Sample Solutions for Tutorial 1

Question 1.
(a) If we let $P(x)$ be $x$ is a positive integer divisible by 4 and 6 and $Q(x)$ be $x$ is a positive integer divisible by 24, then the proposition is of the form

$$(\forall x)(P(x) \Rightarrow Q(x)).$$

The negation is then

$$(\exists x)\neg(P(x) \Rightarrow Q(x)),$$

which is

$$(\exists x)(P(x) \land \neg Q(x)),$$

or, in English,

There is a positive integer is divisible by 4 and by 6, which is not divisible by 24.

The converse is

$$(\forall x)(Q(x) \Rightarrow P(x)),$$

or, in English,

If a positive integer is divisible by 24, then it is divisible by 4 and by 6.

(b) If we write $P(x)$ for $x$ is a prime number, $Q(x)$ for $x$ is an odd number, then the first proposition is

$$(\forall x)(P(x) \Rightarrow Q(x)).$$

Its negation is thus

$$(\exists x)(P(x) \land \neg Q(x)),$$

or, in English,

There is a prime number which is not odd.

If we write $C(x)$ for $x$ is a clever person, $D(x)$ for $x$ does dumb things, then the second proposition is

$$(\exists x)(C(x) \land D(x)).$$

Its negation is thus

$$\neg(\exists x)(C(x) \land D(x)),$$

which is

$$(\forall x)\neg(C(x) \land D(x)),$$

or

$$(\forall x)((\neg C(x)) \lor (\neg D(x)))$$

or, in English,

No clever person does dumb things.

Question 2.
Let $a$ be a real number and suppose that for every real number, $b$,

$$(a + b)^2 = a^2 + b^2.$$ 

Then, since $\frac{1}{2}$ is a real number, we must have

$$a^2 + (\frac{1}{2})^2 = (a + \frac{1}{2})^2$$

$$= a^2 + a + (\frac{1}{2})^2$$

Subtracting $a^2 + (\frac{1}{2})^2$ from both sides of the equation, we see that $a = 0$. 
Question 3.

Let $P(n)$ be the proposition $\sum_{j=1}^{n} j^3 = \left(\frac{1}{2} \frac{1}{2} 1(1 + 1)\right)^2$

$n = 1$:

$$\sum_{j=1}^{1} j^3 = 1^3 = 1 = \left(\frac{1}{2} \times 2\right)^2 = \left(\frac{1}{2} \frac{1}{2} 1(1 + 1)\right)^2.$$ 

Hence $P(1)$ is true.

$n \geq 1$: We make the Inductive hypothesis that $P(n)$ is true, that is,

$$\sum_{j=1}^{n} j^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{1}{2} n(n + 1)\right)^2.$$ 

Then

$$\sum_{j=1}^{n+1} j^3 = \left(\sum_{j=1}^{n} j^3\right) + (n + 1)^3$$
$$= \left(\frac{1}{2} n(n + 1)\right)^2 + (n + 1)^3 \quad \text{by the Inductive Hypothesis}$$
$$= (n + 1)^2\left(\frac{1}{2} n^2 + (n + 1)\right)$$
$$= (n + 1)^2\left(\frac{n^2 + 4n + 4}{4}\right)$$
$$= \frac{(n + 1)^2(n + 2)^2}{4}$$
$$= \left(\frac{1}{2} (n + 1)(n + 1 + 1)\right)^2$$

Hence $P(n + 1)$ is true whenever $P(n)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for every counting number, $n$. That is to say, $\sum_{j=1}^{n} j^3 = \left(\frac{1}{2} \frac{1}{2} 1(1 + 1)\right)^2$ for every counting number, $n$.

Question 4.

Let $a$ be a positive real number, and $P(n)$ the proposition $(1 + a)^n \geq 1 + na$.

$n = 1$: Then

$$(1 + a)^1 = 1 + a = 1 + 1.a,$$

showing that $P(1)$ is true.

$n \geq 1$: We make the Inductive hypothesis that $P(n)$ is true, that is,

$$(1 + a)^n \geq 1 + na.$$ 

Then

$$(1 + a)^{n+1} = (1 + a)(1 + a)^n$$
$$\geq (1 + a)(1 + na) \quad \text{by the Inductive Hypothesis and the fact that } 1 + a > 0$$
$$= 1 + (n + 1)a + na^2$$
$$> 1 + (n + 1)a \quad \text{as } na^2 > 0$$
Hence $P(n + 1)$ is true whenever $P(n)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for every counting number, $n$. That is to say, $(1 + a)^n \geq 1 + na$, for every counting number.

**Question 5.**

Let $P(n)$ the proposition that if $n$ is a counting number, then $3^{2n} - 1$ is divisible by 8.

$n = 1$: Then

$$3^{2\times1} - 1 = 9 - 1 = 8,$$

so that $P(1)$ is true.

$n \geq 1$: We make the Inductive hypothesis that $P(n)$ is true, that is, there is a counting number $k$, such that

$$3^{2n} - 1 = 8k.$$

Then

$$3^{2(n+1)} - 1 = 3^{2n+2} - 1$$
$$= 9\times3^{2n} - 1$$
$$= 9\times3^{2n} - 9 + 8$$
$$= 9\times(3^{2n} - 1) + 8$$
$$= 9\times8k + 8$$
$$= 8\times(9k + 1),$$

which is plainly divisible by 8.

Hence $P(n + 1)$ is true whenever $P(n)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for every counting number, $n$. That is to say, if $n$ is a counting number, $3^{2n} - 1$ is divisible by 8.

**Question 6.**

We illustrate, successively $A \setminus B, A, B'$ and $A \cap B'$

![Venn Diagram](image)
Since the shaded areas in the first and last diagrams agree, the sets they depict coincide.