Sample Solutions

Question 1. Let \( p \) be a prime number and suppose that \( \sqrt{p} \) is rational.

Then \( \sqrt{p} = \frac{m}{n} \), for some counting numbers, \( m, n \), with no common factors, whence

\[
m^2 = pn^2, \tag{1}
\]

Thus, \( p \) divides \( m^2 \).

Since \( p \) is prime, \( p \) must divide \( m \), so that \( m = pk \) for some counting number, \( k \).

Substituting in Equation (1), we see that \( p^2k^2 = pn^2 \), or

\[
n^2 = pk^2,
\]

from which it follows that \( p \) divides \( n^2 \).

Since \( p \) is prime, \( p \) must divide \( n \), so that \( p \) is a common factor of \( m \) and \( n \), contradicting the choice of \( m \) and \( n \).

Hence \( \sqrt{p} \) is irrational for every prime number \( p \).

Question 2. If \( k = 1 \), then \( 9^k - 1 = 9 - 1 = 8 \), which is plainly divisible by 8.

Now suppose that for some counting number, \( k \), there is a counting number \( m \) with \( 9^k - 1 = 8m \).

Then

\[
9^{k+1} - 1 = 9^{k+1} - 9^k + 9^k - 1
= 9^k(9 - 1) + 8m \quad \text{by hypothesis}
= 8(9^k + m),
\]

showing that 8 divides \( 9^{k+1} - 1 \).

So, by the Principle of Mathematical Induction, 8 divides \( 9^k - 1 \) for all counting numbers \( k \).

Question 3. Let \( a > 1 \) be a real number. If \( n = 1 \), then

\[
(1 + a)^n = 1 + a = 1 + na.
\]

Suppose that for some counting number, \( n \), \( (1 + a)^n \geq 1 + na \). Then

\[
(1 + a)^{n+1} = (1 + a)(1 + a)^n
\geq (1 + a)(1 + na) \quad \text{as } 1 + a > 0
= 1 + (n + 1)a + na^2
> 1 + (n + 1)a \quad \text{as } na^2 > 0.
\]

So, by the Principle of Mathematical Induction \( (1 + a)^n \geq 1 + na \) for all counting numbers \( n \).

Question 4. Let \( A, B \) be subsets of \( X \). We use Venn diagrams to illustrate, successively, the subsets \( A \cup B, A', B', A' \cap B' \) and \( (A' \cap B)' \).
Since the first and last diagrams agree, the sets they depict also agree.