Sample Solutions for Tutorial 5

Question 1.
Take \( a \in \mathbb{R} \). Then, for all \( x \in \mathbb{R} \),
\[
\cos x - \cos a = -2 \sin \left( \frac{x+a}{2} \right) \sin \left( \frac{x-a}{2} \right)
\]
Thus, taking \( f : \mathbb{R} \to \mathbb{R}, \ x \mapsto \cos x \), we have
\[
f'(a) = \lim_{x \to a} \frac{\cos x - \cos a}{x - a}
= \lim_{x \to a} \frac{-2 \sin \left( \frac{x+a}{2} \right) \sin \left( \frac{x-a}{2} \right)}{x - a}
= -\lim_{x \to a} \sin \left( \frac{x+a}{2} \right) \lim_{x \to a} \sin \left( \frac{x-a}{2} \right)
= -\sin a
\]
as the sine function is continuous and \( \lim_{u \to 0} \frac{\sin u}{u} = 1 \).

Question 2.
(a) \( f \) is the sum of the polynomial function given by \( 3x^5 - 6x^3 \) and the cosine function. Since both of these are differentiable everywhere, so is \( f \) and its derivative is the sum of there derivatives. Thus, for every \( x \in \mathbb{R} \),
\[
f'(x) = 15x^4 - 18x^2 - \sin x.
\]
(b) Since \( g(x) = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases} \), it is immediate that \( g \) is differentiable at \( a \) whenever \( a \neq 0 \), with
\[
f'(a) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}
\]
Now
\[
g'(0) = \lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}
\]
But
\[
\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^-} 1 = 1
\]
whereas
\[
\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^-} (-1) = -1.
\]
Since \( \lim_{x \to 0^+} \frac{g(x) - g(0)}{x} \neq \lim_{x \to 0^-} \frac{g(x) - g(0)}{x} \), \( g \) is not differentiable at 0.

(c) Note that \( h = \beta \circ \alpha \), with \( \alpha, \beta \) given by

\[
\alpha: [-1, 1] \to [0, 1], \quad x \mapsto 1 - x^2
\]

\[
\beta: [0, 1] \to [0, 1], \quad u \mapsto \sqrt{u}
\]

Since \( \alpha \) is a polynomial function, it is differentiable, with derivative is given by

\[
\alpha'(x) = -2x.
\]

\( \beta \) is differentiable at \( u \) when \( u > 0 \), with

\[
\beta'(u) = \frac{1}{2\sqrt{u}} = \frac{1}{2\beta(u)}
\]

Take \( u \geq 0 \). Then

\[
\frac{\beta(u) - \beta(0)}{u - 0} = \frac{\sqrt{u}}{u} = \frac{1}{\sqrt{u}}
\]

\[\to \infty \quad \text{as} \quad u \to 0^+.\]

This shows that \( \beta \) is not differentiable at 0.

Since \( \alpha(x) = 0 \) if and only if \( x = 0, \ell 1 \), we see that \( h \) is differentiable at every \( a \in ]0, 1[ \), but not at \( a = 0, 1 \), and

\[
h'(a) = \beta'(\alpha(a))\alpha'(a)
\]

\[
= \frac{1}{2\beta(\alpha(a))}(-2a)
\]

\[
= -\frac{a}{\sqrt{1 - a^2}}
\]

(d) Since \( k \) is not continuous at \( a \neq 0 \), \( k \) cannot be differentiable at \( a \neq 0 \).

\[
\frac{k(x) - k(0)}{x - 0} = \frac{k(x)}{x}
\]

\[
= \begin{cases} 
  x & \text{if } x \in \mathbb{Q} \\
  0 & \text{otherwise,}
\end{cases}
\]

\[\to 0 \quad \text{as} \quad x \to 0.
\]

Hence \( k \) is differentiable only at 0 and \( k'(0) = 0 \).

Question 3.

Recall that there is a best approximation of the form \( p_1(x) = c_0 + c_1 x \) near \( x = a \) to the function, \( f \), if and only if \( f \) is differentiable at \( x = a \), and then

\[
p_1(x) = f'(a)(x - 1) + f(a).
\]

This is then the equation of the line in the Cartesian plane which is tangent to the graph of \( f \) at the point with coordinates \( (a, f(a)) \).
\[
\tan x := \frac{\sin x}{\cos x}
\]
is differentiable everywhere on \( ] - \frac{\pi}{2}, \frac{\pi}{2} [ \), with
\[
\frac{d}{dx} \tan x = \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right)
\]
\[
= \frac{d}{dx} \left( \sin x \frac{1}{\cos x} \right)
\]
\[
= \cos x \frac{1}{\cos x} + \sin x (-\sin x) \left( \frac{-1}{\cos^2 x} \right)
\]
\[
= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}
\]
\[
= \frac{1}{\cos^2 x}
\]
Now \( \cos 0 = 1 \) and \( \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}} \), so that the equation of the line tangent to the graph at the point \((0, 0)\) is \( y = 1(x - 0) = 0 \), or
\[
x - y = 0,
\]
and that of the line tangent to the graph at the point \( (\frac{\pi}{4}, 1) \) is \( y = \frac{1}{\frac{1}{2}} \left( x - \frac{\pi}{4} \right) + 1 \), or
\[
2x - y = \frac{\pi}{2} - 1
\]