1. Exponents

We begin by defining $a^n$ for suitable $a$ and $n$.

If we restrict $n$ to being a counting number, we can define exponentiation as *repeated multiplication* and this is then a well defined operation.

Informally, we define the $n^{th}$ power of the real number, $a$, to be the real number obtained by taking $n$ copies of $a$ and multiplying them together:

$$a^n = a \cdots a$$

$n$ copies

We formulate this in a recursive definition.

**Definition 1.1.** Let $a$ be any real number. The $n^{th}$ power of $a$, $a^n$, is given by

$$a^0 := a$$

$$a^{n+1} := a \cdot a^n$$

for any counting number, $n$.

**Lemma 1.2.** Take $a \in \mathbb{R}$ and $m, n \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Then

$$a^m a^n = a^{m+n}$$

$$(a^m)^n = a^{mn}$$

**Proof.** The claims follow by repeated application of the associative law for multiplication.

$$a^m a^n = \underbrace{a \cdots a}_{m \text{ copies}} \underbrace{a \cdots a}_{n \text{ copies}}$$

$$= a \cdots a \cdots a$$

$m+n$ copies

$$= a^{m+n}$$

$$(a^m)^n = \underbrace{a^m \cdots a^m}_{n \text{ copies}}$$

$$= \underbrace{a \cdots a \cdots a}_{m \text{ copies}} \underbrace{a \cdots a}_{m \text{ copies}}$$

$$= \underbrace{a \cdots a \cdots a}_{mn \text{ copies}}$$

$$= a^{mn}$$

**Remark 1.3.** If we restrict attention to $a > 0$, then for every counting number, $n$,

$$a^{n+1} = \begin{cases} 
  < a^n & \text{if } a < 1 \\
  = a^n = 1 & \text{if } a = 1 \\
  > a^n & \text{if } a > 1 
\end{cases}$$

In particular, if $a > 0$ and $a \neq 1$, then

$$e : \mathbb{N} \setminus \{0\} \longrightarrow \mathbb{R}, \quad n \longmapsto a^n$$

is an injective function.

We now seek to extend the definition of the $n^{th}$ power of the real number $a$ to all natural numbers, $n$, not just counting numbers, while preserving Lemma 1.2.

In particular, we wish to have $a^m a^n = a^{m+n}$ continue to hold even if either $m$ or $n$ is 0. In other words, we want

$$a^n = a^{0+n} = a^0 a^n$$
to hold. But this is equivalent to

\[(a^0 - 1)a^n = 0,\]

from which we deduce that either \(a^n = 0\) or \(a^0 = 1\).

In the first case, we must have \(a = 0\), and nothing can be inferred about \(a^0\).

If, on the other hand, \(a \neq 0\), then we must have \(a^0 = 1\).

If \(a \neq 0\), then \(a^0 := 1\). We do not define \(0^0\).

This allows us to define \(a^n\) recursively for any non-zero real number, \(a\), and any natural number:

**Definition 1.4.** Let \(a\) be a non-zero real number. Then

\[
a^0 := 1
\]

\[
a^{n+1} := a.na^n \quad \text{for any } n \in \mathbb{N}
\]

We verify that all of Lemma 1.2 remains true as long as \(a \neq 0\).

\[
a^m a^0 = a^{m+0} = a^m
\]

\[
(a^0)^n = 1^n = 1 = a^0 = a^0 n
\]

\[
(a^m)^0 = 1 = a^0 = a^{m0}
\]

We next extend the definition to all integers. The problem is to define \(a^{-n}\) for \(n \in \mathbb{N}\). We let Lemma 1.2 and Definition 1.4 guide us.

We want

\[
1 = a^0 = a^{(-n)+n} = a^{-n} a^n
\]

Since \(a \neq 0\) and \(n \in \mathbb{N}\), \(a^n \neq 0\), whence the above equality implies \(a^{-n}\) must be \(\frac{1}{a^n}\).

**Definition 1.5.** Given \(a \in \mathbb{R} \setminus \{0\}\) and \(n \in \mathbb{N}\),

\[
a^{-n} := \frac{1}{a^n}
\]

**Remark 1.6.** Since \((-(-n)) = n\) for every integer \(n\), \(\frac{1}{x} = x\) and \((\frac{1}{x})^n = \frac{1}{x^n}\) or every non-zero real number, we see that Lemma 1.2 still holds.

We extend the definition to all rational numbers, the problem being to define \(a^{\frac{n}{m}}\) where \(n\) is a counting number. We let Lemma 1.2 guide us again.

