(a) \( S = \{ x \in \mathbb{N} : x^3 \geq 25 \} = \{ x \in \mathbb{N} : x \geq 5 \}\)
Hence \( S \) has infimum 2, which belongs to \( S \), but it has no supremum, since it is not bounded above.

(b) \( T = \{ x \in \mathbb{R} : \sqrt[3]{16 - x} > 0 \} = \{ x \in \mathbb{R} : x < 2^3 \} = [-2, 2)\)
Thus \( \inf(T) = -2, \sup(T) = 2 \) and neither is an element of \( T \).

(c) Let \( P(n) \) be the proposition \( 5^n > n^2 \) \((n \in \mathbb{N})\).
Since \( 5^0 = 5 > 0 = 0^2 \), \( P(0) \) is true.

Suppose that for some \( n \in \mathbb{N} \), \( P(n) \) is true, i.e., \( 5^n > n^2 \).
Then \( (n+1)^2 = n^2 + 2n + 1 \)
\( \leq n^2 + 2n + n^2 \) as \( n \geq 1 \)
\( < 5n^2 \)
\( < 5 \cdot 5^n \) by the inductive hypothesis.

So, by the PMI, \( 5^n > n^2 \) for all \( n \geq 1 \).

(a) \( a^2 - 3a \leq -3 \implies (a - \frac{3}{2})^2 \leq -\frac{3}{4} \)
So if \( a^2 - 3a \leq -3 \), then \((a - \frac{3}{2})^2 \leq -\frac{3}{4}\)
which is a contradiction, since the square of a real number cannot be negative.
(b) Put $Z = re^{i\theta}$ with $r > 0$ and $0 \leq \theta < 2\pi$

Since $i = e^{i\pi/2}$ and $Z^4 = r^4 e^{i4\theta}$,

$$Z^4 = i \iff r^4 e^{i4\theta} = e^{i\pi/2} \iff r^4 \cos 4\theta = 0 \land r^4 \sin 4\theta = 1$$

with $r > 0$ and $0 \leq 4\theta < 2\pi$

$\implies r = 1$ and $4\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}$

$\implies r = 1$ and $\theta = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

i.e. $Z = t\left(\cos\left(\frac{\pi}{8}\right) + i \sin\left(\frac{\pi}{8}\right)\right), \ t\left(\cos\left(\frac{5\pi}{8}\right) + i \sin\left(\frac{5\pi}{8}\right)\right)$
(a) \( i \ f(x) = \frac{x}{\sqrt{1-x^2}} \) is well-defined \( \iff \ \sqrt{1-x^2} > 0 \iff \ -1 < x < 1 \)

So consider \( \mathbb{S} \) in 1, 1 \( \to \mathbb{R} \).

As \( \ x \to 1^- \), \( f(x) \to -\infty \)

As \( \ x \to 1^+ \), \( f(x) \to +\infty \)

So since \( \text{im}(f) \) is an interval, \( \text{im}(f) = \mathbb{R} \).

(ii) \( f(x) = \frac{x^2}{x^2+1} \) is well-defined for all \( x \in \mathbb{R} \) as \( x^2 + 1 > 0 \)

So consider \( f: \mathbb{R} \to \mathbb{R}, \ x \to \frac{x}{x^2+1} \)

As \( f(-x) = -f(x) \), \( f \) is an odd function

\[ \lim_{x \to 0} f(x) = 0 \]

\[ f'(x) = \frac{(x^2+1)2x - x^2(2x)}{(x^2+1)^2} = \frac{2x}{x^2+1} > 0 \] for \( \ |x| > 1 \)

As \( x^2 = 0 \), \( f(x) = 0 \) if \( \ |x| < 1 \)

Thus \( f(1) = \frac{1}{2} \) is the (global) maximum

and \( f(-1) = -\frac{1}{2} \) is the (global) minimum

Of \( \text{im}(f) = [-\frac{1}{2}, \frac{1}{2}] \)

(b) Note that for \( x \to 0 \), \( \ f(x) = \frac{x}{1+x^2} \)

Hence \( f \) is not injective

As \( \text{im}(f) \neq \mathbb{R} \), \( f \) is not surjective

\( \inf(f) = -\frac{1}{2} \), \( \sup(f) = \frac{1}{2} \) by part (a) (ii)

