Question 1. (a) Since 
\[ u_{n+1} - u_n = \frac{n + 1}{(n + 1)^2 + 1} - \frac{n}{n^2 + 1} \]
\[ = \frac{(n + 1)(n^2 + 1) - n((n + 1)^2 + 1)}{(n + 1)^2 + 1)(n^2 + 1)} \]
\[ = \frac{-(n + 1)(n^2 + 1)}{((n + 1)^2 + 1)(n^2 + 1)} \]
\[ \leq 0, \]
the sequence \((u_n)_{n \in \mathbb{N}^*}\) is monotonically decreasing.

Since \(0 \leq \frac{n}{n^2 + 1} < \frac{n}{n^2} = \frac{1}{n}\) for \(n \geq 1\), and \(\lim_{n \to \infty} \frac{1}{n} = 0\),
\[ \lim_{n \to \infty} \frac{n}{n^2 + 1} = 0 \]

(b) Since 
\[ u_{n+1} - u_n = (n + 1)^2 - n^2 \]
\[ = 2n + 1 \]
\[ > 0, \]
the sequence \((u_n)_{n \in \mathbb{N}^*}\) is monotonically increasing.

Moreover, since \(n^2 > n\) for \(n > 1\) and \(n \to \infty\) as \(n \to \infty\),
\[ n^2 \to \infty \text{ as } n \to \infty \]

(c) Since \(u_n > 0\) for every \(n \in \mathbb{N}^*\) and 
\[ \frac{u_{n+1}}{u_n} = \frac{(n + 1)^{n+1}}{n^n} \]
\[ = (n + 1)(1 + \frac{1}{n})^n \]
\[ > 1, \]
the sequence \((u_n)_{n=1}^{\infty}\) is monotonically increasing.

Moreover, since \(n^n > n\) for \(n > 1\) and \(n \to \infty\) as \(n \to \infty\),
\[ \frac{n^2 + 1}{n} \to \infty \text{ as } n \to \infty \]

(d) Since \(\frac{n^2 + 1}{n} = n + \frac{1}{n}\),
\[ u_{n+1} - u_n = 1 + \frac{1}{n + 1} - \frac{1}{n} \]
\[ = 1 - \frac{1}{n(n + 1)} \]
\[ > 0, \]
so that the sequence \((u_n)_{n \in \mathbb{N}^*}\) is monotonically increasing.

Since \(u_n = \frac{n^2 + 1}{n} = n + \frac{1}{n} > n\),
\[ \frac{n^2 + 1}{n} \to \infty \text{ as } n \to \infty. \]
(e) Since
\[ u_{n+1} - u_n = 2(n+1) + (-1)^{n+1} - 2^n - (-1)^n \]
\[ = 2(1 - (-1)^n) \]
\[ = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases} \]

\((u_n)_{n \in \mathbb{N}^*}\) is monotonically non-decreasing.

Since \(u_n \geq 2n - 1 \geq n\),
\[ 2^n + (-1)^n \to \infty \text{ as } n \to \infty. \]

(f) Note that in this case, the first term of the sequence is \(u_2\).

Since \(u_n = \frac{n^3 - 1}{n^3 - 1} = \frac{n + 1}{n^2 + n + 1} \) for \(n > 1\),
\[ u_{n+1} - u_n = \frac{n + 2}{(n+1)^2 + (n+1) + 1} - \frac{n + 1}{n^2 + n + 1} \]
\[ = \frac{(n+2)(n^2 + n + 1) - (n+1)(n^2 + 3n + 3)}{(n^2 + 3n + 3)(n^2 + n + 1)} \]
\[ = -\frac{(n^2 + 3n + 1)}{(n^2 + 3n + 3)(n^2 + n + 1)} < 0, \]

the sequence \((u_n)_{n=2}^{\infty}\) is monotonically decreasing.

Since \(u_n = \frac{n^2 - 1}{n^3 - 1} < \frac{2n}{n^2 + n + 1} < \frac{2}{n}\),
\[ \lim_{n \to \infty} \frac{n^2 - 1}{n^3 - 1} = 0. \]

(g)
\[ u_{n+1} - u_n = \frac{(n+1)^3 + 2(n+1) + 1}{1 - 10(n+1)^2 - (n+1)^3} - \frac{n^3 + 2n + 1}{1 - 10n^2 - n^3} \]

We are only interested in whether this is positive or negative. Since \(m^3 + 10m - 1 > 0\) for all \(m > 1\), it is enough to determine whether the numerator obtained when \(u_{n+1} - u_n\) is written as a rational function of \(n\),
\[ ((n+1)^3 + 2(n+1) + 1)(1 - 10n^2 - n^3) - (n^3 + 2n + 1)((1 - 10(n+1)^2 - (n+1)^3), \]
is positive or negative.

Since \((n+1)^3 = n^3 + 3n^2 + 3n + 1\) and \((n+1)^2 = n^2 + 2n + 1\), we obtain, successively,
\[ (n+1)^3 + 2(n+1) + 1 = n^3 + 3n^2 + 5n + 4 \]
\[ 1 - 10(n+1)^2 - (n+1)^3 = -(n^3 + 13n^2 + 23n + 10) \]
\[ ((n+1)^3 + 2(n+1) + 1)(1 - 10n^2 - n^3) = -(n^6 + 13n^5 + 53n^4 + 53n^3 + 37n^2 - 5n - 4) \]
\[ (n^3 + 2n + 1)((n^3 + 13n^2 + 23n + 10) = -(n^6 + 13n^5 + 25n^4 + 37n^3 + 59n^2 + 43n + 11) \]
\[ \nu = -10n^4 + 16n^3 - 22n^2 + 38n + 7 \]
\[ = -(10n^2(n^2 - 1) + 12n^2(n - 1) + 4n^3 + 38n + 7) < 0 \quad \text{for all } n \in \mathbb{N} \]
Thus \((u_n)_{n \in \mathbb{N}}\) is monotonically decreasing. Moreover,
\[
    u_n = \frac{n^3 + 2n + 1}{1 - 10n^2 - n^3} = \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{1 + 10\frac{1}{n} - \frac{1}{n^3}}
\]
So, since \(\lim_{n \to \infty} \frac{1}{n^k} = 0\) for \(k > 0\),
\[
    \lim_{n \to \infty} \frac{n^3 + 2n + 1}{1 - 10n^2 - n^3} = -1
\]

