

Question 4.
Given \(f: \mathbb{R} \rightarrow \mathbb{R}\) with \(f'(x) = 0\) for all \(x \in \mathbb{R}\), take \(a, b \in \mathbb{R}\) with \(a < b\).
Since \(f'(x) = 0\) for all \(x \in \mathbb{R}\), \(f\) satisfies the hypotheses of the Mean Value Theorem.
Thus, there is a \(c \in [a,b]\) with \(f(b) - f(a) = f'(c)(b - a)\).
But \(f'(c) = 0\), whence \(f(b) = f(a)\), showing that \(f\) is constant.

Take \(f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}, \ x \mapsto \frac{x}{|x|} = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}\)
Since \(f(x) = -1\) for all \(x < 0\), we have \(f'(x) = 0\) for all \(x < 0\).
Similarly, since \(f(x) = 1\) for all \(x > 0\), we have \(f'(x) = 0\) for all \(x > 0\).
Hence \(f'(x) = 0\) for all \(x \in \text{dom}(f)\), even though \(f\) is not constant.