(c) \[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin(x^2)}{x} = 0 \]

\[ \lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x^2 + x) = x = f(0) \]

Hence \( f \) is continuous at \( 0 \iff k = 0 \)
(a) \( u_n := r^{n+1} \quad (r \in \mathbb{R}) \)

If \( r = 0, i \), then \((u_n)\) is a constant sequence.
If \( r < 0 \), then \((u_n)\) is an alternating sequence.
If \( 0 < r < 1 \), then \((u_n)\) is monotonically decreasing.
If \( r > 1 \), then \((u_n)\) is monotonically increasing.

\( u_n \to \infty \) if \( r > 1 \)
\( u_n \to 0 \) if \( |r| < 1 \)

An oscillates without converging if \( r = \pm 1 \)

(ii) \( u_n := \frac{n+1}{n^{2+1}} \quad u_0 = 1, \ u_1 = 1 \)

\[ u_{n+1} = u_n = \frac{n+2}{n+2} - \frac{p+1}{n^{2+2}} \]

\[ = \frac{r(n^2+4n+2)}{(n^2+4n+2)} \]

\[ = \frac{n^2+2n+1}{n^2+4n+2} \]

Thus, \((u_n)\) is monotonically decreasing.

\[ \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n^2+4n+2} = 0 \]

(b) (i) \( u_n := \frac{5^n}{n^2} \quad \text{So} \quad \frac{u_{n+1}}{u_n} = \frac{5^{n+1}}{5^n} \to 0 \quad \text{as} \quad n \to \infty. \)

Thus, by the ratio test, \( \sum u_n \) converges.

(ii) \( u_n := \frac{n}{n^{2+1}} = \frac{1}{n^{1+\frac{1}{2}}} = \frac{1}{\sqrt{n}} \quad \text{for} \quad n > 1 \)

Thus, \( \lim_{n \to \infty} u_n = 1, \) whence \( \sum u_n \) cannot converge.

(iii) \( u_n := \frac{1}{n^{2+1}} \)

\[ \lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} 2 \frac{n^{2+1}}{(n+1)^{2+1}} = 2 > 1 \]

So, by the ratio test, \( \sum u_n \) diverges.
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(f) \( f : \mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto \sqrt{1 + \cos x} \)

g(\mathbb{R} \rightarrow \mathbb{R}, \quad x \mapsto e^{-x^2} \)

\( f'(x) = -\frac{\sin x}{\sqrt{1 + \cos x}} \), so that \( f \) is differentiable whenever 

\( \cos x + 1 \neq 0 \quad \Rightarrow \quad x \neq (2n+1)\pi \quad (n \in \mathbb{Z}) \)

Now \( f'(2n+1\pi) = \lim_{h \to 0} \frac{f((2n+1)\pi + h) - f((2n+1)\pi)}{h} \)

\[ = \lim_{h \to 0} \frac{\sqrt{1 - \cos h}}{h} \]

as \( f((2n+1)\pi) = 0 \)

and \( \cos((2n+1)\pi) = -1 \)

\[ = \lim_{h \to 0} \frac{\sqrt{1 - \cos^2 h}}{h} \]

Now \( \lim_{h \to 0} \frac{1 - \cos h}{h} = -\lim_{h \to 0} \frac{\sin^2 h}{2h} = -\frac{1}{\sqrt{2}} \)

but \( \lim_{h \to 0} \frac{\sqrt{1 - \cos h}}{h} \)

Thus \( \lim_{h \to 0} \frac{f((2n+1)\pi + h) - f((2n+1)\pi)}{h} \) does not exist.

\( g'(x) = -2x e^{-x^2} \), so that \( g \) is differentiable everywhere.

