Question 1.
(a) For $x \in \mathbb{R}_0^+$

$$f(x) - f(1) = \begin{cases} \frac{\sqrt{x} - 1}{x - 1} & \text{if } x < 1 \\ \frac{x - 1}{2(x - 1)} & \text{if } x > 1 \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{x} + 1} & \text{if } x < 1 \\ \frac{1}{2} & \text{if } x > 1 \end{cases}$$

$$\to \frac{1}{2} \text{ as } x \to 1$$

Hence, $f$ is differentiable at 1, with $f'(1) = \frac{1}{2}$.

(b) For $x \neq 0$

$$g(x) - g(0) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ \frac{x^2 - 0}{x - 0} & \text{otherwise} \end{cases}$$

$$= \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x & \text{otherwise} \end{cases}$$

$$\to 0 \text{ as } x \to 0$$

Hence, $g$ is differentiable at 0, with $g'(0) = 0$.

Remark. The function $g$ is an example of a function which is differentiable at exactly one point, for it is easy to see that $g$ is not continuous at any $x$ other than $x = 0$.

(c) For $x \neq 1$

$$h(x) - h(1) = \begin{cases} \frac{x^4 + x^2 - 2}{x - 1} & \text{if } x < 1 \\ \frac{4x^2 - 2x - 2}{x - 1} & \text{if } x > 1 \end{cases}$$

$$= \begin{cases} x^3 + x^2 + 2x - 2 & \text{if } x < 1 \\ 4x - 2 & \text{if } x > 1 \end{cases}$$

$$\to 2 \text{ as } x \to 1$$

Hence, $h$ is differentiable at 1, with $h'(1) = 2$.

Question 2.
For $f: \mathbb{R} \to \mathbb{R}$, $t \mapsto \sqrt{t^2 + 1} = (t^2 + 1)^{\frac{1}{2}}$

$$f'(t) = t(t^2 + 1)^{-\frac{1}{2}}$$

$$= \frac{t}{f(t)}$$
\[ f''(t) = \frac{f(t) - tf'(t)}{(f(t))^2} \]  
by the quotient rule
\[ = \frac{(f(t))^2 - t^2}{(f(t))^4} \]  
\[ = (f(t))^{-3} \]  
as \((f(t))^2 = t^2 + 1\)
\[ f'''(t) = -3(f(t))^{-4} f'(t) \]  
by the chain rule
\[ = -3t (f(t))^3 \]  
by the above calculation

Thus,
\[ f'(t) = \frac{t}{\sqrt{t^2 + 1}} \]
\[ f''(t) = \frac{t}{(\sqrt{t^2 + 1})^3} \]
\[ f'''(t) = \frac{-3t}{(\sqrt{t^2 + 1})^5} \]

**Question 3.**

(a) \( f(x) = \frac{1}{1 + \tan x} \) defines a real-valued function of a real variable whenever \( \tan x \neq -1 \).

Since \( \tan: \mathbb{R} \rightarrow \mathbb{R} \) is a bijection and since \( \tan(x + \pi) = \tan x \),
\[ \tan x = -1 \] if and only if \( x = \frac{(4n - 1)\pi}{4} \) for some integer \( n \).

Hence
\[ D = \mathbb{R} \setminus \left\{ \frac{(4n - 1)\pi}{4} \mid n \in \mathbb{Z} \right\} \]
is the largest subset of \( \mathbb{R} \) on which \( f(x) = \frac{1}{1 + \tan x} \) defines a function, and then
\[ f'(x) = \frac{-1}{(1 + \tan x)^2} \frac{d}{dx} (\tan x) \]
\[ = \frac{1 + \tan^2 x}{(1 + \tan x)^2} \]

(b) Since it is the product of the differentiable functions
\( \alpha: \mathbb{R} \rightarrow \mathbb{R}, \ x \mapsto x^2 \)
\( \beta: \mathbb{R} \rightarrow \mathbb{R}, \ x \mapsto \cos x, \)
\( g(x) = x^2 \cos x \)
defines a differentiable real-valued function on all of \( \mathbb{R} \), and, by the Leibniz (product) rule,
\[ g'(x) = 2x \cos x - x^2 \sin x \]

**Question 4.**

Since \( V(t) = (\alpha(t))^3 \), with \( \alpha(t) = a_0 e^{Ht} \), it follows by the chain rule that
\[ V'(t) = 3 (\alpha(t))^2 \alpha'(t) \]
\[ = 3 (\alpha(t))^2 (a_0 H e^{Ht}) \]
\[ = 3HV(t) \]