Question 1.
(a) Differentiating \( xy + x^2y^2 - 3x + 1 = 0 \) with respect to \( x \), we obtain
\[
\frac{dy}{dx} + 2xy^2 + 2x^2y\frac{dy}{dx} - 3 = 0
\]
Hence
\[
\frac{dy}{dx} = \frac{3 - y - 2xy^2}{x(1 + 2xy)}
\]
as long as \( xy \neq -\frac{1}{2} \) and \( x \neq 0 \).

(b) Differentiating \( y^2\sin x + \cos y = 1 \) with respect to \( x \), we obtain
\[
2y\sin x \frac{dy}{dx} + y^2\cos x - \sin y\frac{dy}{dx} = 0
\]
Hence
\[
\frac{dy}{dx} = \frac{y^2\cos x}{\sin y - \cos y}
\]
as long as \( \sin y \neq y \cos x \).

Question 2.
(a) Since \( f: \mathbb{R} \rightarrow \mathbb{R}, \ u \mapsto 1 + u^2 \) is a polynomial function, it is differentiable everywhere. Hence it satisfies the hypotheses of the Mean Value Theorem on any closed interval \([a, b]\). Moreover, since
\[
f'(u) = 2u,
\]
\[
f'(c) = \frac{f(b) - f(a)}{b - a} = b + a \text{ if and only if } c = \frac{b + a}{2}.
\]
In particular, if \( a = 0 \) and \( b = 1 \), then \( c = \frac{1}{2} \).

(b) Since \( f: \mathbb{R} \rightarrow \mathbb{R}, \ y \mapsto y^3 + y - 4 \) is a polynomial function, it is differentiable everywhere. Hence it satisfies the hypotheses of the Mean Value Theorem on any closed interval \([a, b]\). Moreover, since
\[
f'(y) = 3y^2 + 1
\]
\[
f'(c) = \frac{f(b) - f(a)}{b - a} = b + a \text{ if and only if } 3c^2 = b^2 + ba + a^2.
\]
In particular, if \( a = -1 \) and \( b = 2 \), then \( 3c^2 = 3 \), or \( c = \pm 1 \).
Thus, in this case, the only solution, \( c \), with \( a < c < b \) is \( c = 1 \).

Question 3.
Consider the function \( f: [0, \pi] \setminus \{\frac{\pi}{2}\} \rightarrow \mathbb{R}, \ \theta \mapsto \tan \theta \).
Then \( f(0) = f(\pi) = 0 \), by the definition of \( \tan \theta \).
As \( \frac{\sin x}{x} = \sec^2 x = 1 + \tan^2 x \geq 1 \), there is no real number \( c \) with \( f'(c) = f(\pi) - f(0) \).
This does not contradict Rolle’s Theorem since \([0, \pi]\) is not a subset of the domain of \( f \).

Question 4.
Let \( n \) be a counting number, and consider the function
\[
f: \mathbb{R}^+ \rightarrow \mathbb{R}, \ x \mapsto x^\frac{1}{n} - 1
\]
Then \( f \) is differentiable on \( \mathbb{R}^+ \) with derivative \( f'(x) = \frac{x^\frac{1}{n} - 1}{n} \).
Take \( x > 1 \). By the Mean Value Theorem there is a \( c \in ]1, x[ \) with
\[
\frac{f(x) - f(1)}{x - 1} = f'(c)
\]
that is
\[
\frac{x^n - 1}{x - 1} = c^{\frac{1-n}{n}}
\]
Since \(n\) is a counting number, \(1 - n \leq 0\).
Since \(c > 1\), \(c^{\frac{1-n}{n}} > 1\), whence \(c^{\frac{1-n}{n}} \leq 1\).
Thus
\[
\frac{x^n - 1}{x - 1} \leq \frac{1}{n}.
\]
or, equivalently,
\[
x^n \leq 1 + \frac{1}{n}(x - 1).
\]