(b) Let \( y \) be a differentiable function of \( x \) with

\[ xe^{-xy} + x^2 + y^4 = 7 \]

Then \[ e^{-xy} + x \left( -2xy - 2y \frac{dy}{dx} \right) e^{-xy} + 2x + 2y \frac{dy}{dx} = 0 \]

i.e. \( -2xy e^{-xy} - 2y \frac{dy}{dx} = -e^{-xy} - 2x + 2x^2 e^{-xy} \)

or \[ \frac{dy}{dx} = \frac{(2x^2-1)e^{-xy} - 2x}{2y (1 - e^{xy})} \]

(Whenever this is defined)
(a) \( f : \mathbb{R} \rightarrow \mathbb{R}, \; x \mapsto x^4 - 4x^3 + 4x \)

(i) \( f'(x) = 4x^3 - 12x^2 + 8x \)

\[
= 4x(x^2 - 3x + 2) = 4x(x - 1)(x - 2)
\]

\[
\begin{align*}
\text{< 0 if } & \; x < 0 \text{ or } 1 < x < 2 \\
\text{= 0 if } & \; x = 0, 1, \text{ or } 2 \\
\text{> 0 if } & \; 0 < x < 1 \text{ or } x > 2
\end{align*}
\]

So \( f \) is decreasing on \( (-\infty, 0) \) and on \( (1, 2) \)

and increasing on \( (0, 1) \) and on \( (2, \infty) \)

(ii) \( f''(x) = 12x^2 - 24x + 8 \)

\[
= 12(x^2 - 2x + 1) - 4 = 12(x - 1)^2 - 12
\]

\[
\begin{align*}
\text{< 0 if } & \; 1 - \frac{1}{\sqrt{3}} < x < 1 + \frac{1}{\sqrt{3}} \\
\text{= 0 if } & \; x = 1 \pm \frac{1}{\sqrt{3}} \\
\text{> 0 if } & \; x < 1 - \frac{1}{\sqrt{3}} \text{ or } x > 1 + \frac{1}{\sqrt{3}}
\end{align*}
\]

So \( f \) is concave up on \( (-\infty, 1 - \frac{1}{\sqrt{3}}) \) and on \( (1 + \frac{1}{\sqrt{3}}, \infty) \)

and concave down on \( (1 - \frac{1}{\sqrt{3}}, 1 + \frac{1}{\sqrt{3}}) \)

(b) (i) Boundary points \( f(-1) = f(3) = 10 \)

(ii) Where \( f \) is not differentiable - no such points

(iii) Where \( f'(x) = 0 \)

\( x = 0 \quad f(0) = 1 \)

\( x = 2 \quad f(2) = 2 \)

\( x = 1 \quad f(1) = 1 \)

So abs max: 10 at \( x = -1, 3 \)

abs min: 1 at \( x = 0, 2 \)

rel max: 2 at \( x = 1 \)
(a) 
\[ 4x + 3y + 2z = 19 \quad \text{(i)} \]
\[ 2x - y + 2z = 6 \quad \text{(ii)} \]
\[ x + y + z = 6 \quad \text{(iii)} \]
\[ 2x \text{(iii)} - (\text{iii}) \]
\[ 3y = 6 \quad \text{or} \quad y = 2 \quad \text{(iv)} \]
Substitute in (i)
\[ 4x + 2 \times 2 = 1 \begin{array}{c}
\text{(ii)} \\
2x = 9 \\
\therefore x = 3
\end{array} \quad \text{(vii)} \]
\[ \sqrt{(-\text{iv})} \]
Substitute in (v):
\[ x = 3 \quad \therefore \begin{array}{c}
y = 2 \\
z = 1
\end{array} \quad \text{(vii)} \]

(b) 
\[ A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \]
\[ A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \]
\[ \text{(ii)} \]
\[ 2A + C = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 1 & -3 \\ 3 & 3 & 2 \end{bmatrix} \]
\[ \text{BA is not defined} \quad \text{as} \quad B \text{ is} \ 3 \times 2 \text{a} \quad A \text{ is} \ 3 \times 3 \]

(c) 
\[ 2 -3 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \]
\[ R_5 - R_1 \]
\[ 2 -3 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \]
\[ \text{or} \]
\[ 2 -3 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \]
\[ R_5 \text{or} 2 \]
Q8 \( P: (1,0,0), \ Q: (1,1,0), \ R: (1,1,1), \ S: (2,1,2) \)

(a) \( \overrightarrow{PA} = (2,0,0) \)
\( \overrightarrow{PR} = (2,1,1) \)