We want

\[
\left(a^{\frac{1}{m}}\right)^n = a^{\frac{n}{m}} = a^1 = a.
\]

In other words, \(a^{\frac{1}{m}}\) should solve the equation \(x^n = a\).

Since this equation has no real solutions if \(a < 0\) and \(n\) is even, we restrict attention to \(a > 0\).

**Definition 1.7.** Given \(a > 0\) and a counting number \(n\), \(a^{\frac{1}{m}}\) is the unique positive solution of the equation \(x^n = a\).

We leave it to the reader to verify that Lemma 1.2 still holds.

The next theorem summarises our investigations.

**Theorem 1.8.** Let \(a\) be any positive real number and \(m, n\) rational numbers. Then

\[
a^0 := 1 \quad (1)
\]

\[
a^1 := a \quad (2)
\]

\[
a^m a^n = a^{m+n} \quad (3)
\]

\[
(a^m)^n = a^{mn} \quad (4)
\]

\[
a^{-m} := \frac{1}{a^m} \quad (5)
\]

\[
a^{\frac{m}{n}} := \sqrt[n]{a^m} \quad \text{if } n \neq 0 \quad (6)
\]
Moreover, if \( a > 1 \), then
\[
\begin{align*}
  f : \mathbb{Q} &\rightarrow \mathbb{R}^+, \\
  x &\mapsto a^x
\end{align*}
\]
is monotonically strictly increasing (and hence injective).

**Remark 1.9.** We lose nothing by restricting attention to \( a > 1 \). For if \( 0 < a < 1 \), then \( a^x = b^{-x} \), where \( b := \frac{1}{a} > 1 \) and \( 1^x = 1 \) for all \( x \in \mathbb{Q} \) since we require \( a^x \) to be a real number.

**Remark 1.10.** It is, in fact, possible to extend the definition of \( a^x \) to all real numbers \( x \), but this requires theory not yet developed in this course, on way of doing so uses the properties of the (Riemann) integral as developed in MATH102. We provide an outline of this below.

### 2. Logarithms

Given \( a > 1 \), the function
\[
\begin{align*}
  f : \mathbb{Q} &\rightarrow \mathbb{R}^+, \\
  x &\mapsto a^x
\end{align*}
\]
is monotonically strictly increasing. Moreover, \( a^n \rightarrow \infty \) as \( n \rightarrow \infty \), and, as will be proved later, \( f \) is continuous. Thus \( a^x \rightarrow \infty \) as \( x \rightarrow \infty \) and \( a^x \rightarrow 0 \) as \( x \rightarrow -\infty \).

Hence, by the Intermediate Value Theorem, \( \text{im}(f) = \mathbb{R}^+ \).

Since \( f \) is bijective, it has an inverse, namely the **logarithm to the base** \( a \).

**Definition 2.1.** Take \( a > 1 \) and \( x \in \mathbb{R}^+ \). Then \( u \) is the logarithm of \( x \) to the base \( a \), written \( u = \log_a x \), if and only if \( x = a^u \).

**Theorem 2.2.** Take \( a > 1 \), \( x, y \in \mathbb{R}^+ \) and \( n \in \mathbb{R} \). Then
\[
\begin{align*}
  \log_a 1 &:= 0 \quad (7) \\
  \log_a a &:= 1 \quad (8) \\
  \log_a (xy) &= \log_a x + \log_a y \quad (9) \\
  \log_a (x^y) &= y \log_a x \quad (10)
\end{align*}
\]
**Proof.** The proofs are direct consequences of the definitions and the corresponding results on exponents.

We illustrate this by proving (9).

Put \( u := \log_a x \) and \( v := \log_a y \), so that \( x = a^u \) and \( y = a^v \). Then
\[
xy = a^u a^v = a^{u+v},
\]
from which it follows that
\[
\log_a(xy) = u + v = \log_a x + \log_a y.
\]

\qed

### 3. Natural Logarithms and the Exponential Function

We outline here the definitions and main results, omitting the proofs which involve the theory of (Riemann) integration.

