Sample Solutions for Tutorial 2

**Question 1.** Let \( K := \inf(A) \), \( L := \sup(A) \), \( M := \inf(B) \) and \( N := \sup(B) \).

**A \cup B:** Take \( x \in A \cup B \).
Then \( x \in A \) or \( x \in B \).
If \( x \in A \), then \( x \leq L \leq \max\{L, N\} \).
Otherwise, \( x \in B \), whence \( x \leq N \leq \max\{L, N\} \).
Thus, \( x \leq \max\{L, N\} \) for all \( x \in A \cup B \).
Hence, \( A \cup B \) is bounded above by \( \max\{L, N\} \).
To show that no smaller number can be an upper bound, take \( S < \max\{L, N\} \).
Then either \( S < L \) or \( S < N \).
In the former case, there is an \( x \in A \subseteq A \cup B \), with \( S < x \leq L \), so that \( S \) is not an upper bound for \( A \cup B \).
In the latter case, there is an \( x \in B \subseteq A \cup B \), with \( S < x \leq N \), so that \( S \) is not an upper bound for \( A \cup B \).
Hence \( \max\{L, N\} \) is the least upper bound for (supremum of) \( A \cup B \).

**A \cap B:** Take \( x \in A \cap B \).
Then \( x \in A \) and \( x \in B \).
Since \( x \in A \), we have \( x \geq K \).
Since \( x \in B \), we have \( x \geq M \).
Since \( x \geq K, M \), we have \( x \geq \max\{K, M\} \).
Thus, \( A \cap B \) is bounded below by \( \max\{K, M\} \).
Since \( A \cap B \) is a set of real numbers, that is bounded below, it has has an infimum.
Since \( K, M \) are both lower bounds for \( A \cap B \), \( \inf(A \cap B) \geq \max\{K, M\} \).
To see that equality need not hold, consider \( A := \{0, 2\} \) and \( B := \{1, 2\} \).
Then \( \inf(A) = 0 \), \( \inf(B) = 1 \), so that \( \max\{\inf(A), \inf(B)\} = \max\{0, 1\} = 1 \).
On the other hand, \( \inf(A \cap B) = \inf\{2\} = 2 \).

**Question 2.** As the sum of two real numbers is the same as the sum of their maximum and their minimum, given \( a, b \in \mathbb{R} \),
\[
\max\{a, b\} + \min\{a, b\} = a + b \tag{*}
\]
Similarly, the absolute value of their difference is the larger less the smaller, or,
\[
\max\{a, b\} - \min\{a, b\} = |a - b| \tag{**}
\]
Adding (**) to (*) yields \( 2 \max\{a, b\} = a + b + |a - b| \), or
\[
\max\{a, b\} = \frac{a + b + |a - b|}{2}.
\]
Subtracting (**) from (*) yields \( 2 \min\{a, b\} = a + b - |a - b| \), or
\[
\min\{a, b\} = \frac{a + b - |a - b|}{2}.
\]
Question 3.

(i): Take \( n \in \mathbb{N} \). Then
\[
\frac{1}{2^{n+1}} = \frac{1}{2} \cdot \frac{1}{2^n} < \frac{1}{2^n}
\]
as \( 0 < \frac{1}{2} < 1 \).

Hence we can arrange the elements of \( A \) in strictly decreasing order as
\[
A := \{ 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \}
\]

It follows immediately that 1 is the largest element of \( A \), whence \( A \) is bounded above and has a supremum, 1, which is actually its maximum.

Since every element of \( A \) is positive, it follows immediately that \( A \) is bounded below by 0.

We show that 0 is actually the infimum (greatest lower bound) of \( A \). To show this we must show that given any positive real number, there is an element of \( A \) smaller than that positive real number.

We first use the Principle of Mathematical Induction to show that, for every \( n \in \mathbb{N} \), \( 2^n > n \).

\( n = 0, 1 : \) In these cases we have
\[
2^0 = 1 > 0 \quad \text{and} \quad 2^1 = 2 > 1.
\]

\( n \geq 1 : \) We make the inductive hypothesis that \( 2^n > n \). Then
\[
2^{n+1} = 2 \cdot 2^n > 2n \quad \text{by the Inductive Hypothesis}
\]
\[
= n + n = n + 1 \quad \text{as} \ n \geq 1.
\]

This completes the proof by mathematical induction.

An immediate consequence is that for every counting number, \( n \),
\[
0 < \frac{1}{2^n} \leq \frac{1}{n}.
\]

Take \( a > 0 \). Then \( \frac{1}{a} > 0 \).

By the Archimedean property of the real numbers, there is a counting number, \( n \), with
\[
0 < n - 1 \leq \frac{1}{a} < n.
\]

Since \( 2^n > n \), we have \( 0 < \frac{1}{a} < 2^n \), so that
\[
0 < \frac{1}{2^n} < a
\]

Since \( \frac{1}{2^n} \in A \), we have shown that \( a \) is not a lower bound for \( A \). Thus \( \inf(A) = 0 \).