(b) The orthogonal projection of \( \overrightarrow{PA} \) onto \( \overrightarrow{PR} \) is

\[
\text{proj}_{\overrightarrow{PR}} \overrightarrow{PA} = \left( \frac{\overrightarrow{PA} \cdot \overrightarrow{PR}}{\overrightarrow{PR} \cdot \overrightarrow{PR}} \right) \overrightarrow{PR} = \left( \frac{(2,0,0) \cdot (2,1,1)}{(2,1,1) \cdot (2,1,1)} \right) (2,1,1)
\]

\[
= \frac{4}{4+1+1} (2,1,1) = \frac{2}{3} (2,1,1)
\]

(c) The area is half the length of \( \overrightarrow{PA} \times \overrightarrow{PR} \)

Now \( \overrightarrow{PA} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 2 & 0 & 0 \\ 2 & 1 & 1 \end{vmatrix} = -2i + 2j \), with length \( 2/\sqrt{2} \)

So the area is \( \sqrt{2} \) square units.

(d) \( \overrightarrow{PS} = (3,1,2) \) The volume is the absolute value of

\( \text{det} \left[ \begin{array}{ccc} 2 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{array} \right] = 2 \)

So the volume is 2 cubic units.
(a) For \( f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x^2 \) take \( a \in \mathbb{R} \) and \( \varepsilon > 0 \). Put
\[
\delta := \sqrt{\frac{1}{|a|^2+\varepsilon}} - |a|.
\]
Suppose that \( |x-a| < \delta \). Then \( |x-a| < |a| + \delta \) and
\[
|f(x) - f(a)| = |x^2 - a^2| = |x-a| |x+a|
\leq |x-a| (2|a|+\delta)
< \delta (2|a|+\delta)
= \left(\sqrt{\frac{1}{|a|^2+\varepsilon}} - |a|\right)(\sqrt{\frac{1}{|a|^2+\varepsilon}} + |a|)
= \varepsilon.
\]
Thus \( f \) is continuous everywhere.

(b) Take \( f, g: \mathbb{R} \rightarrow \mathbb{R} \). Suppose that for some \( K > 0 \) and all \( x \in \mathbb{R} \),
\[
|f(u) - f(a)| \leq K |g(u) - g(a)|.
\]
Suppose \( g \) is continuous at \( a \in \mathbb{R} \).
Take \( \varepsilon > 0 \). Then \( \frac{\varepsilon}{K} > 0 \).
So, there is a \( \delta > 0 \) with \( |g(u) - g(a)| < \frac{\varepsilon}{K} \) if \( |u-a| < \delta \).
Then, if \( |u-a| < \delta \) then
\[
|f(u) - f(a)| < K |g(u) - g(a)|
\leq K \frac{\varepsilon}{K} = \varepsilon.
\]
So \( f \) is continuous at \( a \).
(a) Consider \( f : \mathbb{R}_0^+ \rightarrow \mathbb{R}, \ x \mapsto \ln(x+1) \)

Take \( c > 0 \). Then \( f \) is continuous on \([0, c]\)

Moreover, since \( f'(c) = \frac{1}{x+1} \), \( f \) is differentiable on \([0, c]\).

So we may apply the M.V.T.

For \( x > 0 \), there is a \( \Theta \in [0, 1] \) with

\[
f'(\Theta x) = \frac{f(x) - f(0)}{x - 0} = \frac{\ln(x+1)}{x}
\]

i.e.

\[
\frac{1}{1+\Theta x} = \frac{\ln(x+1)}{x}
\]

So, since \( x > 0 \),

\[
x = (\ln(x+1))(1+\Theta x)
\]

\[
< \ln(x+1) \left( 1 + \frac{1}{x} \right)
\]

\[
\Rightarrow \quad \frac{\Theta}{1+x} < \ln(x+1)
\]

So \( 0 < \Theta < 1 \)

as \( x \to 0 \).

(b) If \( f : \mathbb{R} \to \mathbb{R} \) is differentiable on \( a \leq b \) and \( f(a) = 0 \), then, by M.V.T., we can find a \( c \in (a, b) \) with

\[
f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{0-0}{b-a} = 0
\]
Let the length of each side of the equilateral triangle be \( x \). Its perimeter is then 3\( x \) and its area \( \frac{\sqrt{3}}{4} x^2 \).