Recall that for \( n \in \mathbb{Z} \), the function
\[
\begin{align*}
  f : \mathbb{R}^+ &\rightarrow \mathbb{R}, \\
  x &\mapsto x^n
\end{align*}
\]
is differentiable with Theorem 2.2
\[
f'(x) = nx^{n-1}.
\]

Using elementary properties and techniques of integration that for all \( u, v \in \mathbb{R}^+ \) and \( w \in \mathbb{R} \),
\[
\begin{align*}
  L(1) &= 0 \\
  L(uv) &= L(u) + L(v) \\
  L(u^w) &= wL(u) \\
  L\left(\frac{1}{u}\right) &= -L(u),
\end{align*}
\]
which the reader is invited to compare with...
Looked at another way, $f$ is the derivative of

$$F : \mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \frac{x^{n+1}}{n+1},$$

as long as $n \neq -1$.

However, since

$$f : \mathbb{R}^+ \to \mathbb{R}, \quad a \mapsto \frac{1}{x},$$

is a continuous function, by the Fundamental Theorem of Calculus

$$L : \mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \int_1^x \frac{1}{t} dt$$

is a differentiable function with

$$L'(x) = \frac{1}{x} = f(x).$$

Since $L'(x) > 0$ for all $x$, $L$ is monotonically strictly increasing, and hence injective.

Moreover, as indicated in the next diagram, if $n > 1$ is an integer,

$$\int_1^n \frac{1}{t} dt > \sum_{j=2}^n \frac{1}{j}.$$

Since $\sum \frac{1}{n}$ diverges, $L(x) \to \infty$ as $x \to \infty$. Since $L\left(\frac{1}{x}\right) = -L(x)$, $L(x) \to -\infty$ as $x \to 0$.

Thus, $\text{im}(L) = \mathbb{R}$.

Because $L$ is bijective, it has an inverse,

$$E : \mathbb{R} \to \mathbb{R}^+, \quad u \mapsto x,$$

where $u = L(x)$.

It follows that for all $u, v, w \in \mathbb{R}$,

$$E(1) = 0$$

$$E(u + v) = E(u)E(v)$$

$$E(uv) = (E(u))^v,$$  \hspace{1cm} (11)

which the reader is invited to compare with Theorem 1.8.

This motivates the next definition

**Definition 3.1.**

$$e := E(1)$$

Equivalently $e$ is the (uniquely determined) real number such that

$$L(e) = \int_1^e \frac{1}{t} dt = 1$$

It then follows from Equation (11) and Theorem 1.8 that

$$E(x) = e^x.$$
and this allows us to define the natural logarithm.

**Definition 3.2.** The natural logarithm is the function

\[ \ln : \mathbb{R}^+ \rightarrow \mathbb{R}, \quad x \mapsto \ln x := \log_e x \]

and the exponential function

\[ \exp : \mathbb{R} \rightarrow \mathbb{R}^+, \quad x \mapsto e^x \]

Since \( L'(x) = \frac{1}{x} \neq 0 \) for all \( x \in \mathbb{R}^+ \), \( E \) is differentiable and

\[ E'(u) = \frac{1}{L'(E(u))} = \frac{1}{E(u)} = E(u). \]

Thus, we have

\[ \frac{d}{dx} \ln x = \frac{1}{x} \]

\[ \frac{d}{dx} e^x = e^x \]

**Remark 3.3.** The exponential function is one of the most important functions in mathematics.

One reason is that there is a generalisation of it to the theory of Lie groups and Lie algebras, which is at the heart of so much theoretical physics, particularly to quantum theory.

Another reason is that the exponential function in the case of functions of a complex variable encompassed the trigonometric functions, as we indicate.

Recall that each non-zero complex number \( z = x + iy \) can be written in the form \( r \cos \theta + i \sin \theta \), which is unique if we require that \( 0 \leq \theta < 2\pi \).

Thus \( z = r \cos \theta + i \sin \theta \). If we fix the modulus of \( z \) and treat \( i \) as just a constant, we may regard \( z \) as a function of \( \theta \) and then

\[ \frac{dz}{d\theta} = -r \sin \theta + i r \cos \theta = i(r \cos \theta + i \sin \theta) = iz. \]

whence, from the theory of integration, we must have

\[ z = A e^{i\theta}. \]

Using the fact that \( z = r(\neq 0) \) when \( \theta = 0 \), we see that \( A = 1 \) and we have arrived at the Argand form of the complex number \( z \) viz. \( z = re^{i\theta} \). In particular, we obtain Euler’s Formula

\[ e^{i\pi} = -1. \]

We also obtain an “obvious” reason why

\[ (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta, \]

for this is just the equality

\[ (e^{i\theta})^n = e^{in\theta}. \]

[The above is a heuristic argument. Formally rigorous justification is provided in MATH102 and in the study of functions of a complex variable.]