Since \( 0 \notin A \), \( A \) has an infimum, but no minimum;

(ii): Every integer \( n \) can be written in precisely one of the forms \( 4k, 4k + 1, 4k + 2 \) or \( 4k + 3 \), where \( k \) is itself an integer. Then there are four possibilities for \( \cos(n \frac{\pi}{2}) \), namely:
\( n = 4k : \) \( \cos(n \frac{\pi}{2}) = \cos(2k\pi) = \cos 0 = 1 \) as \( \cos(x + 2\pi) = \cos x. \)

\( n = 4k + 1 : \) \( \cos(n \frac{\pi}{2}) = \cos(2k\pi + \frac{\pi}{2}) = \cos \frac{\pi}{2} = 0. \)

\( n = 4k + 2 : \) \( \cos(n \frac{\pi}{2}) = \cos(2k\pi + \pi) = \cos \pi = -1. \)

\( n = 4k + 3 : \) \( \cos(n \frac{\pi}{2}) = \cos(2k\pi + \frac{3\pi}{2}) = \cos \frac{3\pi}{2} = 0. \)

Thus \( B = \{-1, 0, 1\} \) which is bounded, with \(-1\) as minimum and \(1\) as maximum.

(iii): In order for the inequality \( \frac{x}{1+x} \geq 0 \) to make sense, we must have \( 1 + x \neq 0 \), or, equivalently, \( x \neq -1. \)

If \( x \neq -1 \), then either \( x < -1 \) or \( x > -1. \)

In the former case, \( 1 + x < 0 \), so that \( \frac{x}{1+x} \geq 0 \) if and only if \( x \leq 0(1 + x) = 0, \)

which is always the case, since \( x < -1 < 0. \)

In the latter case, \( 1 + x > 0 \), so that \( \frac{x}{1+x} \geq 0 \) if and only if \( x \geq 0(1 + x) = 0, \)

which is only the case when \( x \geq 0. \)

Hence \( C = \{x \in \mathbb{R} \mid x < -1 \text{ or } x \geq 0\} \), and this is bounded neither below nor above.

(iv): Observe that for \( x \neq -1 \)

\[
\frac{x}{1+x} = \frac{1+x-1}{1+x} = \frac{1+x}{1+x} - \frac{1}{1+x} = 1 - \frac{1}{1+x}.
\]

Since \( x \geq 0, 1 + x \geq 1 > 0, \) whence \( 0 < \frac{1}{1+x} \leq 1. \)

Thus, \( 0 \leq 1 - \frac{1}{1+x} < 1, \) or, equivalently,

\[ 0 \leq \frac{x}{1+x} < 1. \]

It follows immediately that \( D \) is bounded above by \( 1 \) and below by \( 0. \)

Moreover, since \( \frac{0}{1+r} = 0 \) and \( 0 \geq 0, \) we see that \( 0 \in D, \) whence \( \min(D) = 0. \)

Choose a real number, \( r, \) with \( 0 < r < 1. \) Then \( 0 < 1 - r < 1, \) whence \( \frac{1}{1-r} > 1. \)

Choose a real number, \( t, \) with \( t > \frac{1}{1-r} - 1 > 0. \)

Then \( 1 + t > \frac{1}{1-r} > 1. \)

Hence \( 0 < \frac{1}{1-r} < 1 - r < 1, \) or equivalently,

\[ -1 < -(1 - r) < -\frac{1}{1+t} < 0, \]
whence
\[ 0 < r = 1 - (1 - r) < 1 - \frac{1}{1 + t} = \frac{t}{1 + t} < 1. \]

Since \( \frac{t}{1 + t} \in D \), we see that \( r \) is not an upper bound for \( D \).
Hence, \( \sup(D) = 1 \), but \( D \) has no maximum.

**Question 4.**

(i) \((2 - i)(2 + i) = 2^2 - i^2 = 4 - (-1) = 5 + 0i.\)

Thus \(|(2 - i)(2i)| = |5 + 0i| = 5\) and
\((2 - i)(2i) = 5 + i0 = 5 - 0i = 5.\)

(ii) \((6 + 5i)(2 - 7i) = (12 - (-35)) + i(-42 + 10) = 47 - 32i.\)

Thus \(|(6 + 5i)(2 - 7i)| = \sqrt{47^2 + (-32)^2} = \sqrt{3233}\) and
\((6 + 5i)(2 - 7i) = 47 + 32i.\)

(iii) \(\frac{2 - i}{1 + 2i} \quad \text{and} \quad \frac{2 - i}{1 + 2i} = i\)

Thus \(|\frac{2 - i}{1 + 2i}| = |i| = 1\) and
\(|\frac{2 - i}{1 + 2i}| = |i| = 1\)

Alternatively, note that since \(i^2 = -1\), we have \(-i^2 = 1.\)

Then \(2 - i = -i^2 - i = -i(1 + 2i)\), so that \(\frac{2 - i}{1 + 2i} = -i(1 + 2i) = -i\)

(iv) \(\frac{1 - 3i}{(2 + i)^2} + \frac{1 + i^3}{1 + i} = \frac{(1 - 3i)(2 - i)^2}{(2 + i)(2 - i)^2} + \frac{(1 - i)(1 - i)}{(1 + i)(1 - i)}\)

\[ = \frac{(1 - 3i)(5 - 4i)}{25} + \frac{2 - 2i}{2} \]

\[ = \frac{-7 - 19i}{25} + 1 - i \]

\[ = \frac{18}{25} - \frac{44}{25}i \]

Thus \(|\frac{1 - 3i}{(2 + i)^2} + \frac{1 + i^3}{1 + i}| = \frac{18}{25} - \frac{44}{25}i = \frac{2}{25}|9 - 22i| = \frac{2\sqrt{565}}{25} = 2\sqrt{\frac{113}{5}}\) and

\(|\frac{1 - 3i}{(2 + i)^2} + \frac{1 + i^3}{1 + i}| = \frac{18}{25} + \frac{44}{25}\)