Let the length of each side of the square be \( y \). Its perimeter is then 4\( y \) and its area \( y^2 \).

Thus, \( 0 < x, y < L \) and

\[ 3x + 4y = L \quad \text{--- (i)} \]

The total area is

\[ A = \frac{\sqrt{3}}{4} x^2 + y^2 \quad \text{--- (iii)} \]

From (i), \( \frac{d}{dx} (3x + 4y) = 0 \) \( \text{--- (iv)} \)

From (ii), \( \frac{d}{dx} \left( \frac{\sqrt{3}}{2} x + 2y \right) = \frac{\sqrt{3}}{2} \]

For an extreme value, we must have \( \frac{dA}{dx} = 0 \).

In

\[ \sqrt{3} x + 4y \frac{dy}{dx} = 0 \quad \text{--- (v)} \]

Then (iii) \( \times \) (iv) have a non-trivial solution if and only if

\[ 12y - 4\sqrt{3} x = 0 \quad \text{ie} \quad x = \frac{\sqrt{3}}{4} y \]

So, from (i), \( y = \frac{L}{3\sqrt{3} + 4} \)

\[ x = \frac{\sqrt{3}}{3\sqrt{3} + 4} \]

From (iii), \( \frac{d^2 A}{dx^2} = 0 \)

From (iv),

\[ \left( \frac{d^2 A}{dx^2} \right)^2 = \frac{\sqrt{3}}{2} + 2 \left( \frac{dy}{dx} \right)^2 + 2 \frac{d^2 y}{dx^2} \]

So, by (iii) \( \times \) (iv),

\[ \frac{d^2 A}{dx^2} = \frac{\sqrt{3}}{2} + \frac{3}{2} > 0 \]

Hence the total area \( \frac{d^2 A}{dx^2} \) is a minimum when

\[ x = \frac{L\sqrt{3}}{3\sqrt{3} + 4} \quad \text{and} \quad y = \frac{L}{3\sqrt{3} + 4} \]

There is no maximum.
(a) Let \( u = (u_1, u_2, u_3, u_4) \) \( v = (v_1, v_2, v_3, v_4) \).

Then \( u \times v = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) \).

So \( \lvert u \times v \rvert^2 = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \)

\[= u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 \]
\[+ u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 \]
\[+ u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \]

On the other hand,

\[\lvert u \rvert^2 \lvert v \rvert^2 = (u_1^2 + u_2^2 + u_3^2 + u_4^2) \cdot (v_1^2 + v_2^2 + v_3^2 + v_4^2) \]
\[= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_1^2v_4^2 \]
\[+ u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_2^2v_4^2 \]
\[+ u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2 + u_3^2v_4^2 \]
\[+ 2u_1u_2v_1v_2 + 2u_1u_3v_1v_3 + 2u_1u_4v_1v_4 + 2u_2u_3v_2v_3 + 2u_2u_4v_2v_4 + 2u_3u_4v_3v_4 \]

Then \( \lvert u \rvert^2 \lvert v \rvert^2 - (u \times v)^2 = u_2^2v_3^2 - 2u_2u_3v_2v_3 + u_3^2v_2^2 \]
\[+ u_3^2v_1^2 - 2u_1u_3v_1v_3 + u_1^2v_3^2 \]
\[+ u_1^2v_2^2 - 2u_1u_2v_1v_2 + u_2^2v_1^2 \]
\[= \lvert u \times v \rvert^2 \]

(b) Let \( P \in \mathbb{R}^3 \) be any point and \( A \neq B \) two distinct points in \( \mathbb{R}^3 \). Then the area of the triangle with vertices \( A, B, P \) is

\[ \frac{1}{2} \left| \overrightarrow{PA} \times \overrightarrow{PB} \right| \]

If the perpendicular distance from \( P \) to the line through \( A \) and \( B \) is \( d \), then the area of the triangle is \( \frac{1}{2} d \cdot \lvert AB \rvert \)

\[ \Rightarrow \quad d = \frac{\left| \overrightarrow{PA} \times \overrightarrow{PB} \right|}{\lvert \overrightarrow{AB} \rvert} \]