1. Applying the Mean Value Theorem

One of the themes of these notes is to show when and how functions can be approximated using polynomial functions. The Mean Value Theorem provides a powerful tool, as we illustrate.

We illustrate the power of the Mean Value Theorem by providing successive approximations to sine and cosine.

1.1. We know, from definition, that for $x \in \mathbb{R}$,
\[
\cos x \leq 1 \tag{1}
\]

1.2. The function
\[
f: \mathbb{R}^+ \rightarrow \mathbb{R}, \quad x \mapsto \sin x - x
\]
satisfies the conditions for the Mean Value Theorem, with
\[
f'(x) = \cos x - 1.
\]
Thus, for each $x \in \mathbb{R}^+$, there is a $c \in [0, x]$ with
\[
f'(c)(x - 0) = f(x) - f(0) = \sin x - x
\]
By Equation (1), $f'(c) = \cos c - 1 \leq 0$ whence, given $x > 0$,
\[
\sin x \leq x \tag{2}
\]

1.3. The function
\[
g: \mathbb{R}^+ \rightarrow \mathbb{R}, \quad x \mapsto \cos x - 1 + \frac{x^2}{2}
\]
satisfies the conditions for the Mean Value Theorem, with
\[
g'(x) = -\sin x + x
\]
Thus, for each $x \in \mathbb{R}^+$, there is a $c \in [0, x]$ with
\[
g'(c)(x - 0) = g(x) - g(0) = \cos x - 1 + \frac{x^2}{2}
\]
By Equation (2), $g'(c) = -\sin c + c \geq 0$ whence, given $x > 0$,
\[
1 - \frac{x^2}{2} \leq \cos x \leq 1 \tag{3}
\]

1.4. The function
\[
h: \mathbb{R}_0^+ \rightarrow \mathbb{R}, \quad x \mapsto \sin x - x + \frac{x^3}{6}
\]
satisfies the conditions for the Mean Value Theorem, with
\[
h'(x) = \cos x - 1 + \frac{x^2}{2}.
\]
Thus, for each $x \in \mathbb{R}^+$, there is a $c \in [0, x]$ with
\[
h'(c)(x - 0) = h(x) - h(0) = \sin x - x + \frac{x^3}{6}
\]
By Equation (3), $h'(c) = \cos c - 1 + \frac{c^2}{2} \geq 0$ whence, given $x > 0$,
\[
x - \frac{x^3}{6} \leq \sin x \leq x \tag{4}
\]
1.5. The function
\[ j : \mathbb{R}^+ \rightarrow \mathbb{R}, \quad x \mapsto \cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24} \]
satisfies the conditions for the Mean Value Theorem, with
\[ j'(x) = -\sin x + x - \frac{x^3}{6} \]
Thus, for each \( x \in \mathbb{R}^+ \), there is a \( c \in [0, x] \) with
\[ j'(c)(x - 0) = j(x) - j(0) = \cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24} \]
By Equation (4), \( j'(c) = -\sin c + c - \frac{c^3}{6} \leq 0 \) whence, given \( x > 0 \),
\[ 1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24} \tag{5} \]
1.6. Continuing this way, we find that for every counting number \( n \)
\[ x - \frac{x^3}{3!} + \ldots - \frac{x^{4n+3}}{(4n+3)!} \leq \sin x \leq x - \frac{x^3}{3!} + \ldots + \frac{x^{4n+1}}{(4n+1)!} \]
\[ 1 - \frac{x^2}{2} + \ldots - \frac{x^{4n+2}}{(4n+2)!} \leq \cos x \leq 1 - \frac{x^2}{2} + \ldots + \frac{x^{4n}}{(4n)!} \]
Here \( k! \) is \( k \) factorial, the product of the first \( k \) counting numbers. It is defined recursively by
\[ 0! := 1 \]
\[ (k + 1)! := (k + 1)(k!) \quad \text{for } k \in \mathbb{N} \]
We reformulate the inequalities above, using “\( \Sigma \)”-notation.
\[ \sum_{j=0}^{2n} (-1)^j \frac{x^{2j+1}}{(2j+1)!} \leq \sin x \leq \sum_{j=0}^{2n} (-1)^j \frac{x^{2j+1}}{(2j+1)!} \]
\[ \sum_{j=0}^{2n+1} (-1)^j \frac{x^{2j}}{(2j)!} \leq \sin x \leq \sum_{j=0}^{2n} (-1)^j \frac{x^{2j}}{(2j)!} \]
Comment. While it has not been difficult to verify that the polynomials above approximate sine and cosine, the reader must surely wonder how to come up with such polynomials. When the functions to be approximated can be differentiated often enough, the coefficients are determined by their derivatives, as is shown in the section on Taylor series.