(h) Since \(2^{-n} > 0\) for all \(n \in \mathbb{N}^*\), and \(0 < u_{n+1} := 2^{-(n+1)} = \frac{1}{2^{n+1}} u_n\), \((u_n)_{n \in \mathbb{N}^*}\) is monotonically decreasing.

Moreover, since \(2^n > n\) for all \(n \in \mathbb{N}\), \(u_n := \frac{1}{n^n} < \frac{1}{n}\), whence
\[
    \lim_{n \to \infty} 2^{-n} = 0
\]

(i) Since \(u_n > 0\) for all \(n \in \mathbb{N}^*\) and
\[
    \frac{u_{n+1}}{u_n} = \frac{(n+1)!n^n}{(n+1)^{n+1}n!} = \left(\frac{n}{n+1}\right)^n < 1,
\]
the sequence \((u_n)_{n \in \mathbb{N}^*}\) is monotonically decreasing.

Moreover, since \(\frac{1}{n} < 1\) for \(1 < j < n\),
\[
    u_n = \frac{n!}{n^n} = \frac{n(n-1) \ldots 1}{n.n \ldots n} < \frac{1}{n}
\]
whence
\[
    \lim_{n \to \infty} \frac{n!}{n^n} = 0.
\]

**Question 2.**

Since \((n)_{n \in \mathbb{N}}\) is monotonically increasing, so is \(\left(n + \sqrt{n^2 - \frac{\gamma^2}{c^2}}\right)_{n \in \mathbb{N}}\).

Since this is a sequence of positive terms, each of
\[
    \left(\frac{\gamma^2}{c^2}\right)_{n \in \mathbb{N}}
\]
\[
    \left(1 + \frac{\gamma^2}{c^2}\right)_{n \in \mathbb{N}}
\]
\[
    \left(\frac{1}{n} + \frac{\gamma^2}{n^2 - \frac{\gamma^2}{c^2}}\right)_{n \in \mathbb{N}^*}
\]
is monotonically decreasing.

Hence, \((E_n)_{n \in \mathbb{N}}\) is a monotonically increasing sequence.

Since \(n \to \infty\) as \(n \to \infty\), we have \(\frac{\gamma^2}{n + \sqrt{n^2 - \frac{\gamma^2}{c^2}}} \to 0\) and so
\[
    \lim_{n \to \infty} E_n = mc^2
Question 3.

As $2^n > n$ for every $n \in \mathbb{N}$, we see that $0 < 2^{-n} < \frac{1}{n}$ for all $n \in \mathbb{N}$.

(a) Take $K \in \mathbb{R}$.

Since $\mathbb{N}$ is not bounded above, there is an $N \in \mathbb{N}$ with $N > K$.

Take $n \geq N$.

Then $2^n > n \geq N > K$, proving that $2^n \to \infty$ as $n \to \infty$.

(b) Take $\varepsilon > 0$.

Then $\frac{1}{\varepsilon} > 0$.

Since $\mathbb{N}$ is not bounded above, there is an $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$, or, equivalently, $0 < \frac{1}{n} < \varepsilon$.

If $n \geq N$, then $0 < 2^{-n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

Thus, given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ with $|2^{-n} - 0| < \varepsilon$ whenever $n \geq N$, proving that $\lim_{n \to \infty} 2^{-n} = 0$.

Question 4.

Let $h_n$ be the height of the ball after the $n^{th}$ bounce, and $d_n$ the distance travelled by the ball up to the $n^{th}$ bounce.

$$h_0 = 20$$

$$h_{n+1} = \frac{4}{5}h_n \quad \text{for all } n \in \mathbb{N}$$

Thus $h_n = 20 \left(\frac{4}{5}\right)^n$.

In particular, $h_3 = 20 \left(\frac{4}{5}\right)^3 = \frac{512}{125}$, so that the height of the third bounce is 10.24 metres.

Now $d_0 = h_0 = 20$, and, for $n \in \mathbb{N}$

$$d_{n+1} = d_n + h_n + h_{n+1}$$

$$= d_n + \frac{9}{5}h_n$$

$$= d_n + 36 \left(\frac{4}{5}\right)^n,$$

so that

$$d_{n+1} = 36 \left(1 + \frac{4}{5} + \ldots + \left(\frac{4}{5}\right)^n\right)$$

$$= 36 \frac{1 - \left(\frac{4}{5}\right)^{n+1}}{1 - \frac{4}{5}}$$

$$= 180 \left(1 - \left(\frac{4}{5}\right)^{n+1}\right)$$

Since $\lim_{n \to \infty} d_n = 180$, the ball travels 180 metres before coming to rest.

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1 We repeat a proof for the benefit of those who have forgotten.

Since $2^0 = 1 > 0$, and $2^1 = 2 > 1$, the proposition is true for $n = 0, 1$.

Suppose that for some $n \in \mathbb{N}^*$, $2^n > n$. Then $2^{n+1} = 2.2^n > 2n \geq n + 1$.

By the Principle of Mathematical Induction, $2^n > n$ for every $n \in \mathbb{N}$.